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Lifting distributions to the cotangent bundle

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Abstract. A classification of all $\mathcal{M}f_m$-natural operators $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ lifting p-dimensional distributions $D \subset TM$ on m-manifolds M to q-dimensional distributions $A(D) \subset TT^*M$ on the cotangent bundle T^*M is given.

1. A p-dimensional distribution on a manifold M is a smooth (C^∞) vector subbundle $D \subset TM$ of the tangent bundle of M such that $\dim(D_x) = p$ for any point $x \in M$. Thus a p-dimensional distribution D is a smooth (C^∞) section of the Grassmann bundle $\mathrm{Gr}_p M$ of M.

Given a manifold M, let $T^*M = (TM)^*$ be the cotangent bundle of M. Every embedding $f : M \to N$ of m-manifolds induces a vector bundle embedding $T^*f := (T(f^{-1}))^* : T^*M \to T^*N$ covering f, where Tf denotes the differential of f.

A general concept of natural operators can be found in [1]. We need the following partial definition of natural operators only.

Let m, p and q be integers such that $m \geq 1$, $0 \leq p \leq m$ and $0 \leq q \leq 2m$. An $\mathcal{M}f_m$-natural operator $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ lifting p-dimensional distributions D from m-manifolds M to q-dimensional distributions $A(D)$ on the cotangent bundle T^*M is a family of $\mathcal{M}f_m$-invariant regular operators (functions)

$$A : \underline{\mathrm{Gr}_p}(M) \to \underline{\mathrm{Gr}_q}(T^*M)$$

from the set $\underline{\mathrm{Gr}_p}(M)$ of all p-dimensional distributions on M (sections of $\mathrm{Gr}_p M \to M$) into the set $\underline{\mathrm{Gr}_q}(T^*M)$ of all q-dimensional distributions on T^*M (sections of $\mathrm{Gr}_q(T^*M) \to T^*M$) for any $\mathcal{M}f_m$-object M, where $\mathcal{M}f_m$ is the category of m-dimensional manifolds and their embeddings. The *invariance* means that if $D_1 \in \underline{\mathrm{Gr}_p}(M)$ and $D_2 \in \underline{\mathrm{Gr}_p}(N)$ are f-*related* (i.e. $\mathrm{Gr}_p f \circ D_1 = D_2 \circ f$) for some $\mathcal{M}f_m$-map $f : M \to N$ then $A(D_1)$ and $A(D_2)$ are T^*f-related. (We recall that $\mathrm{Gr}_p f : \mathrm{Gr}_p M \to \mathrm{Gr}_p N$ is the map induced by f given by $\mathrm{Gr}_p f(W) = Tf(W)$ for any $W \in (\mathrm{Gr}_p M)_x$, i.e. for

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any p-dimensional subspace $W \subset T_xM$, $x \in M$.) The *regularity* means that A transforms smoothly parametrized families of distributions into smoothly parametrized families of distributions.

We have the following $\mathcal{M}f_m$-natural operators $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$.

Example 1. $A^{[1]}(D)_\omega = \{0\}$ for all $\omega \in (T^*M)_x$, $x \in M$, $D \in \underline{\mathrm{Gr}_p}(M)$.

Example 2. $A^{[2]}(D)_\omega = \big\{\frac{d}{dt}(\omega + t\sigma)_{t=0} \in T_\omega T^*M : \sigma \in \mathrm{Ann}(D_x)\big\}$ for all $\omega \in (T^*M)_x$, $x \in M$, $D \in \underline{\mathrm{Gr}_p}(M)$, where $\mathrm{Ann}(W) = \{\sigma \in (T_xM)^* \mid \langle \sigma, w\rangle = 0$ for all $w \in W\}$ is the annihilator of a vector subspace $W \subset T_xM$.

Example 3. $A^{[3]}(D)_\omega = \ker(T_\omega \pi_M)$ for all $\omega \in (T^*M)_x$, $x \in M$, $D \in \underline{\mathrm{Gr}_p}(M)$, where $\pi_M : T^*M \to M$ is the cotangent bundle projection.

Example 4. $A^{[4]}(D)_\omega = (T_\omega \pi_M)^{-1}(D_x)$ for all $\omega \in (T^*M)_x$, $x \in M$, $D \in \underline{\mathrm{Gr}_p}(M)$.

Example 5. $A^{[5]}(D)_\omega = T_\omega T^*M$ for all $\omega \in (T^*M)_x$, $x \in M$, $D \in \underline{\mathrm{Gr}_p}(M)$.

