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On the Euler characteristic of the links of a set determined by smooth definable functions

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Abstract. The purpose of this paper is to carry over to the o-minimal settings some results about the Euler characteristic of algebraic and analytic sets. Consider a polynomially bounded o-minimal structure on the field $\mathbb{R}$ of reals. A (C^∞) smooth definable function $\varphi : U \to \mathbb{R}$ on an open set U in $\mathbb{R}^n$ determines two closed subsets

$$W := \{u \in U : \varphi(u) \le 0\}, \quad Z := \{u \in U : \varphi(u) = 0\}.$$

We shall investigate the links of the sets W and Z at the points $u \in U$, which are well defined up to a definable homeomorphism. It is proven that the Euler characteristic of those links (being a local topological invariant) can be expressed as a finite sum of the signs of global smooth definable functions:

$$\chi(\mathrm{lk}(u;W)) = \sum_{i=1}^{r} \operatorname{sgn} \sigma_i(u), \quad \frac{1}{2}\,\chi(\mathrm{lk}(u;Z)) = \sum_{i=1}^{s} \operatorname{sgn} \zeta_i(u).$$

We also present a version for functions depending smoothly on a parameter. The analytic case of these formulae has been worked out by Nowel. As an immediate consequence, the Euler characteristic of each link of the zero set Z is even. This generalizes to the o-minimal setting a classical result of Sullivan about real algebraic sets.

1. Preliminaries. Throughout this paper we deal with a polynomially bounded o-minimal structure $\mathcal{R}$ on the field $\mathbb{R}$ of reals. For rudiments of o-minimal geometry we refer the reader to e.g. [11]. Polynomial boundedness implies many properties typical of the classical semi- and subanalytic geometry, as for instance the Łojasiewicz inequality or regular separation (see e.g. [18, 19, 6, 12]). We have at our disposal even a stronger version of the inequality with parameter.

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Łojasiewicz inequality with parameter. *Consider two definable functions* $f, g : A \to \mathbb{R}$ *on a set* $A \subset \mathbb{R}^m_u \times \mathbb{R}^n_x$. *Assume that all sections* $A_u := \{x \in \mathbb{R}^n : (u, x) \in A\}$, $u \in \mathbb{R}^m$, *are compact and that all functions*

$$f_u, g_u : A_u \to \mathbb{R}, \quad f_u(x) := f(u, x), \quad g_u(x) := g(u, x),$$

are continuous. If $\{f = 0\} \subset \{g = 0\}$, *then there exist a finite number of exponents* $\lambda_1, \ldots, \lambda_k > 0$ *and definable functions* $c_1, \ldots, c_k : \mathbb{R}^m \to (0, \infty)$ *such that*

$$|f(u, x)| \geq c_i(u)|g(u, x)|^{\lambda_i}, \quad (u, x) \in A,$$

for some $i = 1, \ldots, k$.

We may regard the functions f above as a definable family of functions f_u with parameter u.

Define the *Łojasiewicz exponent* $\lambda(0; f)$ of a continuous definable function germ

$$f : (\mathbb{R}^n, 0) \to \mathbb{R} \quad \text{at } 0 \in \mathbb{R}^n$$

as the infimum of the set Λ of those exponents $\lambda > 0$ for which there exists a constant $c > 0$ such that

$$|f(x)| \geq c \operatorname{dist}(x, \{f = 0\})^\lambda$$

in the vicinity of $0 \in \mathbb{R}^n$; here $\operatorname{dist}(x, V)$ denotes the distance of a point x from a set V; we put $\operatorname{dist}(x, \emptyset) = 1$. The set Λ is not empty by the Łojasiewicz inequality. We always have $\lambda(0; f) \in \Lambda$; moreover, $\lambda(0; f)$ belongs to the exponent field K of a polynomially bounded structure $\mathcal{R}$ (K consists of those exponents $r \in \mathbb{R}$ for which the power function t^r is definable in $\mathcal{R}$).

