Lifting distributions to the cotangent bundle

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Abstract. A classification of all \( M_f \)-natural operators \( A : \text{Gr}_p \rightsquigarrow \text{Gr}_q T^* \) lifting \( p \)-dimensional distributions \( D \subset TM \) on \( m \)-manifolds \( M \) to \( q \)-dimensional distributions \( A(D) \subset TT^*M \) on the cotangent bundle \( T^*M \) is given.

1. A \( p \)-dimensional distribution on a manifold \( M \) is a smooth \((C^\infty)\) vector subbundle \( D \subset TM \) of the tangent bundle of \( M \) such that \( \dim(D_x) = p \) for any point \( x \in M \). Thus a \( p \)-dimensional distribution \( D \) is a smooth \((C^\infty)\) section of the Grassmann bundle \( \text{Gr}_p M \) of \( M \).

Given a manifold \( M \), let \( T^*M = (TM)^* \) be the cotangent bundle of \( M \). Every embedding \( f : M \rightarrow N \) of \( m \)-manifolds induces a vector bundle embedding \( T^*f := (T(f^{-1}))^* : T^*M \rightarrow T^*N \) covering \( f \), where \( Tf \) denotes the differential of \( f \).

A general concept of natural operators can be found in [1]. We need the following partial definition of natural operators only.

Let \( m, p \) and \( q \) be integers such that \( m \geq 1 \), \( 0 \leq p \leq m \) and \( 0 \leq q \leq 2m \). An \( M_f \)-natural operator \( A : \text{Gr}_p \rightsquigarrow \text{Gr}_q T^* \) lifting \( p \)-dimensional distributions \( D \) from \( m \)-manifolds \( M \) to \( q \)-dimensional distributions \( A(D) \) on the cotangent bundle \( T^*M \) is a family of \( M_f \)-invariant regular operators \( A : \text{Gr}_p(M) \rightarrow \text{Gr}_q(T^*M) \) from the set \( \text{Gr}_p(M) \) of all \( p \)-dimensional distributions on \( M \) (sections of \( \text{Gr}_p M \rightarrow M \)) into the set \( \text{Gr}_q(T^*M) \) of all \( q \)-dimensional distributions on \( T^*M \) (sections of \( \text{Gr}_q(T^*M) \rightarrow T^*M \)) for any \( M_f \)-object \( M \), where \( M_f \) is the category of \( m \)-dimensional manifolds and their embeddings. The invariance means that if \( D_1 \in \text{Gr}_p(M) \) and \( D_2 \in \text{Gr}_p(N) \) are \( f \)-related (i.e. \( \text{Gr}_f f \circ D_1 = D_2 \circ f \)) for some \( M_f \)-map \( f : M \rightarrow N \) then \( A(D_1) \) and \( A(D_2) \) are \( T^*f \)-related. (We recall that \( \text{Gr}_f f : \text{Gr}_p M \rightarrow \text{Gr}_p N \) is the map induced by \( f \) given by \( \text{Gr}_f f(W) = Tf(W) \) for any \( W \in (\text{Gr}_p M)_x \), i.e. for

\[2000 \text{Mathematics Subject Classification}: \text{58A20, 58A32.} \]

\textit{Key words and phrases}: distribution, Grassmann bundle, natural operator.

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any \( p \)-dimensional subspace \( W \subset T_x M, x \in M \). The \textit{regularity} means that \( A \) transforms smoothly parametrized families of distributions into smoothly parametrized families of distributions.

We have the following \( \mathcal{M}_f m \)-natural operators \( A : \text{Gr}_p \hookrightarrow \text{Gr}_q T^* \).

**Example 1.** \( A^{[1]}(D)_\omega = \{0\} \) for all \( \omega \in (T^* M)_x, x \in M, D \in \text{Gr}_p(M) \).

**Example 2.** \( A^{[2]}(D)_\omega = \{ \frac{d}{dt}(\omega + t\sigma)_{t=0} \in T\omega T^* M : \sigma \in \text{Ann}(D_\omega) \} \) for all \( \omega \in (T^* M)_x, x \in M, D \in \text{Gr}_p(M) \), where \( \text{Ann}(W) = \{ \sigma \in (T_x M)^* | \langle \sigma, w \rangle = 0 \text{ for all } w \in W \} \) is the annihilator of a vector subspace \( W \subset T_x M \).

**Example 3.** \( A^{[3]}(D)_\omega = \ker(T\omega \pi_M) \) for all \( \omega \in (T^* M)_x, x \in M, D \in \text{Gr}_p(M) \), where \( \pi_M : T^* M \rightarrow M \) is the cotangent bundle projection.

**Example 4.** \( A^{[4]}(D)_\omega = (T\omega \pi_M)^{-1}(D_x) \) for all \( \omega \in (T^* M)_x, x \in M, D \in \text{Gr}_p(M) \).