The main result in this paper is the following theorem.

Theorem 1. *All $\mathcal{M}f_m$-natural operators $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ are described in Examples* 1–5.

From now on let $x^1, \ldots, x^m$ be the usual coordinates on $\mathbb{R}^m$. Let D^p be the integrable p-dimensional distribution on $\mathbb{R}^m$ spanned by $\partial/\partial x^1, \ldots, \partial/\partial x^p$.

2. First we prove the following proposition.

Proposition 1. *All $\mathcal{M}f_m$-natural operators $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ such that $A(D) \subset VT^*M = \ker(T\pi_M)$ for any $D \in \underline{\mathrm{Gr}_p}(M)$ are described in Examples* 1–3.

Lemma 1. *Let $A^1, A^2 : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ be $\mathcal{M}f_m$-natural operators such that $A^1(D) \subset VT^*M$ and $A^2(D) \subset VT^*M$ for any $D \in \underline{\mathrm{Gr}_p}(M)$. Assume $A^1(D^p)_\theta = A^2(D^p)_\theta$, where $\theta \in (T^*\mathbb{R}^m)_0$ is the zero element. Then $A^1(D)_\omega = A^2(D)_\omega$ for any p-dimensional distribution D on an m-manifold M and any $\omega \in (T^*M)_x$, $x \in M$, i.e. $A^1 = A^2$.*

Proof. By the $\mathcal{M}f_m$-invariance of A^1 and A^2 we may assume that $M = \mathbb{R}^m$, $x = 0$, $D_0 = D^p_0$ and $\omega \in (T^*\mathbb{R}^m)_0$.

We have the standard identification $I_\omega : V_\omega T^*\mathbb{R}^m = T_\omega (T^*\mathbb{R}^m)_0 \to (T^*\mathbb{R}^m)_0 = \mathbb{R}^{m*}$, $\omega \in (T^*\mathbb{R}^m)_0$. Using the invariance of A^i with respect to the homotheties $a_t = \big(\frac{1}{t}x^1, \ldots, \frac{1}{t}x^m\big)$ for $t > 0$ we deduce that

$$I_{t\omega}(A^i((\mathrm{Gr}_p((a_t)_*D)_{t\omega}))) = I_\omega(A^i(D)_\omega)$$

for $i = 1, 2$. Letting $t \to 0$ we deduce that $I_\omega(A^i(D)_\omega) = I_\theta(A^i(D^p)_\theta)$ for $i = 1, 2$. Then $A^1(D)_\omega = A^2(D)_\omega$ because of the assumption of the lemma. ∎

LEMMA 2. *Let $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator such that $A(D) \subset VT^*M$ for any $D \in \underline{\mathrm{Gr}_p}(M)$. Then $A(D^p)_\theta = A^{[i]}(D^p)_\theta$ for some $i = 1, 2, 3$.*

Proof. We may assume that $A(D^p)_\theta \neq \{0\}$.

If there is a point $v \in A(D^p)_\theta \setminus A^{[2]}(D^p)_\theta$, then using the invariance of A with respect to linear isomorphisms preserving D^p we see that $V_\theta T^*\mathbb{R}^m \setminus A^{[2]}(D^p)_\theta \subset A(D^p)_\theta$ (if $w \in V_\theta T^*\mathbb{R}^m \setminus A^{[2]}(D^p)_\theta$ then there exists a linear isomorphism ψ preserving D^p such that $w = TT^*\psi(v)$, and thus $w \in A(D^p)_\theta$ because of the invariance of A with respect to ψ and $v \in A(D^p)_\theta$); consequently, $A(D^p)_\theta = A^{[3]}(D^p)_\theta$.

So we may assume $A(D^p)_\theta \subset A^{[2]}(D^p)_\theta$. There is an element $v \in A(D^p)_\theta$, $v \neq 0$. Then using the invariance of A with respect to linear isomorphisms preserving D^p we see that $A^{[2]}(D^p)_\theta \subset A(D^p)_\theta$. Thus $A(D^p)_\theta = A^{[2]}(D^p)_\theta$. ∎

Proof of Proposition 1. Proposition 1 is an immediate consequence of Lemmas 1 and 2. ∎

