

Note on some variational problem related to statistical solutions of differential equations in Banach spaces

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Abstract. The properties of statistical solutions for some general differential equations in Banach spaces are investigated.

1. Introduction. Let X be an infinite-dimensional reflexive real Banach space with a basis $\{x_n\}$, endowed with a norm $\|\cdot\|$. Then $(X^*, \|\cdot\|_*)$ denotes the dual space of X . The value of a functional $u^* \in X^*$ at a point u will be denoted by $\langle u^*, u \rangle$. Let X_n be a subspace of X generated by x_1, \dots, x_n and let P_n denote the projection operator of X onto X_n given by the formula

$$P_n x = \sum_{k=1}^n \langle x_k^*, x \rangle x_k, \quad x \in X.$$

Analogously, for the space X_n^* generated by x_1^*, \dots, x_n^* the projection operator Q_n of X^* onto X_n^* is given by the formula

$$Q_n x^* = \sum_{k=1}^n \langle x^*, x_k \rangle x_k^*, \quad x^* \in X^*.$$

Let $S = [0, T]$ and $F : S \times X \rightarrow X$ be a mapping satisfying the following conditions:

- (A₁) for every $t \in S$ the mapping $F(t, \cdot) : X \rightarrow X$ is continuous with respect to the weak topology,
- (A₂) for every $u \in X$ the mapping $F(\cdot, u) : X \rightarrow X$ is Bochner measurable,
- (A₃) there exist $b_1, b_2 > 0$ such that $\|F(t, u)\| \leq b_1 + b_2 \|u\|$ for $u \in X$, $t \in S$.

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We study the differential equation

$$(1.1) \quad u'(t) = F(t, u(t)).$$

In the classical setting, (1.1) is considered with an initial condition

$$(1.2) \quad u(0) = a,$$

but for physical reasons (e.g. turbulence phenomena) it is interesting to consider a statistical approach to (1.1) replacing a by a probability measure μ_0 .

In [3] we introduced the following definition.

DEFINITION 1.1. Let μ_0 be a given probability measure on $\mathcal{B}(X)$ (the σ -algebra of all Borel subsets of X) with bounded support. We say that a family $\{\mu(t, \cdot)\}_{t \in [0, T]}$ of probability measures is a *statistical solution* of (1.1) with initial measure μ_0 if the following conditions are satisfied:

- (1) For every $u^* \in X^*$ the function $[0, T] \ni t \mapsto \psi(t, u^*) \in \mathbb{C}$ is absolutely continuous ($\psi(t, \cdot)$ denotes the characteristic functional of the measure $\mu(t, \cdot)$).
- (2) For almost all $t \in [0, T]$ and for every $u^* \in X^*$,

$$(1.3) \quad \frac{\partial \psi(t, u^*)}{\partial t} = i \int_X \langle u^*, F(t, u) \rangle e^{i \langle u^*, u \rangle} \mu(t, du).$$

- (3) For every $u^* \in X^*$,

$$(1.4) \quad \psi(0, u^*) = \int_X e^{i \langle u^*, u \rangle} \mu_0(du).$$

Existence of a statistical solution was proven in [3]. Unfortunately, such a solution need not be unique. However, analysis of physical events related to this abstract model shows that frequently, apart from an initial condition, additional constraints should be considered. In what follows, we take into account (inspired by [1], [2], [4]) a functional defined on the set of statistical solutions and investigate under what conditions it has a minimum at exactly one point.