**Example 5.** \( A^{[5]}(D)_\omega = T\omega T^* M \) for all \( \omega \in (T^* M)_x, x \in M, D \in \text{Gr}_p(M) \).

The main result in this paper is the following theorem.

**Theorem 1.** All \( \mathcal{M}_f m \)-natural operators \( A : \text{Gr}_p \hookrightarrow \text{Gr}_q T^* \) are described in Examples 1–5.

From now on let \( x^1, \ldots, x^m \) be the usual coordinates on \( \mathbb{R}^m \). Let \( D^p \) be the integrable \( p \)-dimensional distribution on \( \mathbb{R}^m \) spanned by \( \partial/\partial x^1, \ldots, \partial/\partial x^p \).

2. First we prove the following proposition.

**Proposition 1.** All \( \mathcal{M}_f m \)-natural operators \( A : \text{Gr}_p \hookrightarrow \text{Gr}_q T^* \) such that \( A(D) \subset VT^* M = \ker(T\pi_M) \) for any \( D \in \text{Gr}_p(M) \) are described in Examples 1–3.

**Lemma 1.** Let \( A^1, A^2 : \text{Gr}_p \hookrightarrow \text{Gr}_q T^* \) be \( \mathcal{M}_f m \)-natural operators such that \( A^1(D) \subset VT^* M \) and \( A^2(D) \subset VT^* M \) for any \( D \in \text{Gr}_p(M) \). Assume \( A^1(D^p)_{\theta} = A^2(D^p)_{\theta} \), where \( \theta \in (T^* \mathbb{R}^m)_0 \) is the zero element. Then \( A^1(D)_{\omega} = A^2(D)_{\omega} \) for any \( p \)-dimensional distribution \( D \) on an \( m \)-manifold \( M \) and any \( \omega \in (T^* M)_x, x \in M \), i.e. \( A^1 = A^2 \).

**Proof.** By the \( \mathcal{M}_f m \)-invariance of \( A^1 \) and \( A^2 \) we may assume that \( M = \mathbb{R}^m, x = 0, D_0 = D^p_0 \) and \( \omega \in (T^* \mathbb{R}^m)_0 \).

We have the standard identification \( I_\omega : V_\omega T^* \mathbb{R}^m = T_\omega (T^* \mathbb{R}^m)_0 \rightarrow (T^* \mathbb{R}^m)_0 = \mathbb{R}^m, \omega \in (T^* \mathbb{R}^m)_0 \). Using the invariance of \( A^i \) with respect to the homotheties \( a_t = \left( \frac{1}{t} x^1, \ldots, \frac{1}{t} x^m \right) \) for \( t > 0 \) we deduce that

\[ I_{t\omega}(A^i((\text{Gr}_p((a_t)_* D)_{t\omega}))) = I_\omega(A^i(D)_\omega) \]
for $i = 1, 2$. Letting $t \to 0$ we deduce that $I_{\omega}(A^i(D)_\omega) = I_\theta(A^i(D)_\theta)$ for $i = 1, 2$. Then $A^1(D)_\omega = A^2(D)_\omega$ because of the assumption of the lemma. ■

**Lemma 2.** Let $A : \text{Gr}_p \rightsquigarrow \text{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator such that $A(D) \subset VT^*M$ for any $D \in \text{Gr}_p(M)$. Then $A(D^p)_\theta = A^i(D^p)_\theta$ for some $i = 1, 2, 3$.

**Proof.** We may assume that $A(D^p)_\theta \neq \{0\}$.

If there is a point $v \in A(D^p)_\theta \setminus A^{[2]}(D^p)_\theta$, then using the invariance of $A$ with respect to linear isomorphisms preserving $D^p$ we see that $V_\theta T^*\mathbb{R}^m \setminus A^{[2]}(D^p)_\theta \subset A(D^p)_\theta$ (if $w \in V_\theta T^*\mathbb{R}^m \setminus A^{[2]}(D^p)_\theta$ then there exists a linear isomorphism $\psi$ preserving $D^p$ such that $w = TT^*\psi(v)$, and thus $w \in A(D^p)_\theta$ because of the invariance of $A$ with respect to $\psi$ and $v \in A(D^p)_\theta$); consequently, $A(D^p)_\theta = A^{[3]}(D^p)_\theta$.

