ANNALES
POLONICI MATHEMATICI
93.3 (2008)

Global existence and long-time behavior of solutions to a class of degenerate parabolic equations

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Abstract. We study the global existence and long-time behavior of solutions for a class of semilinear degenerate parabolic equations in an arbitrary domain.

1. Introduction. The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. This is an invariant set that attracts all the trajectories of the system. The existence of a global attractor has been derived for a large class of PDEs (see [5, 7, 14] and references therein). One of the most studied gradient partial differential equations is the reaction-diffusion equation, which models several physical phenomena like heat conduction, population dynamics, etc. There is an extensive literature concerning the existence and asymptotic behavior of solutions of reaction-diffusion equations and systems, in both bounded and unbounded domains (see e.g. [2, 3, 5, 7–9, 13–16]). However, to the best of our knowledge, little seems to be known about the asymptotic behavior of solutions of degenerate equations.

In this paper we study the following semilinear degenerate parabolic equation with variable, nonnegative coefficients, defined on an arbitrary domain (bounded or unbounded) $\Omega \subset \mathbb{R}^N$, $N \geq 2$:

$$\begin{aligned} u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) + g(x) &= 0, && x \in \Omega,\, t > 0, \\ u(x,0) &= u_0, && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega,\, t > 0. \end{aligned} \tag{1.1}$$

Problem (1.1) can be derived as a simple model for neutron diffusion (feed-

2000 *Mathematics Subject Classification*: 35B41, 35K65.

Key words and phrases: semilinear degenerate parabolic equation, global solution, global attractor, Lyapunov functional.

back control of a nuclear reactor) (see [6]). In this case u and σ stand for the neutron flux and neutron diffusion respectively.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes. Motivated by [4], where a degenerate elliptic problem is studied, we assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumptions: when the domain Ω is bounded,

($\mathcal{H}_\alpha$) $\sigma \in L^1_{\rm loc}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$;

and when the domain Ω is unbounded,

($\mathcal{H}_\beta^\infty$) σ satisfies condition ($\mathcal{H}_\alpha$) and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$.

The physical motivation of assumption ($\mathcal{H}_\alpha$) comes from modelling reaction-diffusion processes in composite materials occupying a bounded domain Ω, in which at some points they behave as *perfect insulators*. Following [6, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\sigma(x)$ vanishes at those points. On the other hand, when condition ($\mathcal{H}_\beta^\infty$) is satisfied, it is easy to see that the diffusion coefficient has to be unbounded. Physically, this situation corresponds to a nonhomogeneous medium, occupying the unbounded domain Ω, which behaves as a perfect conductor in $\Omega \setminus B_R(0)$ (see [6, p. 79]) and as a perfect insulator in a finite number of points in $B_R(0)$. Note that in various diffusion processes, the equation involves a diffusion $\sigma(x) \sim |x|^\alpha$, $\alpha \in (0,2)$, in the case of a bounded domain, and $\sigma(x) \sim |x|^\alpha + |x|^\beta$, $\alpha \in (0,2)$, $\beta > 2$, in the case of an unbounded domain.

In order to study problem (1.1) we use the natural energy space $\mathcal{D}_0^1(\Omega,\sigma)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^1} := \Big(\int_\Omega \sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.$$

This is a Hilbert space with respect to the scalar product

$$(u,v) := \int_\Omega \sigma(x)\nabla u \nabla v\,dx.$$

The qualitative behavior of solutions of problem (1.1), in the case of the given diffusion coefficient and

$$f(u) = -\lambda u + |u|^{2\gamma}u, \qquad g(x) \equiv 0, \tag{1.2}$$

was recently studied in [10, 11]. Under the condition $0 < \gamma \le \frac{2-\alpha}{2(N-2+\alpha)}$, the authors of [11] proved the existence of a global attractor in $\mathcal{D}_0^1(\Omega,\sigma)$. Note that the critical exponent of the embedding $\mathcal{D}_0^1(\Omega,\sigma) \hookrightarrow L^p(\Omega)$ is

$2^*_\alpha = \frac{2N}{N-2+\alpha}$, so the condition $\gamma \le \frac{2-\alpha}{2(N-2+\alpha)}$ in (1.2) is necessary to ensure that the Nemytskiĭ operator f maps $\mathcal{D}^1_0(\Omega,\sigma)$ into $L^2(\Omega)$. In the case $0 < \gamma \le \frac{2-\alpha}{N-2+\alpha}$, however, they were still able to prove the existence of a global attractor, but only in $L^2(\Omega)$ (see [10]). Here the existence of a global solution was proved by the Galerkin method (see [12]), which does not work in the general case of the nonlinear term $f(u)$.