3. Now, we prove the following proposition.

PROPOSITION 2. *Let $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator such that $A(D) \subset A^{[4]}(D)$ for any $D \in \underline{\mathrm{Gr}_p}(M)$ and $A(D^p)_\theta \setminus V_\theta T^*\mathbb{R}^m \neq \emptyset$. Then $A = A^{[4]}$.*

LEMMA 3. *Under the assumptions of Proposition 2 we have the inclusion $A(D^p)_\theta \supset V_\theta T^*\mathbb{R}^m$.*

Proof. There exists $a > 0$ such that $A(D^p)_{ad_0x^1} \setminus V_{ad_0x^1}T^*\mathbb{R}^m \neq \emptyset$. Then using the invariance of A with respect to the homotheties $b_t = (tx^1, \ldots, tx^m)$ for $t > 0$ we find that $A(D^p)_{d_0x^1} \setminus V_{d_0x^1}T^*\mathbb{R}^m \neq \emptyset$.

There exist real numbers $a_1, \ldots, a_m, b_1, \ldots, b_p \in \mathbb{R}$ such that

$$\begin{aligned} Y = a_1 \frac{d}{dt}[d_0x^1 + td_0x^1]_{t=0} + \cdots + a_m \frac{d}{dt}[d_0x^1 + td_0x^m]_{t=0} \\ + b_1\mathcal{T}^*\Big(\frac{\partial}{\partial x^1}\Big)_{d_0x^1} + \cdots + b_p\mathcal{T}^*\Big(\frac{\partial}{\partial x^p}\Big)_{d_0x^1} \in A(D^p)_{d_0x^1} \end{aligned}$$

and $b_q \neq 0$ for some $q \in \{1, \ldots, p\}$, where $\mathcal{T}^*(X)$ denotes the flow lifting of a vector X on M to T^*M.

Consider $k \in \{1, \ldots, m\}$. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism such that $\varphi^{-1}(y^1, \ldots, y^m) = (y^1 + y^q y^k, y^2, \ldots, y^m)$ on some open neighborhood of $0 \in \mathbb{R}^m$. Then φ preserves d_0x^1 and D^p. By standard verification one can show that

$$TT^*\varphi(Y) = Y + b_q \frac{d}{dt}[d_0x^1 + td_0x^k]_{t=0} + b_k \frac{d}{dt}[d_0x^1 + td_0x^q]_{t=0}.$$

By the invariance of A with respect to φ we have $TT^*\varphi(Y) \in A(D^p)_{d_0x^1}$, and so

$$b_q \frac{d}{dt}[d_0x^1 + td_0x^k]_{t=0} + b_k \frac{d}{dt}[d_0x^1 + td_0x^q]_{t=0} \in A(D^p)_{d_0x^1}.$$

Putting $k = q$ we find $\frac{d}{dt}[d_0x^1 + td_0x^q]_{t=0} \in A(D^p)_{d_0x^1}$, and hence $\frac{d}{dt}[d_0x^1 + td_0x^k]_{t=0} \in A(D^p)_{d_0x^1}$. Thus $A(D^p)_{d_0x^1} \supset V_{d_0x^1}T^*\mathbb{R}^m$. Then using the invariance of A with respect to the homotheties $a_t = \left(\frac{1}{t}x^1, \ldots, \frac{1}{t}x^m\right)$ for $t > 0$ and next letting $t \to 0$ we deduce that $A(D^p)_\theta \supset V_\theta T^*\mathbb{R}^m$. ∎

Proof of Proposition 2. By Lemma 3 and the assumptions of the proposition, there exists a constant vector field $W = a_1\partial/\partial x^1 + \cdots + a_p\partial/\partial x^p \neq 0$ on $\mathbb{R}^m$ such that $\mathcal{T}^*(W)_\theta \in A(D^p)_\theta$. Using the invariance of A with respect to linear isomorphisms preserving D^p we deduce that $\mathcal{T}^*(\partial/\partial x^i)_\theta \in A(D^p)_\theta$ for $i = 1, \ldots, p$. Then using Lemma 3 we get $A(D^p)_\theta = A^{[4]}(D^p)_\theta$. Thus $q = \dim(A(D^p)_\theta) = m+p$. Hence $A(D) \subset A^{[4]}(D)$ is an $(m+p)$-dimensional distribution for any $D \in \underline{\mathrm{Gr}_p}(M)$. Therefore $A(D) = A^{[4]}(D)$ for any $D \in \underline{\mathrm{Gr}_p}(M)$, i.e. $A = A^{[4]}$. ∎

