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Coefficient inequalities for concave and meromorphically starlike univalent functions

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Abstract. Let $\mathbb{D}$ denote the open unit disk and $f : \mathbb{D} \to \overline{\mathbb{C}}$ be meromorphic and univalent in $\mathbb{D}$ with a simple pole at $p \in (0,1)$ and satisfying the standard normalization $f(0) = f'(0) - 1 = 0$. Also, assume that f has the expansion

$$f(z) = \sum_{n=-1}^{\infty} a_n (z-p)^n, \quad |z-p| < 1-p,$$

and maps $\mathbb{D}$ onto a domain whose complement with respect to $\overline{\mathbb{C}}$ is a convex set (starlike set with respect to a point $w_0 \in \mathbb{C}, w_0 \neq 0$ resp.). We call such functions concave (meromorphically starlike resp.) univalent functions and denote this class by $\mathrm{Co}(p)$ ($\Sigma^{\mathrm{s}}(p, w_0)$ resp.). We prove some coefficient estimates for functions in these classes; the sharpness of these estimates is also established.

1. Introduction. One of the most interesting questions in the theory of univalent functions is to find the region of variability of the nth Taylor (Laurent resp.) coefficient for functions f that are analytic (meromorphic resp.) and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. The leading example is the Bieberbach conjecture settled by de Branges in 1985 for the class $\mathcal{S}$ of normalized analytic univalent functions f in $\mathbb{D}$ although corresponding results for important subclasses of $\mathcal{S}$ are relatively easy and were settled positively much earlier.

In this paper, we consider the family $\mathrm{Co}(p)$ of functions $f : \mathbb{D} \to \overline{\mathbb{C}}$ that satisfy the following conditions:

(i) f is meromorphic in $\mathbb{D}$ and has a simple pole at the point $p \in (0,1)$ with the standard normalization $f(0) = f'(0) - 1 = 0$.
(ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\overline{\mathbb{C}}$ is convex.

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Each $f \in \mathrm{Co}(p)$ has the power series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n(f) z^n, \qquad |z| < p. \tag{1.1}$$

For our investigation, we consider the Laurent expansion of $f \in \mathrm{Co}(p)$ about the pole $z = p$:

$$f(z) = \sum_{n=-1}^{\infty} a_n (z-p)^n, \qquad z \in \Delta_p, \tag{1.2}$$

where $\Delta_p = \{z \in \mathbb{C} : |z-p| < 1-p\}$. Motivated by the works of Pfaltzgraff and Pinchuk [8], Miller [7], and Livingston [6], the class $\mathrm{Co}(p)$ has been investigated recently in [4, 1, 2, 3, 10]. A necessary and sufficient condition for a function f to be in $\mathrm{Co}(p)$ ([6]) is that $\mathrm{Re}\,\phi(z,f) > 0$ for all $z \in \mathbb{D}$, where

$$\phi(z,f) = -(1+p^2) + 2pz - \frac{(z-p)(1-pz)f''(z)}{f'(z)}, \qquad z \in \mathbb{D}.$$

Livingston [6] determined some estimates of the real part of $A_n(f)$ for $n = 2, 3$ when $f \in \mathrm{Co}(p)$ has the expansion (1.1). In the same article he conjectured an estimate for the real part of the general coefficient $A_n(f)$ $(n \geq 2)$ for $f \in \mathrm{Co}(p)$. After a long gap of ten years, positive developments have occurred in this line of work. For example, the recent work of Avkhadiev and Wirths [4] settles the conjecture of Avkhadiev, Pommerenke and Wirths [1] which, in particular, provides a proof of the Livingston conjecture. For ease of reference, we now recall it here.

Theorem A ([4]). *Let $n \geq 2$ and $p \in (0,1)$. For each $f \in \mathrm{Co}(p)$ with the expansion* (1.1) *the inequality*

$$\left| A_n(f) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)} \right| \leq \frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)} \tag{1.3}$$

is valid. Equality is attained in (1.3) *if and only if f is one of the functions f_θ, $\theta \in [0, 2\pi)$, where*

$$f_\theta(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\theta})z^2}{(1-z/p)(1-zp)}. \tag{1.4}$$

For each complex number in the disk described in (1.3) *there exists a function $f \in \mathrm{Co}(p)$ such that this number occurs as the nth Taylor coefficient of f.*

Interestingly, Wirths [10] established the following representation formula for functions in $\mathrm{Co}(p)$.