Consider now a definable family $f = (f_u)_{u \in B}$ of continuous function germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ at $0 \in \mathbb{R}^n$ with parameter set $B \subset \mathbb{R}^m$. In other words, $f : A \to \mathbb{R}$ is a definable function on a set $A \subset B \times \mathbb{R}^n$ such that every section A_u, $u \in B$, is a neighbourhood of $0 \in \mathbb{R}^n$, and every function $f_u : A_u \to \mathbb{R}$ is continuous. We shall need the following

Theorem on Łojasiewicz exponents of a definable family. *Under the above assumptions*, *the family of Łojasiewicz exponents* $\lambda(0; f_u)$, $u \in B$, *is finite*, *and a fortiori bounded.*

It is well known (cf. [25]) that the $\mathbb{R}$-algebra $\mathcal{D}(U)$ of smooth definable functions on an open connected subset U of $\mathbb{R}^n$ is quasi-analytic, i.e. each function $f \in \mathcal{D}(U)$ is uniquely determined by its Taylor series at any point $u \in U$. A definable open set U may be endowed with the Zariski topology induced by the algebra $\mathcal{D}(U)$; the Zariski closed subsets of U are the zero sets of families of functions from $\mathcal{D}(U)$.

We shall show in the Appendix that, for a polynomially bounded structure $\mathcal{R}$, U with the above smoothly definable Zariski topology is a noetherian space. This is a crucial fact which enables noetherian induction, in analogy to algebraic geometry. Actually, our line of reasoning adapts to the o-minimal setting the induction procedure worked out by Nowel [26] for the case of analytic sets associated with noetherian families, and based on a technique developed by Tougeron and El Khadiri [14]. Let us mention, however, that the analytic Zariski topology with respect to the $\mathbb{R}$-algebra $\mathcal{A}(U)$ of analytic definable functions on U remains noetherian even for o-minimal structures $\mathcal{R}$ that are not polynomially bounded (see Appendix).

To investigate Euler characteristic, we shall apply herein, as in papers [35, 36, 27, 26], the two formulae below.

Khimshiashvili's Formula (cf. [17]). *If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a smooth function germ with isolated critical point at $0 \in \mathbb{R}^n$, then the gradient $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ has an isolated zero at $0 \in \mathbb{R}^n$, and*

$$\chi(\mathrm{lk}(0; \{f \le 0\})) = 1 - \deg_0(\nabla f),$$

where $\deg_0$ denotes local topological degree at $0 \in \mathbb{R}^n$.

We say that a smooth map germ $F = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ *is finite* or *has an algebraically isolated zero at* $0 \in \mathbb{R}^n$ if one of the four equivalent (via Nakayama's lemma) conditions holds:

(i) *the ideal $(f_1, \ldots, f_n)$ contains some power of the maximal ideal of the local ring $C_0^\infty(\mathbb{R}^n)$ of smooth function germs at $0 \in \mathbb{R}^n$*;

(ii) *the local algebra of F,*

$$Q(F) := C_0^\infty(\mathbb{R}^n)/(f_1, \ldots, f_n),$$

is a finite-dimensional real vector space;

(iii) *the ideal $(T_0 f_1, \ldots, T_0 f_n)$ generated by the Taylor series of the germs f_i contains some power of the maximal ideal of the formal power series ring $\mathbb{R}[[x_1, \ldots, x_n]]$*;

(iv) *the factor $\mathbb{R}$-algebra $\mathbb{R}[[x_1, \ldots, x_n]]/(T_0 f_1, \ldots, T_0 f_n)$ is a finite-dimensional real vector space.*

Remark. In view of the preparation theorems (the formal version and the Malgrange version for differentiable algebras; see for instance [20]), the above conditions imply that the algebras $\mathbb{R}[[x_1, \ldots, x_n]]$ and $C_0^\infty(\mathbb{R}^n)$ are—via the homomorphisms induced by $F = (f_1, \ldots, f_n)$—finite modules over $\mathbb{R}[[x_1, \ldots, x_n]]$ and $C_0^\infty(\mathbb{R}^n)$, respectively.