4. Quite similarly to Section 3 we prove the next proposition.

Proposition 3. *Let $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator such that $A(D^p)_\theta \setminus A^{[4]}(D^p)_\theta \neq \emptyset$. Then $A = A^{[5]}$.*

Lemma 4. *Under the assumptions of Proposition* 4 *we have the inclusion $A(D^p)_\theta \supset V_\theta T^*\mathbb{R}^m$.*

Proof. We modify the proof of Lemma 3 by replacing p by m wherever appropriate. ∎

Proof of Proposition 3. By Lemma 4 and the assumptions of the proposition, there exists a constant vector field $W = a_1\partial/\partial x^1 + \cdots + a_m\partial/\partial x^m$ with $a_i \neq 0$ for some $i = p+1, \ldots, m$ such that $\mathcal{T}^*(W)_\theta \in A(D^p)_\theta$. Then using the invariance of A with respect to the linear isomorphisms preserving D^p we deduce that $\mathcal{T}^*(U)_\theta \in A(D^p)_\theta$ for any constant vector field $U = c_1\partial/\partial x^1 + \cdots + c_m\partial/\partial x^m$ with $c_j \neq 0$ for some $j = p+1, \ldots, m$. Hence using Lemma 4 we get $A(D^p)_\theta = T_\theta T^*\mathbb{R}^m$. Thus $q = 2m$ and consequently $A = A^{[5]}$. ∎

5. Proof of Theorem 1. Let $A : \mathrm{Gr}_p \rightsquigarrow \mathrm{Gr}_q T^*$ be an $\mathcal{M}f_m$-natural operator. By the $\mathcal{M}f_m$-invariance of A and Proposition 1 we may assume that $A(D)_\omega \setminus V_\omega T^*\mathbb{R}^m \neq \emptyset$ for some $D \in \underline{\mathrm{Gr}_p}(\mathbb{R}^m)$ and $\omega \in (T^*\mathbb{R}^m)_0$. If $p \geq 1$, by a density argument we may assume that $\omega \notin \mathrm{Ann}(D_0)$, and then using the invariance of A with respect to linear isomorphisms we may additionally assume that $\omega = d_0x^1$ and $D_0 = D_0^p$. (The case $p = 0$ is of of course "equivalent" to the case $p = m$. In these two cases we have only canonical distributions on T^*M—as trivial distributions give no information.) Let $v \in A(D)_{d_0x^1} \setminus V_{d_0x^1}T^*\mathbb{R}^m$. Then standardly one can show that there exist a constant vector field $Y^0 \neq 0$ and a vector field Y^1 with $Y^1(0) = 0$ such that $\mathcal{T}^*(Y)_{d_0x^1} = v$, where $Y = Y^0 + Y^1$. There exists a local diffeomorphism ψ such that $j^1_0\psi = \mathrm{id}$ and $\psi_*Y = Y^0$ near 0. Using the invariance of A with respect to ψ we may additionally assume $v = \mathcal{T}^*(Y^0)_{d_0x^1}$. Then using the invariance of A with respect to the homotheties $a_t = \left(\frac{1}{t}x^1, \ldots, \frac{1}{t}x^m\right)$ for $t > 0$ and letting $t \to 0$ we have $\mathcal{T}^*(Y^0)_\theta \in A(D^p)_\theta \setminus V_\theta T^*\mathbb{R}^m$. Thus $A(D^p)_\theta \setminus V_\theta T^*\mathbb{R}^m \neq \emptyset$.

If $A(D) \subset A^{[4]}(D)$ for any $D \in \underline{\mathrm{Gr}_p}(M)$, then $A = A^{[4]}$ because of Proposition 2.

Otherwise we may assume that $A(D)_\omega \setminus A^{[4]}(D)_\omega \neq \emptyset$ for some $D \in \underline{\mathrm{Gr}_p}(\mathbb{R}^m)$ with $D_0 = D_0^p$ and some $\omega \in (T^*\mathbb{R}^m)_0$. Then taking $v \in A(D)_{d_0x^1} \setminus A^{[4]}(D)_{d_0x^1}$ we obtain (as above) $\mathcal{T}^*(Y^0)_\theta \in A(D^p)_\theta \setminus A^{[4]}(D^p)_\theta$. Thus $A(D^p)_\theta \setminus A^{[4]}(D^p)_\theta \neq \emptyset$. Finally, $A = A^{[5]}$ because of Proposition 3. ∎

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