2. Variational problem. Let $\mathcal{C}_{b,w}(X)$ denote the vector space of all real functions on X which are weakly continuous and bounded. Let further $\mathcal{M}(X)$ denote the vector space of all bounded real measures on $\mathcal{B}(X)$. As is known, the space $\mathcal{M}(X)$ may be identified with a subspace of $[\mathcal{C}_{b,w}(X)]'$ (the algebraic dual). The space $\mathcal{M}(X)$ separates points in $\mathcal{C}_{b,w}(X)$ so there exists a locally convex topology in $\mathcal{C}_{b,w}(X)$ such that the topological dual $[\mathcal{C}_{b,w}(X)]^*$ is equal to $\mathcal{M}(X)$. Consequently, $\mathcal{M}(X)$ may be regarded as a locally convex topological space with the weak* topology. Let $\mathcal{C}(S, \mathcal{M}(X))$ denote the space of all continuous functions from S into $\mathcal{M}(X)$. This space is considered with the pointwise convergence topology.

Let W be the set of all families $\mu = \{\mu(t, \cdot)\}_{t \in S}$ of probability measures on $\mathcal{B}(X)$ which are statistical solutions of (1.1) with initial measure μ_0 and satisfy the conditions:

(A4) there exists $d > 0$ such that for all $\mu \in W, t \in S, R \in \mathbb{R}$,

$$\int_{E_R} \|u\| \mu(t, du) \leq d \int_{E_{R/d-1}} (1 + \|u\|) \mu_0(du),$$

where $E_R = \{u \in X : \|u\| > R\}$,

(A5) the mapping $J: S \ni t \mapsto \int_X f(u) \mu(t, du)$ belongs to $L^\infty(S, \mathbb{R})$ for all $f \in \mathcal{C}(X)$ such that $|f(u)| \leq c(1 + \|u\|)$ for $u \in X$.

Let $\mathcal{F}: \mathcal{C}(S, \mathcal{M}(X)) \supset W \rightarrow \mathbb{R}$ be a functional which is

- (A6) sequentially lower semicontinuous,
- (A7) strongly convex.

THEOREM 2.1. *Assume that $W \neq \emptyset$. Then there exists exactly one $\bar{\mu} \in W$ such that*

$$\mathcal{F}(\bar{\mu}) \leq \mathcal{F}(\mu), \quad \mu \in W.$$

Our proof of Theorem 2.1 is based upon the following two lemmas.

LEMMA 2.2. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathcal{B}(X)$ such that*

$$(1) \quad \int_X \|u\| \mu_n(du) \leq M, \quad n \in \mathbb{N}.$$

Assume that the sequence $\{\psi_n\}$ of characteristic functionals of μ_n is point-wise convergent to a continuous functional $\psi: X^ \rightarrow \mathbb{C}$. Then there exists a probability measure μ on $\mathcal{B}(X)$ such that*

$$\int_X \|u\| \mu(du) < \infty, \quad \psi(u^*) = \int_X e^{i\langle u^*, u \rangle} \mu(du), \quad u^* \in X^*.$$

Moreover, for every $f \in \mathcal{C}_b(X, \mathbb{R})$ and $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \int_X f(P_k u) \mu_n(du) = \int_X f(P_k u) \mu(du).$$

In the case of a Hilbert space the proof of this lemma can be found in [4]. If X is a reflexive Banach space with basis the proof is similar.

LEMMA 2.3. *The set W is sequentially compact in $\mathcal{C}(S, \mathcal{M}(X))$.*

Proof. Let $\psi_n(t, \cdot)$ be the sequence of characteristic functionals of measures $\mu_n(t, \cdot), t \in S, n \in \mathbb{N}$. In view of the definition of statistical solution and (A4) we have

$$(2.1) \quad |\psi_n(t_1, u^*) - \psi_n(t_2, u^*)| \leq m_1 |t_1 - t_2|, \quad u^* \in X^*, t_1, t_2 \in S,$$

where $m_1 > 0$ is independent of n .

Directly from the definition of characteristic functional we have

$$(2.2) \quad |\psi_n(t, u^*) - \psi_n(t, v^*)| \leq m_2 \|u^* - v^*\|, \quad t \in S, u^*, v^* \in X^*,$$

where $m_2 > 0$ is independent of n and t .