So we may assume $A(D^p)_\theta \subset A^{[2]}(D^p)_\theta$. There is an element $v \in A(D^p)_\theta$, $v \neq 0$. Then using the invariance of $A$ with respect to linear isomorphisms preserving $D^p$ we see that $A^{[2]}(D^p)_\theta \subset A(D^p)_\theta$. Thus $A(D^p)_\theta = A^{[2]}(D^p)_\theta$. ■

**Proof of Proposition 1.** Proposition 1 is an immediate consequence of Lemmas 1 and 2. ■

3. Now, we prove the following proposition.

**Proposition 2.** Let $A : \text{Gr}_p \rightsquigarrow \text{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator such that $A(D) \subset A^{[4]}(D)$ for any $D \in \text{Gr}_p(M)$ and $A(D^p)_\theta \setminus V_\theta T^*\mathbb{R}^m \neq \emptyset$. Then $A = A^{[4]}$.

**Lemma 3.** Under the assumptions of Proposition 2 we have the inclusion $A(D^p)_\theta \supset V_\theta T^*\mathbb{R}^m$.

**Proof.** There exists $a > 0$ such that $A(D^p)_{ad_0x^1} \setminus V_{ad_0x^1}T^*\mathbb{R}^m \neq \emptyset$. Then using the invariance of $A$ with respect to the homotheties $b_t = (tx^1, \ldots, tx^m)$ for $t > 0$ we find that $A(D^p)_{d_0x^1} \setminus V_{d_0x^1}T^*\mathbb{R}^m \neq \emptyset$.

There exist real numbers $a_1, \ldots, a_m, b_1, \ldots, b_p \in \mathbb{R}$ such that

\[ Y = a_1 \frac{d}{dt}[d_0x^1 + td_0x^1]_{t=0} + \cdots + a_m \frac{d}{dt}[d_0x^1 + td_0x^m]_{t=0} + b_1 T^* \left( \frac{\partial}{\partial x^1} \right)_{d_0x^1} + \cdots + b_p T^* \left( \frac{\partial}{\partial x^p} \right)_{d_0x^1} \in A(D^p)_{d_0x^1} \]

and $b_q \neq 0$ for some $q \in \{1, \ldots, p\}$, where $T^*(X)$ denotes the flow lifting of a vector $X$ on $M$ to $T^*M$.

Consider $k \in \{1, \ldots, m\}$. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism such that $\varphi^{-1}(y^1, \ldots, y^m) = (y^1 + y^q y^k, y^2, \ldots, y^m)$ on some open neighborhood of $0 \in \mathbb{R}^m$. Then $\varphi$ preserves $d_0x^1$ and $D^p$. By standard verification one can
show that
\[ TT^* \varphi(Y) = Y + b_q \frac{d}{dt} \left[ d_0 x^1 + td_0 x^k \right]_{t=0} + b_k \frac{d}{dt} \left[ d_0 x^1 + td_0 x^q \right]_{t=0}. \]
By the invariance of \( \varphi \) with respect to \( \varphi \) we have \( TT^* \varphi(Y) \in A(D^p)_{d_0x^1} \), and so
\[ b_q \frac{d}{dt} \left[ d_0 x^1 + td_0 x^k \right]_{t=0} + b_k \frac{d}{dt} \left[ d_0 x^1 + td_0 x^q \right]_{t=0} \in A(D^p)_{d_0x^1}. \]

Putting \( k = q \) we find \( \frac{d}{dt} \left[ d_0 x^1 + td_0 x^q \right]_{t=0} \in A(D^p)_{d_0x^1} \), and hence \( \frac{d}{dt} \left[ d_0 x^1 + td_0 x^k \right]_{t=0} \in A(D^p)_{d_0 x^1} \). Thus \( A(D^p)_{d_0 x^1} \supset V_{d_0 x^1} T^* \mathbb{R}^m \). Then using the invariance of \( \varphi \) with respect to the homotheties \( a_t = (\frac{1}{t} x^1, \ldots, \frac{1}{t} x^m) \) for \( t > 0 \) and next letting \( t \to 0 \) we deduce that \( A(D^p)_{\varphi} \supset V_{\varphi} T^* \mathbb{R}^m \).

**Proof of Proposition 2.** By Lemma 3 and the assumptions of the proposition, there exists a constant vector field \( W = a_1 \partial/\partial x^1 + \cdots + a_p \partial/\partial x^p \neq 0 \) on \( \mathbb{R}^m \) such that \( T^*(W)_{\varphi} \in A(D^p)_{\varphi} \). Using the invariance of \( \varphi \) with respect to linear isomorphisms preserving \( D^p \) we deduce that \( T^*(\partial/\partial x^i)_{\varphi} \in A(D^p)_{\varphi} \) for \( i = 1, \ldots, p \). Then using Lemma 3 we get \( A(D^p)_{\varphi} = A^{[4]}(D^p)_{\varphi} \). Thus \( q = \dim(A(D^p)_{\varphi}) = m + p \). Hence \( A(D) \subset A^{[4]}(D) \) is an \((m + p)\)-dimensional distribution for any \( D \in \text{Gr}_p(M) \). Therefore \( A(D) = A^{[4]}(D) \) for any \( D \in \text{Gr}_p(M) \), i.e. \( A = A^{[4]} \).