In this paper we extend those results in the sense that we prove the existence of a global solution and a global attractor in $\mathcal{D}^1_0(\Omega,\sigma)$, with quite general assumptions on the nonlinear term f, even if the Nemytskiĭ operator f does not map $\mathcal{D}^1_0(\Omega,\sigma)$ into $L^2(\Omega)$. More precisely, for a bounded domain, to study the global existence and long-time behavior of solutions we assume in (1.1) that $g \in L^2(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$\begin{gathered} |f(u) - f(v)| \le C_1|u-v|(1+|u|^\varrho + |v|^\varrho), \\ \frac{2-\alpha}{N-2+\alpha} < \varrho < \frac{4-2\alpha}{N-2+\alpha}, \end{gathered} \tag{1.3}$$

$$f(u)u \ge -\mu u^2 - C_2, \tag{1.4}$$

$$F(u) \ge -\frac{1}{2}\mu u^2 - C_2, \tag{1.5}$$

where $C_1, C_2 \ge 0$, F is the primitive $F(y) = \int_0^y f(s)\,ds$ of f, $\mu < \lambda_1$, and λ_1 is the first eigenvalue of the operator $-\mathrm{div}(\sigma(x)\nabla)$ in Ω with homogeneous Dirichlet condition.

Let $A = -\mathrm{div}(\sigma(x)\nabla)$ with the domain of definition

$$D(A) = \{u \in \mathcal{D}^1_0(\Omega,\sigma) : Au \in L^2(\Omega)\}$$

(see Sec. 2) and define the Nemytskiĭ operator f corresponding to the above function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(u)(x) = f(u(x)), \qquad u \in \mathcal{D}^1_0(\Omega,\sigma).$$

Then problem (1.1) can be formulated as an abstract evolutionary equation

$$\frac{du}{dt} + Au + f(u) + g = 0, \quad u(0) = u_0.$$

The main purpose of this paper is to study the existence of a global solution and a global attractor for the dynamical system generated by (1.1).

Note that if the exponent ϱ in (1.3) satisfies $0 \le \varrho \le \frac{2-\alpha}{N-2+\alpha}$, then the Nemytskiĭ operator f is locally Lipschitzian from $X^{1/2} = \mathcal{D}^1_0(\Omega,\sigma)$ to $X = L^2(\Omega)$. This combined with the fact that A is a sectorial operator in X ensures the existence of a local classical solution $u \in C([0,T),\mathcal{D}^1_0(\Omega,\sigma)) \cap C((0,T),D(A)) \cap C^1((0,T),L^2(\Omega))$. Moreover, by direct computation, the local solution u satisfies

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2_X, \qquad t \in (0,T), \tag{1.6}$$

where

$$\Phi(u) = \frac{1}{2}\,\|u\|^2_{X^{1/2}} + \int\limits_{\Omega} (F(u) + gu)\,dx.$$

This implies that Φ is a strict Lyapunov functional. Then the proof of existence of a global solution is quite straightforward by using Φ. Therefore, in this paper we only focus on the case $\frac{2-\alpha}{N-2+\alpha} < \varrho < \frac{4-2\alpha}{N-2+\alpha}$.

Let us describe the methods we use. First, under assumption (1.3), one can check that the Nemytskiĭ operator f is locally Lipschitzian from $\mathcal{D}^1_0(\Omega,\sigma)$ to $L^q(\Omega)$, $q = 2^*_\alpha/(\varrho+1)$. Secondly, by the fixed point method, we prove the existence of a unique local mild solution u, i.e. $u \in C([0,T), \mathcal{D}^1_0(\Omega,\sigma))$ is a solution of the integral equation

$$u(t) = e^{-At}u_0 - \int\limits_0^T e^{-A(t-s)}(f(u(s)) + g)\,ds.$$

In this case, however, it is not easy to show that $\Phi(u)$ is a strict Lyapunov functional. Indeed, (1.6) is obtained, at least formally, by taking the scalar product of the equation with u_t. Note that we only have $u_t \in \mathcal{D}^{-1}(\Omega,\sigma)$, the dual space of $\mathcal{D}^1_0(\Omega,\sigma)$, and so one cannot multiply the equation by u_t. Hence we have to study the regularity of u_t. We show that, in particular, $u_t \in \mathcal{D}^1_0(\Omega,\sigma)$. This enables us to use the natural Lyapunov functional $\Phi(u)$ and condition (1.3) to prove that the solution exists globally in time. We also show that orbits of bounded sets are bounded. Finally, by proving the asymptotical compactness property of the semigroup $S(t)$ generated by (1.1) and using the dissipativeness condition (1.4) to prove the boundedness of the set E of equilibrium points, we obtain the existence of a global attractor $\mathcal{A}$ in $\mathcal{D}^1_0(\Omega,\sigma)$. Obviously, the above method can also be applied for $0 \le \varrho \le \frac{2-\alpha}{N-2+\alpha}$. Therefore, in fact, we obtain the existence of global solutions and global attractors for $0 \le \varrho < \frac{4-2\alpha}{N-2+\alpha}$.

The paper is organized as follows. In Section 2, we recall some results on function spaces and sectorial operators, and some basic concepts and results of the theory of infinite-dimensional dissipative dynamical systems. For clarity, in Sections 3 and 4, we only consider the case of a bounded domain and the diffusion coefficient σ satisfying condition $(\mathcal{H}_\alpha)$. In Section 3, we prove the existence of a local solution by the fixed point method, then we study the regularity of u_t, proving that in fact $u_t \in X^\delta$ $(1/2 < \delta < 1-\gamma)$. This enables us to use the natural Lyapunov functional to prove that the solutions exist globally. In Section 4, the existence of a global attractor of the semigroup generated by (1.1) is proved. As a consequence, we deduce that every solution tends to the set of equilibrium points as $t \to +\infty$. In the last section, we give some remarks on similar results for an unbounded domain and σ satisfying condition $(\mathcal{H}^\infty_\beta)$.