Theorem B ([10]). *For each $f \in \mathrm{Co}(p)$, there exists a function ω holomorphic in $\mathbb{D}$ such that $\omega(\mathbb{D}) \subset \overline{\mathbb{D}}$ and*

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{(1-z/p)(1-zp)}, \quad z \in \mathbb{D}. \tag{1.5}$$

The above representation formula has been used by the authors in [5] to obtain some other kind of coefficient estimates for functions in the class $\mathrm{Co}(p)$ with the Laurent expansion of the form (1.2).

In the present article, we first obtain certain coefficient estimates for functions in $\mathrm{Co}(p)$ with the same expansion of the form (1.2). Next we discuss a related class of meromorphically starlike functions, namely, the class $\Sigma^{\mathrm{s}}(p, w_0)$, and obtain a simple and easily applicable representation formula for this class. Using this formula, we also obtain some sharp coefficient estimates for functions in this class. As a consequence, we rectify a mistake in [6, Theorem 9].

Now, we state our first result.

THEOREM 1.1. *Let* $p \in (0, (\sqrt{5}-1)/2]$ *and* $f \in \mathrm{Co}(p)$ *have the expansion* (1.2). *Then*

$$\left| p - (1-p^2)\frac{a_0}{a_{-1}} \right| \le \frac{p}{|a_{-1}|}, \quad \textit{i.e.} \quad \left| a_{-1} - \frac{1-p^2}{p} a_0 \right| \le 1. \tag{1.6}$$

The inequality is sharp.

REMARK. In [10] Wirths has obtained the region of variability for $a_{-1}(f)$, namely, the inequality

$$\left| a_{-1} + \frac{p^2}{1-p^4} \right| \le \frac{p^4}{1-p^4} \quad \text{for } 0 < p < 1.$$

In [5], the domain of variability of $a_0(f)$ is determined by the inequality

$$\left| \frac{(1-p^2)a_0}{p} + \frac{1-p^2+p^4}{1-p^4} \right| \le \frac{p^2(2-p^2)}{1-p^4} \quad \text{for } p \in (0, \sqrt{3}-1].$$

Equality in each of the above two inequalities is attained if and only if f is one of the functions given in (1.4). ∎

The next result presents sharp coefficient estimates for all $n \ge 3$ if $f \in \mathrm{Co}(p)$ has the expansion (1.2).

THEOREM 1.2. *If* $f \in \mathrm{Co}(p)$ *with* $p \in (0,1)$ *and has the expansion* (1.2), *then for* $n \ge 3$ *we have*

$$\left| a_{n-2} - \frac{(1-p^2)a_{n-1}}{p} \right| \le \frac{p}{(1-p^4)(1-p)^{n-1}} \left[1 - \left(\frac{1-p^4}{p^4} \right)^2 \left| a_{-1} + \frac{p^2}{1-p^4} \right|^2 \right]. \tag{1.7}$$

Equality holds for the functions f_θ $(0 \le \theta \le 2\pi)$ *of the form* (1.4).

2. Proofs of Theorems 1.1 and 1.2

(2.1) *Proof of Theorem 1.1.* Let $f \in \mathrm{Co}(p)$. Then, by Theorem B, there exists a function ω holomorphic in $\mathbb{D}$ with $\omega(\mathbb{D}) \subset \overline{\mathbb{D}}$ satisfying the representation formula (1.5).

Now, let $f \in \mathrm{Co}(p)$ have the Laurent expansion (1.2) and let ω have the Taylor expansion

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \quad z \in \Delta_p. \tag{2.2}$$

Using these two expansions, the series formulation of (1.5) takes the form

$$\begin{aligned}(z-p)\left((z-p) - \frac{1-p^2}{p}\right) \sum_{n=-1}^{\infty} a_n (z-p)^n \\ = p + (z-p) - \frac{p}{1+p^2}\left(1 + \sum_{n=0}^{\infty} c_n (z-p)^n\right)((z-p)^2 + 2p(z-p) + p^2).\end{aligned} \tag{2.3}$$

Comparing the coefficients of $z-p$ on both sides of (2.3), we see that

$$a_{-1} - \frac{1-p^2}{p}\, a_0 = \frac{1-p^2}{1+p^2} - \frac{p^2}{1+p^2}\,(2c_0 + pc_1). \tag{2.4}$$

Using the classical Schwarz–Pick lemma, it follows that

$$|\omega'(p)| \le \frac{1-|\omega(p)|^2}{1-p^2}, \quad \text{i.e.} \quad |c_1| \le \frac{1-|c_0|^2}{1-p^2}.$$