Consider a finite smooth map germ $F = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with Jacobian J. We have, of course, the canonical isomorphism

$$Q(F) \cong \mathbb{R}[[x_1, \ldots, x_n]]/(T_0 f_1(x), \ldots, T_0 f_n(x)).$$

For any linear form $\phi : Q \to \mathbb{R}$, one can define a symmetric bilinear form Φ on Q by putting

$$\Phi : Q \times Q \to \mathbb{R}, \quad \Phi(p, q) := \phi(pq).$$

Eisenbud–Levin Formula (cf. [13] or [3, Part I, Chap. 5]). *The residue class $\bar{J}$ of J in the local algebra $Q(F)$ is non-zero and the bilinear form Φ is non-degenerate iff $\phi(\bar{J}) \neq 0$. Moreover, if $\phi(\bar{J}) > 0$, then*

$$\deg_0(F) = \operatorname{sign} \Phi.$$

2. Local degree of a definable family of smooth map germs. We keep the notation from the previous section. Consider a definable family $F = (f_1, \ldots, f_n) = (F_u)_{u \in U}$ of smooth map germs $(\mathbb{R}^n, 0) \to \mathbb{R}^n$ with an open parameter set $U \subset \mathbb{R}^m$. In other words, $F : A \to \mathbb{R}^n$ is a smooth definable map on an open neighbourhood A of the set $U \times \{0\} \subset U \times \mathbb{R}^n$, $F \in \mathcal{D}(A)^n$. Suppose that V is an irreducible Zariski closed subset of U, and the map germs

$$F_u = (f_{1,u}, \ldots, f_{n,u}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \quad u \in V,$$

have an isolated zero at $0 \in \mathbb{R}^n$. Let Λ be the family of Łojasiewicz exponents for the families $(f_{1,u})_{u \in V}, \ldots, (f_{n,u})_{u \in V}$ (which is a finite set of positive real numbers), and take $k_0 := \max \Lambda$. It is clear that for every $u \in V$ there is a constant $c_u > 0$ for which

$$\|F_u(x)\| \geq c_u \|x\|^{k_0}$$

in the vicinity of zero. Therefore—following [36, 26]—we shall modify the family F by adding the map $(ax_1^k, \ldots, ax_n^k)$ with any $a \in \mathbb{R}$, $a \neq 0$, and $k \in \mathbb{N}$, $k > k_0$, without changing its local topological degree at zero for $u \in V$:

$$\deg_0 F_u(x) = \deg_0 G_u(x) \quad \text{for } u \in V,$$

where

$$G(u, x) = F(u, x) + a(x_1^k, \ldots, x_n^k);$$

instead, we may obviously consider the family

$$G(u, x) = aF(u, x) + (x_1^k, \ldots, x_n^k).$$

For all but a finite number of $a \in \mathbb{R}$, the germs $G_u(x)$ are finite (or, in other words, have an algebraically isolated zero at $0 \in \mathbb{R}_x^n$) for all u from a Zariski open and dense subset of V, i.e. for $u \in V \setminus \Sigma$ where Σ is a proper Zariski closed subset of V. We now sketch a proof. We first recall a direct consequence of Nakayama's lemma (cf. [37, Chap. II.5]):

Claim. *Let A be a local ring whose maximal ideal $\mathfrak{m}$ is finitely generated, and let M be a finite A-module. Then*

$$\operatorname{length} M \leq h \quad \textit{iff} \quad \operatorname{length} M/\mathfrak{m}^{h+1} M \leq h.$$

Remark. If A contains a field k canonically isomorphic to the residue field $A/\mathfrak{m}$, then $\operatorname{length} M = \dim_k M$.