4. Quite similarly to Section 3 we prove the next proposition.

**Proposition 3.** Let \( A : \text{Gr}_p \sim \text{Gr}_q T^* \) be an \( M_{f_m} \)-natural operator such that \( A(D^p)_{\varphi} \setminus A^{[4]}(D^p)_{\varphi} \neq \emptyset \). Then \( A = A^{[5]} \).

**Lemma 4.** Under the assumptions of Proposition 4 we have the inclusion \( A(D^p)_{\varphi} \supset V_{\varphi} T^* \mathbb{R}^m \).

**Proof.** We modify the proof of Lemma 3 by replacing \( p \) by \( m \) wherever appropriate.

**Proof of Proposition 3.** By Lemma 4 and the assumptions of the proposition, there exists a constant vector field \( W = a_1 \partial/\partial x^1 + \cdots + a_m \partial/\partial x^m \) with \( a_i \neq 0 \) for some \( i = p + 1, \ldots, m \) such that \( T^*(W)_{\varphi} \in A(D^p)_{\varphi} \). Then using the invariance of \( A \) with respect to the linear isomorphisms preserving \( D^p \) we deduce that \( T^*(U)_{\varphi} \in A(D^p)_{\varphi} \) for any constant vector field \( U = c_1 \partial/\partial x^1 + \cdots + c_m \partial/\partial x^m \) with \( c_j \neq 0 \) for some \( j = p + 1, \ldots, m \). Hence using Lemma 4 we get \( A(D^p)_{\varphi} = T_{\varphi} T^* \mathbb{R}^m \). Thus \( q = 2m \) and consequently \( A = A^{[5]} \).

5. **Proof of Theorem 1.** Let \( A : \text{Gr}_p \sim \text{Gr}_q T^* \) be an \( M_{f_m} \)-natural operator. By the \( M_{f_m} \)-invariance of \( A \) and Proposition 1 we may assume that \( A(D)_{\omega} \setminus V_{\omega} T^* \mathbb{R}^m \neq \emptyset \) for some \( D \in \text{Gr}_p(\mathbb{R}^m) \) and \( \omega \in (T^* \mathbb{R}^m)_{\omega} \). If \( p \geq 1 \),
by a density argument we may assume that $\omega \notin \operatorname{Ann}(D_0)$, and then using
the invariance of $A$ with respect to linear isomorphisms we may additionally
assume that $\omega = d_0x^1$ and $D_0 = D^p_0$. (The case $p = 0$ is of course
“equivalent” to the case $p = m$. In these two cases we have only canonical
distributions on $T^*M$—as trivial distributions give no information.) Let $v \in A(D)_{d_0x^1} \setminus V_{d_0x^1}T^*\mathbb{R}^m$. Then standardly one can show that there exist a
constant vector field $Y_0 \neq 0$ and a vector field $Y_1$ with $Y_1(0) = 0$ such that
$T^*(Y)_{d_0x^1} = v$, where $Y = Y^0 + Y^1$. There exists a local diffeomorphism
$\psi$ such that $j_0^1 \psi = \text{id}$ and $\psi_* Y = Y^0$ near 0. Using the invariance of $A$
with respect to $\psi$ we may additionally assume $v = T^*(Y^0)_{d_0x^1}$. Then using
the invariance of $A$ with respect to the homotheties $a_t = (\frac{1}{t}x^1, \ldots, \frac{1}{t}x^m)$
for $t > 0$ and letting $t \to 0$ we have $T^*(Y^0)_{\theta} \in A(D^p)_{\theta} \setminus V_{\theta}T^*\mathbb{R}^m$. Thus
$A(D)_{\theta} \setminus A[4]_\theta \neq \emptyset$.

If $A(D) \subset A[4](D)$ for any $D \in \text{Gr}_p(M)$, then $A = A[4]$ because of
Proposition 2.

Otherwise we may assume that $A(D)_{\omega} \setminus A[4](D)_{\omega} \neq \emptyset$ for some
$D \in \text{Gr}_p(\mathbb{R}^m)$ with $D_0 = D^p_0$ and some $\omega \in (T^*\mathbb{R}^m)_0$. Then taking $v \in A(D)_{d_0x^1} \setminus A[4](D)_{d_0x^1}$ we obtain (as above) $T^*(Y^0)_{\theta} \in A(D^p)_{\theta} \setminus A[4](D^p)_{\theta}$.
Thus $A(D^p)_{\theta} \setminus A[4](D^p)_{\theta} \neq \emptyset$. Finally, $A = A[5]$ because of Proposition 3. ■

References


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Received 16.10.2007
and in final form 18.1.2008

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