In view of this observation, we have the estimate

$$|2c_0 + pc_1| \le \frac{p(1-|c_0|^2) + 2(1-p^2)|c_0|}{1-p^2}.$$

For convenience, we set $x = |c_0|$ and consider

$$R_p(x) = p(1-x^2) + 2(1-p^2)x.$$

We see that $R_p(x)$ attains a local maximum at $x_m = (1-p^2)/p$. Since $x_m \ge 1$ for $p \in (0, (\sqrt{5}-1)/2]$, we see that

$$|R_p(x)| \le R_p(1) = 2(1-p^2), \qquad x \in [0,1],\, p \in (0, (\sqrt{5}-1)/2],$$

and therefore, we have the estimate $|2c_0 + pc_1| \le 2$ for p in that interval. Now using this we get from (2.4) the estimate (1.6). It is a simple exercise to see that equality is attained in (1.6) for the function

$$f(z) = \frac{-zp}{(z-p)(1-zp)}. \quad \blacksquare$$

(2.5) *Proof of Theorem 1.2.* Let $f \in \mathrm{Co}(p)$ with the expansion (1.2). Next, following the notation of the proof of Theorem 1.1, we compare the coefficients of $(z-p)^n$ $(n \geq 3)$ on both sides of the equation (2.3). This gives

$$a_{n-2} - \frac{1-p^2}{p}\, a_{n-1} = -\frac{p}{1+p^2}\,(c_{n-2} + 2pc_{n-1} + p^2 c_n) \quad (n \geq 3).$$

Now, for a unimodular bounded analytic function ω in the unit disk $\mathbb{D}$ having the expansion (2.2) in Δ_p, we recall the following result due to Ruscheweyh [9, Theorem 2]:

$$(1-p)^n(1+p)|c_n| \leq 1 - |c_0|^2 \quad (n \geq 1)$$

where equality holds for $\omega(z) = e^{i\theta}, \theta \in [0, 2\pi)$. Using this, we easily obtain

$$\left| a_{n-2} - \frac{(1-p^2)a_{n-1}}{p} \right| \leq \frac{p(1-|c_0|^2)}{(1+p^2)(1+p)(1-p)^n} \quad (n \geq 3).$$

Consequently, (1.7) follows since

$$c_0 = \frac{1-p^4}{p^4}\, a_{-1} + \frac{1}{p^2},$$

by comparing the constant terms on both sides of (2.3). Now, equality holds of (1.7) for the functions f_θ, $\theta \in [0, 2\pi)$, of (1.4), since both sides of the inequality are zero. ■

3. Meromorphically starlike functions. Let $\Sigma^{\mathrm{s}}(p, w_0)$ denote the class of meromorphic and univalent functions f in $\mathbb{D}$ (with the standard normalization $f(0) = f'(0) - 1 = 0$) having a simple pole at $p \in (0,1)$ with the expansion (1.2) such that f is starlike with respect to a fixed $w_0 \in \mathbb{C}, w_0 \neq 0$ (i.e. $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike set with respect to w_0). A well-known fact [6] is that $f \in \Sigma^{\mathrm{s}}(p, w_0)$ if and only if $\operatorname{Re} \psi(z, f) > 0$ for all $z \in \mathbb{D}$, where

(3.1) $$\psi(z, f) = \frac{-(z-p)(1-pz)f'(z)}{f(z) - w_0}, \quad z \in \mathbb{D}.$$

We now prove a useful representation formula for functions in the class $\Sigma^{\mathrm{s}}(p, w_0)$.

THEOREM 3.1. *For $0 < p < 1$, let $f \in \Sigma^{\mathrm{s}}(p, w_0)$. Then there exists a function ω holomorphic in $\mathbb{D}$ such that $\omega(\mathbb{D}) \subset \overline{\mathbb{D}}$,*

$$\omega(0) = -\frac{1}{2}\left(\frac{1}{w_0} + p + \frac{1}{p}\right)$$

and

(3.2) $$f(z) = w_0 + \frac{pw_0(1 + z\omega(z))^2}{(z-p)(1-zp)}, \quad z \in \mathbb{D}.$$

Proof. The proof is a direct consequence of [12, Corollary 2] where the notation $\sigma^*(p, w_0)$ is used in place of $\Sigma^{\mathrm{s}}(p, w_0)$. By that corollary, if $f \in \Sigma^{\mathrm{s}}(p, w_0)$, then

$$\left| \left\{ \frac{f(z) - w_0}{pw_0} (z-p)(1-pz) \right\}^{1/2} - 1 \right| \le |z|, \qquad z \in \mathbb{D}.$$

Now writing

$$\omega(z) = \frac{1}{z} \left\{ \frac{f(z) - w_0}{pw_0} (z-p)(1-pz) \right\}^{1/2} - \frac{1}{z}$$

and simplifying the above expression for f we get the desired representation formula for functions in the class $\Sigma^{\mathrm{s}}(p, w_0)$. Here we note that ω is holomorphic in $\mathbb{D}$ and $|\omega(z)| \le 1$. Also since $f'(0) = 1$ we get $\omega(0) = -\frac{1}{2}(1/w_0 + p + 1/p)$. ∎

As a consequence of Theorem 3.1, we have the following result which has been proved in [11] by using a different method.