2. Preliminary results

2.1. *Function spaces.* We recall some basic results on function spaces defined in [4]. Let $N \geq 2$, $\alpha \in (0,2)$, and

$$2^*_\alpha = \begin{cases} 4/\alpha \in (0,2) & \text{if } \alpha \in (0,2),\ N = 2, \\ \dfrac{2N}{N-2+\alpha} \in \left(2, \dfrac{2N}{N-2}\right) & \text{if } \alpha \in (0,2),\ N \geq 3. \end{cases}$$

The exponent 2^*_α has the role of the critical exponent in the classical Sobolev embedding. We have the following generalized version of the Poincaré inequality ([4, Corollary 2.6]):

LEMMA 2.1. *Let Ω be a bounded (resp. unbounded) domain in $\mathbb{R}^N$, $N \geq 2$, and assume that condition $(\mathcal{H}_\alpha)$ (resp. $(\mathcal{H}^\infty_\beta)$) is satisfied. Then there exists a constant $c > 0$ such that*

$$\int_\Omega |u|^2 dx \leq c \int_\Omega \sigma(x)|\nabla u|^2\, dx \quad \textit{for every } u \in C^\infty_0(\Omega). \tag{2.1}$$

We emphasize that conditions $(\mathcal{H}_\alpha)$ and $(\mathcal{H}^\infty_\beta)$ are optimal in the sense that for $\alpha > 2$ there exist functions such that (2.1) is not satisfied (see [4]), and in the case of an unbounded domain, (2.1) does not hold in general if $\beta \leq 2$ in $(\mathcal{H}^\infty_\beta)$. We also refer to the examples of [1].

The natural energy space for the problem (1.1) involves the space $\mathcal{D}^1_0(\Omega, \sigma)$, defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}^1_0} := \Big(\int_\Omega \sigma(x)|\nabla u|^2\, dx\Big)^{1/2}.$$

The space $\mathcal{D}^1_0(\Omega, \sigma)$ is a Hilbert space with respect to the scalar product

$$(u, v) := \int_\Omega \sigma(x)\nabla u \nabla v\, dx.$$

The following lemmas come from [4, Propositions 3.3–3.5].

LEMMA 2.2. *Assume that Ω is a bounded domain in $\mathbb{R}^N, N \geq 2$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold*:

(i) $\mathcal{D}^1_0(\Omega, \sigma) \hookrightarrow L^{2^*_\alpha}(\Omega)$ *continuously*,
(ii) $\mathcal{D}^1_0(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in [1, 2^*_\alpha)$.

LEMMA 2.3. *Assume that Ω is an unbounded domain in $\mathbb{R}^N, N \geq 2$, and σ satisfies $(\mathcal{H}^\infty_\beta)$. Then the following embeddings hold*:

(i) $\mathcal{D}^1_0(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ *continuously for every* $p \in [2^*_\beta, 2^*_\alpha]$,
(ii) $\mathcal{D}^1_0(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in (2^*_\beta, 2^*_\alpha)$.

We now consider the case where Ω is a bounded domain (the unbounded case is considered similarly with $(\mathcal{H}^\infty_\beta)$ instead of $(\mathcal{H}_\alpha)$).

We consider the boundary value problem

$$-\operatorname{div}(\sigma(x)\nabla u) = g(x), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.2}$$

In order to apply the Friedrichs extension of symmetric operators, we set

$$X = L^2(\Omega), \quad D(\widetilde{A}) = C_0^\infty(\Omega), \quad \widetilde{A}u = -\operatorname{div}(\sigma(x)\nabla u).$$

Then problem (2.2) corresponds to the operator equation

$$\widetilde{A}u = g, \quad u \in C_0^\infty(\Omega),\ g \in X.$$

For every $u, v \in C_0^\infty(\Omega)$, we have

$$(\widetilde{A}u, v) = \int_\Omega \sigma(x)\nabla u \nabla v\, dx = (u, \widetilde{A}v).$$

It follows from (2.1) that there exists a constant $C > 0$ such that

$$(\widetilde{A}u, u) \ge C\|u\|_X^2 \quad \text{for every } u \in C_0^\infty(\Omega).$$