By inequalities (2.1), (2.2), the family $\psi_n, n \in \mathbb{N}$, is equicontinuous on $S \times X$. As the sequence ψ_n is bounded, we can choose a subsequence $\psi_{n'}$ which is convergent to a continuous functional ψ .

By Lemma 2.2, we know that ψ is the characteristic functional of a probability measure $\mu(t, \cdot), t \in S$. It is straightforward to show that the mapping $S \ni t \mapsto \mu(t, \cdot) \in \mathcal{M}(X)$ is an element of W such that $\mu_{n'} \mapsto \mu$ in the sense of $\mathcal{C}(S, \mathcal{M}(X))$. ■

Proof of Theorem 2.1. By Lemma 2.3 and assumption (A6), the proof is similar to the proof of the classical extreme value theorem. Uniqueness of $\bar{\mu}$ is an obvious consequence of (A7). ■

In what follows, we study the properties of the statistical solution constructed in [3]. In particular, we show that, assuming some kind of regularity of F , the demand $W \neq \emptyset$ is not necessary.

First, we recall some facts for completeness.

Set

$$(2.3) \quad \mu_n^0 = \mu_0(P_n^{-1}(\omega \cap X_n)), \quad \omega \in \mathcal{B}(X), n \in \mathbb{N}.$$

DEFINITION 2.4. A *statistical approximate solution* of (1.1) is a family $\{\mu_n(t, \cdot)\}_{t \in [0, T]}$ of probability measures on $\mathcal{B}(X)$ such that

- (1) $\text{supp } \mu_n(t, \cdot) \subset X_n, t \in S$.
- (2) For every $u^* \in X^*$ the characteristic functional of the measure $\mu_n(t, \cdot), [0, T] \ni t \mapsto \psi_n(t, u^*) \in \mathbb{C}$, is absolutely continuous.
- (3) For almost all $t \in [0, T]$ and for every $u^* \in X^*$,

$$(2.4) \quad \frac{\partial \psi_n(t, u^*)}{\partial t} = i \int_X \langle Q_n u^*, F(t, u) \rangle e^{i \langle u^*, u \rangle} \mu_n(t, du).$$

- (4) For every $u^* \in X^*$,

$$(2.5) \quad \psi_n(0, u^*) = \int_X e^{i \langle u^*, u \rangle} \mu_n^0(du).$$

For every $n \in \mathbb{N}$ and $t \in S$ we set

$$F_n(t, \cdot) = P_n \circ F(t, \cdot)|_{X_n}.$$

The mapping $F_n(t, \cdot)$, written in the coordinates with respect to the basis of X , gives n scalar functions

$$F_n^i : S \times \mathbb{R}^n \ni (t, u_1, \dots, u_n) \rightarrow F_n^i(t, u_1, \dots, u_n) \in \mathbb{R}, \quad i = 1, \dots, n.$$

It is assumed that

- (i) for all $n \in \mathbb{N}$, $t \in S$ and $i = 1, \dots, n$ the functions $F_n^i(t, \cdot)$ are of class $\mathcal{C}^1(\mathbb{R}^n)$,
- (ii) for all $n \in \mathbb{N}$ and $i = 1, \dots, n$ there exists $k_n^i \in L^2(S, \mathbb{R})$ such that for all $(u_1, \dots, u_n) \in \mathbb{R}^n$ and for a.a. $t \in S$,

$$\left| \frac{\partial F_n^i(t, u_1, \dots, u_n)}{\partial u_k} \right| \leq k_n^i(t), \quad k = 1, \dots, n.$$

THEOREM 2.5. *Let S_n be the translation along solution of the problem*

$$\begin{aligned} u_n'(t) &= F_n(t, u_n(t)), \\ u_n(0) &= P_n a. \end{aligned}$$

Let $\{\mu_n\}$ be the family of measures given by the formula

$$(2.6) \quad \mu_n(t, \omega) = \mu_0(P_n^{-1}(S_n^{-1}(t, \omega))), \quad \omega \in \mathcal{B}(X), t \in S.$$