Corollary 3.1. *For $0 < p < 1$, let $f \in \Sigma^{\mathrm{s}}(p, w_0)$. Then*

$$\left| w_0 + \frac{p(1+p^2)}{(1-p^2)^2} \right| \le \frac{2p^2}{(1-p^2)^2}. \tag{3.3}$$

In particular,

$$\frac{p}{(1+p)^2} \le |w_0| \le \frac{p}{(1-p)^2}.$$

Proof. As $|\omega(0)| \le 1$ and $\omega(0) = -\frac{1}{2}(1/w_0 + p + 1/p)$, it follows that

$$\left| \frac{1}{w_0} + \frac{1+p^2}{p} \right| \le 2,$$

which is easily seen to be equivalent to the inequality (3.3). The other inequality is a simple consequence of (3.3). ∎

Theorem 3.2. *Let $f \in \Sigma^{\mathrm{s}}(p, w_0)$ have the Laurent expansion* (1.2). *Then*

(i) $$\left| a_{-1} - \frac{pw_0}{1-p^2} \right| \le \frac{p|w_0|}{1-p^2} \, \frac{|p/w_0 + p^2 + 1| + 2p^2}{2 + |p/w_0 + p^2 + 1|} \left(\frac{|p/w_0 + p^2 + 1| + 2p^2}{2 + |p/w_0 + p^2 + 1|} + 2 \right)$$

for $p \in (0, 1)$,

(ii) $$\left| a_0 - \frac{1 - p^2 + p^4}{(1-p^2)^2} w_0 \right| \le \frac{p(2 + 2p - p^3)}{(1-p^2)^2} |w_0|, \qquad p \in (0, (\sqrt{5} - 1)/2].$$

Both inequalities are sharp for

$$f(z) = \frac{-zp}{(z-p)(1-pz)} = w_0 + \frac{pw_0}{(z-p)(1-pz)} (1-z)^2 \in \Sigma^s(p, w_0)$$

where $w_0 = -p/(1+p)^2$.

Proof. Consider the Taylor expansion for ω:

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad z \in \Delta_p.$$

Now substituting (1.2) and (2.2) in the representation formula (3.2) we get the following series formulation of (3.2) valid in Δ_p:

$$\begin{aligned} &\sum_{n=-1}^{\infty} a_n (z-p)^n - w_0 \\ &= \frac{pw_0}{1-p^2} \sum_{n\geq 0} \left(\frac{p}{1-p^2}\right)^n (z-p)^{n-1} \\ &\quad \times \Bigg[1 + \{(z-p)^2 + p^2 + 2p(z-p)\} \sum_{n\geq 0} \left(\sum_{k=0}^{n} c_k c_{n-k}\right)(z-p)^n \\ &\quad + \{2p + 2(z-p)\} \sum_{n\geq 0} c_n (z-p)^n\Bigg]. \end{aligned} \tag{3.4}$$

In deriving the above expression, we make use of the relations

$$\begin{aligned} (z-p)(1-pz) &= (1-p^2)(z-p)\left(1 - \frac{p}{1-p^2}(z-p)\right), \\ (1+z\omega(z))^2 &= 1 + 2(z-p+p)w(z) \\ &\qquad + ((z-p)^2 + 2p(z-p) + p^2)w(z)w(z). \end{aligned}$$

Now, we proceed to prove (i). Comparing the coefficients of $1/(z-p)$ on both sides of (3.4) we get

$$a_{-1} = \frac{pw_0}{1-p^2}[1 + p^2 c_0^2 + 2pc_0].$$

The Schwarz–Pick lemma applied to ω shows that

$$|c_0| = |\omega(p)| \leq \frac{|\omega(0)| + p}{1 + p|\omega(0)|},$$

where

$$|\omega(0)| = \frac{1}{2}\left|\frac{1}{w_0} + p + \frac{1}{p}\right|.$$

Using this, we get the desired estimate for a_{-1}. It is also easy to check that (i) is sharp for the function given in the statement.