Hence, $\widetilde{A}$ is symmetric and strongly monotone. Applying the Friedrichs extension theorem (see e.g. [17, Vol. IIA, pp. 126–135]), we find that the energy space X_E equals $\mathcal{D}_0^1(\Omega, \sigma)$ since X_E is the completion of $D(\widetilde{A}) = C_0^\infty(\Omega)$ with respect to the scalar product $(u, v) = \int_\Omega \sigma(x)\nabla u\nabla v\, dx$, and the extensions satisfy

$$\widetilde{A} \subseteq A \subseteq A_E,$$

where $A_E : \mathcal{D}_0^1(\Omega, \sigma) \to \mathcal{D}_0^{-1}(\Omega, \sigma)$ is the energetic extension ($\mathcal{D}_0^{-1}(\Omega, \sigma)$ is the dual of $\mathcal{D}_0^1(\Omega, \sigma)$), and $A = -\operatorname{div}(\sigma(x)\nabla)$ is the Friedrichs extension of $\widetilde{A}$ with the domain of definition

$$D(A) = \{u \in \mathcal{D}_0^1(\Omega, \sigma) : Au \in X\}.$$

By Lemma 2.2 (notice that $2_\alpha^* > 2$), we have the evolution triple

$$\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{D}_0^{-1}(\Omega, \sigma)$$

with compact and dense embeddings. Hence, there exists a complete orthonormal system of eigenvectors (e_j, λ_j) such that

$$(e_j, e_k) = \delta_{jk} \quad \text{and} \quad -\operatorname{div}(\sigma(x)\nabla e_j) = \lambda_j e_j, \quad j, k = 1, 2, \ldots,$$
$$0 < \lambda_1 \le \lambda_2 \le \cdots, \qquad \lambda_j \to \infty \quad \text{as } j \to \infty.$$

From the argument above, the operator A is positive and self-adjoint. We may define its fractional powers as follows (see [8, Chapter 1]):

For $\theta \in (0, +\infty)$, we define, as usual, the operator $A^{-\theta} : X \to X$ as

$$A^{-\theta}u = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-At} u\, dt, \quad u \in X.$$

$A^{-\theta}$ is injective and we define X^θ to be the range of $A^{-\theta}$, and $A^\theta : X^\theta \to X$ to be the inverse of $A^{-\theta}$. We also set $X^0 = X$ and $A^0 = \mathrm{Id}_X$.

$X^\theta = D(A^\theta)$ is a separable Hilbert space endowed with the scalar product

$$(u,v)_{X^\theta} = (A^\theta u, A^\theta v), \qquad \|u\|_{X^\theta} = \|A^\theta u\|_X.$$

For any $\theta, \eta \in (0, +\infty)$ and $\theta \geq \eta$, X^θ is continuously embedded in X^η. Moreover, this embedding is compact if $\theta > \eta$.

Applying the theory of interpolation between Hilbert spaces for the evolution triple above (see [17, Vol. IIB, pp. 1109–1110]), we have

$$\mathcal{D}_0^1(\Omega,\sigma) = [\mathcal{D}_0^1(\Omega,\sigma), L^2(\Omega)]_0 = D(A^{1/2}) = X^{1/2}.$$

For $\theta \in (0, +\infty)$, let $X^{-\theta}$ be the dual space of X^θ. We endow $X^{-\theta}$ with the scalar product

$$(u,v)_{X^{-\theta}} = (R_\theta^{-1} u, R_\theta^{-1} v),$$

where $R_\theta : X^\theta \to X^{-\theta}$ is the Fréchet–Riesz isomorphism $u \mapsto (\cdot, u)_{X^\theta}$.

Notice that since A has a complete orthonormal system of eigenvectors in X with $Ae_j = \lambda_j e_j$, the space X^θ is the set of series $\sum_{k=1}^\infty c_k e_k$ such that $\sum_{k=1}^\infty c_k^2 \lambda_k^{2\theta} < \infty$ (see [17, Vol. IIB, p. 1110]).

We have the following basic result (see [8, Theorem 1.4.3, p. 26]).

THEOREM 2.1. *Suppose that A is sectorial and* $\operatorname{Re}\sigma(A) > \delta > 0$. *For $\theta \geq 0$, there exists a positive number $C_\theta < \infty$ such that*

$$\|A^\theta e^{-At}\| \leq C_\theta t^{-\theta} e^{-\delta t} \quad \textit{for all } t > 0,$$

and if $0 < \theta \leq 1$, $x \in X^\theta$, *then*

$$\|(e^{-At} - I)x\| \leq \frac{1}{\theta} C_{1-\theta} t^\theta \|A^\theta x\|.$$

From this theorem, in particular, we have some results which we will use in the next section:

$$\|e^{-At}\| \leq M e^{-\delta t} \qquad \text{for all } t \geq 0,$$

$$\|(e^{-At} - \mathrm{I})x\| \leq Ct\|Ax\| \qquad \text{for any } x \in X^1,\ t \geq 0,$$

$$e^{-tA}x \in \bigcap_{\theta \in \mathbb{R}} X^\theta \qquad \text{for any } x \in X^\eta,\ t > 0,$$

$$\|e^{-tA}x\|_{X^{1/2}} \leq C_\gamma t^{-1/2-\gamma} \|x\|_{X^{-\gamma}} \qquad \text{for any } x \in X^{-\gamma}, t > 0, \gamma \in (0, 1/2).$$

2.2. *Existence of global attractors.* For the convenience of the reader, we summarize some definitions and results of the theory of infinite-dimensional dynamical dissipative systems in [7, 14, 5] which we will use.

Let Y be a metric space (not necessarily complete) with metric d. If $C \subset Y$ and $b \in Y$ we set $\varrho(b, C) := \inf_{c \in C} d(b, c)$, and if $B, C \subset Y$ we set $\operatorname{dist}(B, C) := \sup_{b \in B} \varrho(b, C)$. Let $S(t)$ be a *continuous semigroup* on the metric space Y.