Then $\{\mu_n(t, \cdot)\}_{t \in [0, T]}$ is a statistical approximate solution of (1.1) with initial measure μ_0 . Moreover, there is $d > 0$ such that for every $R \in \mathbb{R}$,

$$(2.7) \quad \int_{E_R} \|u\| \mu_n(t, du) \leq d \int_{E_{R/d-1}} (1 + \|u\|) \mu_0(du).$$

THEOREM 2.6. *There exists a statistical solution of (1.1) with initial measure μ_0 .*

Sketch of proof. Let $\psi_n : S \times X^* \rightarrow \mathbb{C}$, $n \in \mathbb{N}$, be the characteristic functional of μ_n . In virtue of the Ascoli theorem there exists a subsequence $\{\psi_{n'}\}$ such that

$$\lim_{n' \rightarrow \infty} \psi_{n'}(t, u^*) = \psi(t, u^*), \quad (t, u^*) \in S \times X^*,$$

where $\psi : S \times X^* \rightarrow \mathbb{C}$ is a continuous functional. Without loss of generality we may assume that $\psi_{n'}$ coincides with ψ_n .

By Lemma 2.2 we can show that there exists a family of $\{\mu(t, \cdot)\}$ on $\mathcal{B}(X)$ such that

$$\begin{aligned} \int_X \|u\| \mu(t, du) &< \infty, \quad t \in S, \\ \psi(t, u^*) &= \int_X \exp(i\langle u^*, u \rangle) \mu(t, du), \quad u^* \in X^*. \end{aligned}$$

This family is a statistical solution of (1.1) with initial measure μ_0 . For more details we refer the reader to [3]. ■

THEOREM 2.7. *The statistical solution constructed above satisfies conditions (A4), (A5) (with d taken from (2.7)).*

Proof. It can be shown that for $t \in S$, $k \in \mathbb{N}$, $R \in \mathbb{R}$,

$$(2.8) \quad \liminf_{n \rightarrow \infty} \int_{E_R(k)} \|P_k u\| \mu_n(t, du) \geq \int_{E_R(k)} \|P_k u\| \mu(t, du),$$

where $E_R(k) = \{u \in X : \|P_k u\| > R\}$.

As a consequence of (2.7) and (2.8), we have

$$(2.9) \quad \int_{E_R(k)} \|P_k u\| \mu(t, du) \leq d \int_{E_{R/d-1}} (1 + \|u\|) \mu_0(du).$$

This inequality, together with the observation that

$$\lim_{k \rightarrow \infty} \int_{E_R(k)} \|P_k u\| \mu(t, du) = \int_{E_R} \|u\| \mu(t, du),$$

finishes the proof that condition (A4) is satisfied.

We have

$$J = \lim_{k \rightarrow \infty} J_k,$$

where $J_k: S \ni t \mapsto \int_X f(P_k u) \mu(t, du)$, $k \in \mathbb{N}$, are continuous functions. Hence J is measurable and, by (A4), condition (A5) is easily verified. ■

An important example of a functional \mathcal{F} is related to the average value of a measure. Recall that if μ is a probability measure on $\mathcal{B}(X)$ with

$$\int_X \|u\| \mu(du) < \infty,$$

then there exists $E(\mu) \in X$ such that

$$\langle u^*, E(\mu) \rangle = \int_X \langle u^*, u \rangle \mu(du), \quad u^* \in X^*.$$

$E(\mu)$ is called the *average value* of μ .

For $\{\mu(t, \cdot)\}_{t \in S} \in W$ we set

$$\mathcal{F}(\{\mu(t, \cdot)\}_{t \in S}) = \int_0^T \|E(\mu(t, \cdot))\|^2 dt.$$

It is easily checked that the functional \mathcal{F} is lower semicontinuous and strongly convex on W .

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