For a finite smooth map germ $G = (g_1, \ldots, g_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ we thus get the following equivalences:

$$\dim_{\mathbb{R}} \mathbb{R}[[x]]/(T_0 g_1, \ldots, T_0 g_n) > h$$

iff

$$\dim_{\mathbb{R}} \mathbb{R}[[x]]/(\mathfrak{m}^{h+1} + (T_0 g_1, \ldots, T_0 g_n)) > h$$

iff

$$\dim_{\mathbb{R}} (\mathfrak{m}^{h+1} + (T_0 g_1, \ldots, T_0 g_n))/\mathfrak{m}^{h+1} < \dim_{\mathbb{R}} \mathbb{R}[[x]]/\mathfrak{m}^{h+1} - h$$

iff the collection of partial derivatives $(\partial g_i^{|\alpha|}/\partial x^\alpha(0))$, $i = 1, \ldots, n$, $|\alpha| \le h$, satisfies a polynomial equation.

Hence, for our definable family $G(u, x)$, we get the equivalence

$$\dim_{\mathbb{R}} C_0^\infty(\mathbb{R}^n)/(g_{1,u}, \ldots, g_{n,u}) > h$$

iff the collection of partial derivatives $(\partial g_i^{|\alpha|}/\partial x^\alpha(u, 0))$, $i = 1, \ldots, n$, $|\alpha| \le h$, satisfies a polynomial equation iff $P(u, a) = 0$ for a certain smooth definable function

$$P(u, a) = \sum_{j=0}^{d} \varphi_j(u) a^j$$

which depends polynomially on a.

But for $a = 0$ we have

$$\dim_{\mathbb{R}} C_0^\infty(\mathbb{R}^n)/(g_{1,u}, \ldots, g_{n,u}) = \dim_{\mathbb{R}} \mathbb{R}[[x]]/(x_1^k, \ldots, x_n^k) = k^n,$$

whence the function $\varphi_0(u)$ has no zero on V whenever $h \ge k^n$. Take $h \ge k^n$ and fix any $u_0 \in V$. Then the polynomial $P(u_0, a)$ has at most d zeroes $a_1, \ldots, a_d$. Consequently, $P(u_0, a) \ne 0$ for every $a \in \mathbb{R}$ distinct from $a_1, \ldots, a_d$. Therefore $\Sigma := \{u \in V : P(u, a) = 0\}$ is a proper Zariski closed subset of V, and thus we obtain the desired conclusion:

$$\dim_{\mathbb{R}} C_0^\infty(\mathbb{R}^n)/(g_{1,u}, \ldots, g_{n,u}) \le h$$

whenever $u \in V \setminus \Sigma$.

At this stage we can prove the following

THEOREM. *Suppose that V is an irreducible Zariski closed subset of U, and the map germs*

$$F_u = (f_{1,u}, \ldots, f_{n,u}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \quad u \in V,$$

have an isolated zero at $0 \in \mathbb{R}^n$. Then there exists a proper Zariski closed subset Σ of V such that the local topological degree $\deg_0 F_u(x)$ is over $V \setminus \Sigma$ the sum of the signs of a finite number of smooth definable functions $\sigma_1, \ldots, \sigma_s \in \mathcal{D}(U)$:

$$\deg_0 F_u(x) = \operatorname{sgn} \sigma_1(u) + \cdots + \operatorname{sgn} \sigma_s(u) \quad \textit{for } u \in V \setminus \Sigma.$$

Proof. Similarly to papers [27, 26], the proof makes use of the Eisenbud–Levin formula and an elementary but powerful formal division algorithm of Grauert–Hironaka (cf. [7, 15, 5]).

For a Zariski closed subset V of U, let

$$\mathfrak{I}(V) := \{f \in \mathcal{D}(U) : f|V \equiv 0\} \quad \text{and} \quad \mathcal{D}(V) := \mathcal{D}(U)/\mathfrak{I}(V).$$

Obviously, the set V is irreducible iff the ideal $\mathfrak{I}(V)$ is prime iff the $\mathbb{R}$-algebra $\mathcal{D}(V)$ is an integral domain.