A set $A \subset Y$ is *invariant* if $S(t)A = A$ for any $t \geq 0$.

The positive orbit of $x \in Y$ is the set $\gamma^+(x) = \{S(t)x : t \geq 0\}$. If $B \subset Y$, the *positive orbit* of B is the set

$$\gamma^+(B) = \bigcup_{t\ge 0} S(t)B = \bigcup_{z\in B} \gamma^+(z).$$

More generally, for $\tau \ge 0$, we define the orbit of B after time τ by

$$\gamma_\tau^+(B) = \gamma^+(S(\tau)B).$$

The subset $A \subset Y$ *attracts* a set B if $\mathrm{dist}(S(t)B, A) \to 0$ as $t \to +\infty$.

The subset A is a *global attractor* if A is closed, bounded, invariant, and attracts all bounded sets.

The semigroup $S(t)$ is *asymptotically compact* if, for any bounded subset B of Y such that $\gamma_\tau^+(B)$ is bounded for some $\tau \ge 0$, every set of the form $\{S(t_n)z_n\}$ with $z_n \in B$ and $t_n \ge \tau$, $t_n \to +\infty$ as $n \to \infty$ is relatively compact.

A continuous semigroup $S(t)$ is a *continuous gradient system* if there exists a function $\Phi \in C^0(Y, \mathbb{R})$ such that $\Phi(S(t)u) \le \Phi(u)$ for all $t \ge 0$ and $u \in Y$, and $\Phi(S(t)u) = \Phi(u)$ for all $t \ge 0$ implies that u is an equilibrium point, i.e. $S(t)u = u$ for all $t \ge 0$. The function Φ is called a *strict Lyapunov functional.*

Let E be the set of equilibrium points for the semigroup $S(t)$. The *unstable set* of E is defined by

$$W^{\mathrm{u}}(E) = \{y \in Y : S(-t)y \text{ is defined for } t \ge 0 \text{ and } S(-t)y \to E \text{ as } t \to \infty\}.$$

From Proposition 2.19 and Theorem 4.6 in [14], we have

THEOREM 2.2. *Let $S(t)$, $t \ge 0$, be an asymptotically compact gradient system which has the property that, for any bounded set $B \subset Y$, there exists $\tau \ge 0$ such that $\gamma_\tau^+(B)$ is bounded. If the set E of equilibrium points is bounded, then $S(t)$ has a compact global attractor $\mathcal{A}$ and $\mathcal{A} = W^{\mathrm{u}}(E)$. Moreover, if Y is a Banach space, then $\mathcal{A}$ is connected.*

If the global attractor $\mathcal{A}$ exists, then (see [5, p. 21]) it contains a *global minimal attractor* $\mathcal{M}$ which is defined as a minimal closed positively invariant set with the property

$$\lim_{t\to+\infty} \mathrm{dist}(S(t)y, \mathcal{M}) = 0 \qquad \text{for every } y \in Y.$$

Moreover, if $\mathcal{M}$ is compact then it is invariant and $\mathcal{M} = \bigcup_{z\in V} \omega(z)$.

2.3. *Singular Gronwall inequality.* In order to study the regularity of solutions and prove the asymptotic compactness of the semigroup $S(t)$ generated by problem (1.1), we need the following lemma.

LEMMA 2.4 (see [8, p. 190]). *Assume that $\varphi(t)$ is a continuous nonnegative function on $(0,T)$ such that*

$$\varphi(t) \le c_0 t^{-\gamma_0} + c_1 \int_0^t (t-s)^{-\gamma_1} \varphi(s)\, ds, \qquad t \in (0,T),$$

where $c_0, c_1 \geq 0$ *and* $0 \leq \gamma_0, \gamma_1 < 1$. *Then there exists a constant* $K = K(\gamma_1, c_1, T)$ *such that*

$$\varphi(t) \leq \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T), \qquad t \in (0,T).$$

3. Global existence of a solution in the case of a bounded domain

LEMMA 3.1. *For every* $p \in (2, 2^*_\alpha)$, *there is a* $\gamma \in [0, 1/2)$ *such that* X^γ *is continuously embedded in* $L^p(\Omega)$.

Proof. Using the Hölder inequality we have

$$\|u\|_{L^p} \leq \|u\|_X^\delta \|u\|_{L^{2^*_\alpha}}^{1-\delta}, \quad \text{where} \quad \delta = \frac{2(2^*_\alpha - p)}{p(2^*_\alpha - 2)}.$$

Hence

$$\|u\|_{L^p} \leq C \|u\|_X^\delta \|u\|_{X^{1/2}}^{1-\delta}. \tag{3.1}$$

By the interpolation of fractional power spaces,

$$\|u\|_{X^{1/2}} \leq \|u\|_X^{1/2} \|u\|_{X^1}^{1/2}, \qquad \forall u \in X^1. \tag{3.2}$$

Let B be the inclusion map from $X^{1/2}$ to $Y = L^p(\Omega)$. It follows from (3.1) and (3.2) that

$$\|Bu\|_Y \leq \|u\|_X^\delta (C\|u\|_X^{1/2} \|u\|_{X^1}^{1/2})^{1-\delta} = C_1 \|u\|_{X^1}^{\overline{\delta}} \|u\|_X^{1-\overline{\delta}} = C_1 \|Au\|_X^{\overline{\delta}} \|u\|_X^{1-\overline{\delta}},$$

where $\overline{\delta} = \frac{1}{2}(1-\delta) < 1/2$. By [8, p. 28, Exercise 11], B has a unique extension to a continuous linear operator from X^γ to Y for every γ satisfying $\overline{\delta} < \gamma < 1/2$. Lemma 3.1 is proved.