(ii) Comparing the constant terms on both sides of (3.4) we get

$$a_0 - w_0 = \frac{p^2 w_0}{(1-p^2)^2}(1 + p^2 c_0^2 + 2pc_0) + \frac{2pw_0}{1-p^2}(p^2 c_0 c_1 + p^2 c_0^2 + pc_1 + c_0),$$

or equivalently,

$$a_0-\frac{1-p^2+p^4}{(1-p^2)^2}\,w_0=\frac{p^2w_0}{(1-p^2)^2}\,(p^2c_0^2+2pc_0)+\frac{2pw_0}{1-p^2}\,(p^2c_0c_1+p^2c_0^2+pc_1+c_0).$$

Now, we recall the estimates from the Schwarz–Pick lemma:

$$|c_0|\le 1,\quad |c_1|\le\frac{1-|c_0|^2}{1-p^2}.$$

For convenience, we use the notation $x=|c_0|$. Using the above estimates, it is easy to see that the last equality implies that

$$\left|a_0-\frac{1-p^2+p^4}{(1-p^2)^2}\,w_0\right|\le\frac{p|w_0|}{(1-p^2)^2}\,(2p+2x+2p^2x-2p^2x^3-p^3x^2).$$

Next, we introduce

$$Q_p(x)=2p+2x+2p^2x-2p^2x^3-p^3x^2,\quad 0\le x\le 1.$$

Then Q_p attains a local maximum at

$$x_m=(-p^2+\sqrt{p^4+12(1+p^2)}\,)/(6p).$$

Since $x_m\ge 1$ for $p\in(0,(\sqrt{5}-1)/2]$, we have

$$\max\{Q_p(x):x\in[0,1]\}=Q_p(1)=2+2p-p^3.$$

This proves inequality (ii), and the sharpness can easily be verified for the function given in the statement. ■

Remark. It is a simple exercise to see that

$$\frac{|p/w_0+p^2+1|+2p^2}{2+|p/w_0+p^2+1|}\le p \tag{3.5}$$

is equivalent to

$$\frac{1}{2}\left|\frac{p}{w_0}+p^2+1\right|\le p,\quad\text{i.e.}\quad |\omega(0)|\le 1.$$

Thus, (3.5) holds. If we use the inequality (3.5) then inequality (i) of Theorem 3.2 turns out to be

$$\left|a_{-1}-\frac{pw_0}{1-p^2}\right|\le\frac{p^2}{1-p^2}\,(p+2)|w_0|,\quad p\in(0,1).$$

Applying the triangle inequality and inequality (ii) of Theorem 3.2 we get

$$|a_{-1}|\le\frac{p(1+p)}{1-p}\,|w_0|,\qquad p\in(0,1), \tag{3.6}$$

and

$$|a_0|\le\frac{1}{(1-p)^2}\,|w_0|,\qquad p\in(0,(\sqrt{5}-1)/2],$$

respectively. Both the above estimates are sharp for the function stated in Theorem 3.2.

The estimate (3.6) shows that there was a minor error in one of the results of Livingston, namely Theorem 9 in [6]. Indeed, a counterexample is given by the function

$$g(z)=\frac{-zp}{(z-p)(1-pz)}\in\Sigma^{\mathrm{s}}\left(p,\frac{-p}{1+p^2}\right).$$

Here we note that

$$a_{-1}(g)=\frac{-p^2}{1-p^2}$$

does not belong to the disk stated in Theorem 9 of [6]. Moreover, the error actually occurred in [6, p. 290] where the inequality in the 6th line needs to be reversed, since $\xi-p\leq 0$. We can now formulate a corrected version of [6, Theorem 9] for future use.

THEOREM 3.3. *If* $f\in\Sigma^{\mathrm{s}}(p,w_0)$ *and has the Laurent expansion* (1.2), *then*

$$|a_{-1}|\geq\frac{p(1-p)}{1+p}\,|w_0|.$$

The inequality is sharp for the function

$$g(z)=\frac{-zp}{(z-p)(1-pz)}=w_0+\frac{pw_0}{(z-p)(1-pz)}\,(1-z)^2\in\Sigma^{\mathrm{s}}(p,w_0)$$

where $w_0=-p/(1-p)^2$.

Here we also note that

$$\mathrm{Re}\left(\frac{(z-p)(1-zp)g'(z)}{f(z)+\frac{p}{(1-p)^2}}\right)=-(1-p)^2\mathrm{Re}\left(\frac{1+z}{1-z}\right)<0$$

for all $z\in\mathbb{D}$ and g satisfies the normalization condition $g(0)=0=g'(0)-1$ whenever $w_0=-p/(1-p)^2$.

REMARK. In view of the last theorem, the corollary that follows from Theorem 9 in [6] is also not true since it uses the incorrect estimate for $|a_{-1}|$. ∎

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