Keeping the foregoing notation, $G = (g_1, \ldots, g_n) : A \to \mathbb{R}^n$ is the modified definable family of smooth map germs at $0 \in \mathbb{R}^n$. By means of evaluation homomorphisms, one may look at $\mathcal{D}(U)[[x]]$ as a family of formal power series rings $\mathbb{R}[[x]]$. Consider the ideal $I \subset \mathcal{D}(V)[[x]]$ generated by the Taylor series $T_0 g_{1,u}(x), \ldots, T_0 g_{n,u}(x)$, $u \in V$.

Denote by $\mathfrak{N}(I)$ the Hironaka diagram of initial exponents for I and put

$$\Delta(I) := \mathbb{N}^n \setminus \mathfrak{N}(I).$$

Generally, the division algorithm says that there is a function $\tau(u) \in \mathcal{D}(V)$ such that each $f \in \mathcal{A}(V)[[x]]$ can be represented in a unique fashion in the form

$$f = q + r \quad \text{with } q, r \in \mathcal{D}(V)[1/\tau][[x]],\ q \in I,\ \operatorname{supp}(r) \subset \Delta(I).$$

The division algorithm is compatible with evaluation homomorphisms at the points of the Zariski open set $\{u \in V : \tau(u) \neq 0\}$, i.e.

$$\mathfrak{N}(I_u) = \mathfrak{N}(I), \quad \Delta(I_u) = \Delta(I) \quad \text{if } u \in V,\ \tau(u) \neq 0,$$

and $f_u = q_u + r_u$ is the result of the algorithm carried out at a point $u \in V$ with $\tau(u) \neq 0$. If $\Delta(I_u)$ is a finite set for u in a Zariski open and dense subset of V, then so are $\Delta(I)$ and $\Delta(I_u) = \Delta(I)$ for all $u \in V$, $\tau(u) \neq 0$. Therefore $\mathcal{D}(V)[1/\tau][[x]]/(I \cdot \mathcal{D}(V)[1/\tau][[x]])$ and $\mathbb{R}[[x]]/I_u$ are finite free modules over $\mathcal{D}(V)[1/\tau]$ and $\mathbb{R}$, respectively, with a basis which consists of the monomials x^α with $\alpha \in \Delta(I) = \Delta(I_u)$, provided that $u \in V$, $\tau(u) \neq 0$.

In our case, the map germs G_u at $0 \in \mathbb{R}^n$ are finite for u from a Zariski open and dense subset of V, and thus the set $\Delta(I)$ is finite of some cardinality s. Hence, after enlarging the singular set Σ by the zero set of the function $\tau \in \mathcal{D}(V)$, the local algebras $Q(G_u) = \mathbb{R}[[x]]/I_u$ are finite-dimensional real vector spaces with basis x^α, $\alpha \in \Delta(I)$.

The division algorithm carried out for the Jacobian J of G with respect to the variables x yields a unique representation of the form $J = q + r$, where

$$q_u \in (T_0 g_{1,u}(x), \ldots, T_0 g_{n,u}(x)) \quad \text{if } \tau(u) \neq 0$$

and

$$r = \sum_{\alpha \in \Delta} r_\alpha(u)\, x^\alpha, \qquad r_\alpha \in \mathcal{D}(V)[1/\tau].$$

The functions r_α, $\alpha \in \Delta(I)$, have no common zero in $V \setminus \Sigma$ because $J_u \neq 0 \in Q(G_u)$ for $u \in V \setminus \Sigma$.

Consider now the family of linear forms

$$\phi_u : Q(G_u) \to \mathbb{R}, \qquad \phi_u(x^\alpha) = r_\alpha(u),$$

for $\alpha \in \Delta(I)$ and $u \in V \setminus \Sigma$. Since $\phi_u(J_u) = \sum_{\alpha\in\Delta} r_\alpha^2 > 0$ for all $u \in V \setminus \Sigma$, it follows from the Eisenbud–Levin formula that for all $u \in V \setminus \Sigma$ the local topological degrees $\deg_0 G_u$ coincide with the signatures of the symmetric bilinear forms $\Phi_u(p,q) := \phi_u(pq)$. By means of linear transformations over the quotient field of the integral domain $\mathcal{D}(V)$, the matrix of the bilinear forms Φ_u can be reduced over the set V to the diagonal form with some quotients $\theta_1, \dots, \theta_s$ on the diagonal. Enlarging again the singular set Σ by the zero sets of the denominators of the quotients and multiplying the diagonal matrix by the squares of those denominators, we may assume that $\theta_1, \dots, \theta_s \in \mathcal{D}(V)$. Then