Put

$$q = \frac{2^*_\alpha}{\varrho + 1}, \qquad p = \frac{q}{q-1}.$$

Since $\frac{2-\alpha}{N-2+\alpha} < \varrho < \frac{4-2\alpha}{N-2+\alpha}$ we have $1 < q < 2$. Thus,

$$p = \frac{q}{q-1} = \frac{2^*_\alpha}{2^*_\alpha - (\varrho+1)} \in (2, 2^*_\alpha).$$

It follows from Lemma 3.1 that there exists $\gamma \in (0, 1/2)$ such that X^γ is continuously embedded in $L^p(\Omega)$. Hence $L^q(\Omega) = (L^p(\Omega))'$ is continuously embedded in $X^{-\gamma}$.

LEMMA 3.2. *Let* f *be a map satisfying condition* (1.3). *Then* $f : X^{1/2} \to L^q(\Omega)$ *is Lipschitzian on every bounded subset of* $X^{1/2}$.

Proof. Let $u \in X^{1/2}$. From (1.3) we have $|f(u)| \leq C(1 + |u|^{\varrho+1})$. Hence

$$\int_\Omega |f(u)|^q \, dx \leq C_1 \int_\Omega (1 + |u|^{q(\varrho+1)}) \, dx = C_1 \int_\Omega (1 + |u|^{2^*_\alpha}) \, dx < \infty.$$

This shows that the operator $f : X^{1/2} \to L^q(\Omega)$ is well defined.

Let $u, v \in X^{1/2}$ with $\|u\|_{X^{1/2}}, \|v\|_{X^{1/2}} \le r$. It follows from (1.3) that

$$(3.3)\quad \int_\Omega |f(u)-f(v)|^q\,dx \le C\int_\Omega |u-v|^q(1+|u|^{q\varrho}+|v|^{q\varrho})\,dx$$
$$= C\int_\Omega |u-v|^q\,dx + C\int_\Omega |u-v|^q(|u|^{q\varrho}+|v|^{q\varrho})\,dx.$$

Using the Hölder inequality with $\frac{1}{\varrho+1}+\frac{\varrho}{\varrho+1}=1$ we have

$$(3.4)\quad \int_\Omega |u-v|^q(|u|^{q\varrho}+|v|^{q\varrho})\,dx$$
$$\le C\|\,|u-v|^q\|_{L^{\varrho+1}}(\|\,|u|^{q\varrho}\|_{L^{(\varrho+1)/\varrho}}+\|\,|v|^{q\varrho}\|_{L^{(\varrho+1)/\varrho}})$$
$$= C\|u-v\|^q_{L^{2^*_\alpha}}(\|u\|^{q\varrho}_{L^{2^*_\alpha}}+\|v\|^{q\varrho}_{L^{2^*_\alpha}})$$
$$\le C(r)\|u-v\|^q_{L^{2^*_\alpha}} \le C(r)\|u-v\|^q_{X^{1/2}}.$$

Notice that $q < 2^*_\alpha$ and $X^{1/2}$ is continuously embedded in $L^{2^*_\alpha}(\Omega)$. Thus

$$(3.5)\quad \int_\Omega |u-v|^q\,dx \le C\|u-v\|^q_{X^{1/2}}.$$

From (3.3)–(3.5) we get

$$\|f(u)-f(v)\|_{L^q} \le C(r)\|u-v\|_{X^{1/2}}.$$

Lemma 3.2 is proved.

Since $L^q(\Omega)$ is continuously embedded in $X^{-\gamma}$, we can assume that f, as a map from $X^{1/2}$ to $X^{-\gamma}$, is Lipschitz continuous on every bounded subset of $X^{1/2}$. This property of f is an important tool for proving the local existence of a mild solution of problem (1.1).

From this property of f and properties of the semigroup e^{-tA} generated by the operator $-A$ (see Sec. 2), using the arguments as in [8, Chapter 3, pp. 52–73], we obtain the following proposition on the local existence and smoothness of solutions.

Proposition 3.1. *Assume that f satisfies conditions* (1.3). *Then for any $R > 0$ and $u_0 \in X^{1/2}$ such that $\|u_0\|_{X^{1/2}} \le R$, there exists $T = T(R) > 0$ small enough such that problem* (1.1) *has a unique mild solution $u \in C([0,T); X^{1/2})$. Moreover, u is differentiable on $(0,T)$ and $u_t(t) \in X^\delta$ for any $\delta \in (1/2, 1-\gamma)$ and all $t \in (0,T)$.*

Denote by $\langle\,,\rangle$ the pairing between $X^{-1/2}$ and $X^{1/2}$. From (1.1) we have

$$\langle u_t, u_t\rangle + \langle Au, u_t\rangle + \langle f(u), u_t\rangle + \langle g, u_t\rangle = 0.$$

Hence

$$\|u_t\|^2_X + \frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/2}} + \frac{d}{dt}\int_\Omega (F(u)+gu)\,dx = 0.$$

Putting

$$\Phi(u) = \frac{1}{2}\|u\|^2_{X^{1/2}} + \int_\Omega (F(u) + gu)\,dx \tag{3.6}$$

we obtain

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2_X, \qquad t \in (0,T). \tag{3.7}$$

THEOREM 3.1. *If f satisfies conditions* (1.3), (1.5), *then for any* $u_0 \in X^{1/2}$, *problem* (1.1) *has a unique global solution* $u \in C([0,+\infty), X^{1/2})$.

Proof. Suppose that the solution u is defined on the maximal interval $[0, t_{\max})$. Using hypothesis (1.5) and the Cauchy inequality we get

$$\Phi(u(t)) \ge \frac{1}{2}\|u(t)\|^2_{X^{1/2}} - \frac{\mu}{2}\|u(t)\|^2 - C(\Omega) - \varepsilon\|u(t)\|^2 - \frac{1}{4\varepsilon}\|g\|^2.$$

By choosing ε small enough such that $\mu + 2\varepsilon < \lambda_1$ we obtain

$$\Phi(u(0)) \ge \Phi(u(t)) \ge \frac{1}{2}\left(1 - \frac{\mu + 2\varepsilon}{\lambda_1}\right)\|u\|^2_{X^{1/2}} - C.$$

Hence

$$\|u(t)\|_{X^{1/2}} \le M, \qquad \forall t \in [0, t_{\max}).$$

This implies that $t_{\max} = +\infty$. Indeed, suppose that $t_{\max} < +\infty$ and $\limsup_{t\to t^-_{\max}} \|u(t)\|_{X^{1/2}} < +\infty$. Then there exists a sequence $(t_n)_{n\ge1}$ and a constant K such that $t_n \to t^-_{\max}$ as $n \to \infty$ and $\|u(t_n)\|_{X^{1/2}} < K$, $n = 1, 2, \ldots$. As already shown above, for each $n \in \mathbb{N}$ there exists a unique solution of problem (1.1) with initial data $u(t_n)$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depends on K and is independent of $n \in \mathbb{N}$. Thus, $t_{\max} < t_n + T^*$ for $n \in \mathbb{N}$ large enough. This contradicts the maximality of $t_{\max}$ and the proof of Theorem 3.1 is complete.

4. Long-time behavior of the solution in a bounded domain

THEOREM 4.1. *Under conditions* (1.3)–(1.5), *the semigroup* $S(t)$ *generated by* (1.1) *has a compact connected global attractor* $\mathcal{A} = W^{\mathrm{u}}(E)$ *in* $X^{1/2}$.

Proof. First, from (3.7) and the proof of Theorem 3.1 we see that $\gamma^+(B)$ is bounded for any bounded subset B of $X^{1/2}$, and the function Φ defined by (3.6) is a strict Lyapunov functional.

Notice that the set of equilibrium points is

$$E = \{z \in X^{1/2} : Az + f(z) + g = 0\}.$$

For $z \in E$, we have

$$0 = \|z\|^2_{X^{1/2}} + \int_\Omega (f(z)z + gz)\,dx.$$

Using hypothesis (1.4) and the Cauchy inequality we obtain

$$\|z\|_{X^{1/2}} \le M \quad \text{for all } z \in E,$$

i.e. E is bounded in $X^{1/2}$. Thus, in order to prove the existence of a global attractor, we only need to prove that $S(t)$ is asymptotically compact in $X^{1/2}$.

Let $(u_n)_{n\ge 1}$ be a bounded sequence in $X^{1/2}$ and $t_n \to +\infty$. Fix $T > 0$. Since $\{u_n\}$ is bounded and the orbits of bounded sets are bounded, $\{S(t_n - T)u_n\}$ is bounded in $X^{1/2}$. Since $X^{1/2}$ is compactly embedded in X, there is a subsequence $\{S(t_{n_k} - T)u_{n_k}\}$ and $v \in X^{1/2}$ such that $v_k = S(t_{n_k} - T)u_{n_k} \rightharpoonup v$ weakly in $X^{1/2}$ and $v_k \to v$ strongly in X as $k \to \infty$. We will prove that $S(t_{n_k})u_{n_k} = S(T)v_k$ converges strongly to $S(T)v$ in $X^{1/2}$, and thus $S(t)$ is asymptotically compact.

Define $v_k(t) = S(t)v_k$ and $v(t) = S(t)v$. We have

$$v_k(t) = e^{-At}v_k - \int_0^t e^{-A(t-s)}(f(v_k(s)) + g)\,ds,$$

$$v(t) = e^{-At}v - \int_0^t e^{-A(t-s)}(f(v(s)) + g)\,ds.$$

Hence

$$\|v_k(t)-v(t)\|_{X^{1/2}} \le C_1 t^{-1/2}\|v_k - v\| + C_2\int_0^t (t-s)^{-1/2-\gamma}\|v_k(s)-v(s)\|_{X^{1/2}}\,ds.$$

By the singular Gronwall inequality (see Lemma 2.4), there is a constant C such that, for $t \in (0,T]$,

$$\|v_k(t) - v(t)\|_{X^{1/2}} \le Ct^{-1/2}\|v_k - v\|,$$

in particular,

$$\|v_k(T) - v(T)\|_{X^{1/2}} \le CT^{-1/2}\|v_k - v\|.$$

Since $v_k \to v$ in $L^2(\Omega)$, $v_k(T) \to v(T)$ in $X^{1/2}$ as $k \to \infty$. This implies that $S(t)$ is asymptotically compact. Applying Theorem 2.2, we obtain the conclusion of the theorem.

The following proposition describes the asymptotic behavior of solutions of (1.1) as $t \to +\infty$.

PROPOSITION 4.1. *Under conditions* (1.3)–(1.5), *the semigroup* $S(t)$, $t \ge 0$, *generated by* (1.1) *has a global minimal attractor* $\mathcal{M}$, *given by* $\mathcal{M} = E$, *in the space* $X^{1/2}$. *In particular*,

$$\lim_{t\to+\infty} \operatorname{dist}(S(t)y, E) = 0 \quad \text{for every } y \in X^{1/2}.$$

Proof. The existence of $\mathcal{M}$ immediately follows from the fact that the semigroup $S(t)$ has a compact global attractor (see Sec. 2.3). In order to prove the second statement, we will show that $\mathcal{M} = E$.

It is obvious that $E \subset \mathcal{M}$. We now prove that $\mathcal{M} \subset E$. Indeed, since $\mathcal{M} = \bigcup_{z \in X^{1/2}} \omega(z)$, it suffices to show that $\omega(z) \subset E$ for all $z \in X^{1/2}$. Taking $a \in \omega(z)$ arbitrarily, by the definition of $\omega(z)$, there exists a real sequence $\{t_n\}$, $t_n \to +\infty$, such that $S(t_n)z = u(t_n) \to a$. Since the Lyapunov functional $\varPhi$ is bounded below, this implies that

$$\varPhi(a) = \lim_{t \to +\infty} \varPhi(u(t_n)) = \inf\{\varPhi(S(t)z) = \varPhi(u(t)) : t \geq 0\},$$

i.e. $\varPhi = \text{const}$ on $\omega(z)$. Therefore, as the Lyapunov function is nonincreasing along the orbit $S(t)z$ and $\omega(z)$ is positively invariant, we conclude that $a \in E$. This completes the proof.

5. Some remarks on the case of an unbounded domain. In this section we discuss the case of an unbounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. We assume that the weight function $\sigma(x)$ satisfies condition $(\mathcal{H}_\beta^\infty)$. From Section 2.2 we see that, for condition $(\mathcal{H}_\beta^\infty)$, the operator $A = -\text{div}(\sigma(x)\nabla)$ has the same properties as in the case of a bounded domain (in particular, A is still a sectorial operator in $X = L^2(\Omega)$). On the other hand, we still have the continuous embedding $\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^{2_\alpha^*}(\Omega)$, and in particular the embedding $\mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^2(\Omega)$ is compact. Therefore, we may apply the methods used for a bounded domain to this case with some small changes in the conditions on f.

More precisely, to prove the existence of a global attractor in $\mathcal{D}_0^1(\Omega, \sigma)$, we assume that the nonlinear term $f(u)$ satisfies

$$|f(u) - f(v)| \leq C|u - v|(1 + |u|^\varrho + |v|^\varrho), \quad \frac{2 - \alpha}{N - 2 + \alpha} < \varrho < \frac{4 - 2\alpha}{N - 2 + \alpha}, \tag{5.1}$$

$$f(0) = 0, \quad f(u)u \geq -\mu u^2, \quad F(u) \geq -\frac{1}{2}\mu u^2, \quad \mu < \lambda_1. \tag{5.2}$$

We may now repeat the arguments used in Sections 3 and 4 to obtain

THEOREM 5.1. *Under conditions* $(\mathcal{H}_\beta^\infty)$ *and* (5.1)–(5.2), *problem* (1.1) *defines a semigroup* $S(t) : \mathcal{D}_0^1(\Omega, \sigma) \to \mathcal{D}_0^1(\Omega, \sigma)$, *which has a compact connected global attractor* $\mathcal{A} = W^{\mathrm{u}}(E)$ *in* $\mathcal{D}_0^1(\Omega, \sigma)$.

PROPOSITION 5.1. *Under conditions* (5.1)–(5.2), *the semigroup* $S(t)$, $t \geq 0$, *generated by* (1.1) *has a global minimal attractor* $\mathcal{M}$, *given by* $\mathcal{M} = E$, *in the space* $X^{1/2}$. *In particular*,

$$\lim_{t \to +\infty} \text{dist}(S(t)y, E) = 0 \quad \textit{for every } y \in X^{1/2}.$$

Acknowledgements. The authors would like to thank the referee for the helpful suggestions which improved the presentation of the paper.

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Received 17.11.2007
and in final form 9.1.2008 (1833)