$$\deg_0 F_u = \deg_0 G_u = \operatorname{sign} \Phi_u = \theta_1(u) + \dots + \theta_s(u)$$

for all $u \in V \setminus \Sigma$. To complete the proof, it suffices to take for $\sigma_1(u), \dots, \sigma_s(u) \in \mathcal{D}(U)$ any representatives of the functions $\theta_1, \dots, \theta_s \in \mathcal{D}(V)$.

Via a routine noetherian induction (cf. [26]), one can easily obtain

Corollary. *Let $F = (f_1, \dots, f_n) = (F_u)_{u\in U}$ be a definable family of smooth map germs $(\mathbb{R}^n, 0) \to \mathbb{R}^n$ with an open parameter set $U \subset \mathbb{R}^m$. Suppose the map germ F_u has an isolated zero for every u from a Zariski closed subset V of U. Then the local topological degree $\deg_0 F_u(x)$ is over V the sum of the signs of a finite number of smooth definable functions $\sigma_1, \dots, \sigma_s \in \mathcal{D}(U)$:*

$$\deg_0 F_u(x) = \operatorname{sgn} \sigma_1(u) + \dots + \operatorname{sgn} \sigma_s(u) \quad \text{for } u \in V.$$

3. Euler characteristic of the links of the sets W and Z. Throughout this section, we shall deal with a definable family $f = (f_u)_{u\in U}$ of smooth function germs

$$f_u : (\mathbb{R}^n, 0) \to \mathbb{R}$$

with an open parameter set $U \subset \mathbb{R}^m$. Let

$$\begin{aligned} V &:= \{u \in U : f_u(0) = 0\}, \\ W_u &:= \{x \in \mathbb{R}^n : f_u(x) \le 0\}, \qquad Z_u := \{x \in \mathbb{R}^n : f_u(x) = 0\}, \\ S_\varepsilon &:= \{x \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = \varepsilon\}, \qquad L_{u,\varepsilon} := W_u \cap S_\varepsilon. \end{aligned}$$

We shall regard W_u both as a definable set in the vicinity of $0 \in \mathbb{R}^n$ and as a germ at zero. After Szafraniec [35] (see also [26]), we apply the following modification $g = (g_u)_{u\in U}$ of the definable family f:

$$g_u(x) := f_u(x) - (x_1^2 + \dots + x_n^2)^k.$$

Put

$$Y_u := \{x \in \mathbb{R}^n : g_u(x) \le 0\}, \qquad M_{u,\varepsilon} := Y_u \cap S_\varepsilon.$$

PROPOSITION. *There exists $k(f) > 0$ such that for every integer $k > k(f)$ and every $u \in V$, for all sufficiently small $\varepsilon > 0$ (i.e. for all $\varepsilon \in (0, \varepsilon_{k,u})$ with some $\varepsilon_{k,u} > 0$) the following assertion holds: 0 is a regular value of the restriction $g_u|S_\varepsilon$, $L_{u,\varepsilon}$ is a deformation retract of $M_{u,\varepsilon}$, and the modification g_u has an isolated (if any) critical point at the origin.*

Proof. Fix a $u \in V$ and an $\varepsilon > 0$. We first observe that the set

$$\Sigma_{u,\varepsilon} := \{x \in S_\varepsilon : f_u(x) > 0,\ x \text{ is a critical point of } f_u|S_\varepsilon\}$$

is closed in S_ε, or equivalently, that no accumulation point of $\Sigma_{u,\varepsilon}$ lies in $Z_u \cap S_\varepsilon$. Indeed, suppose a point $a \in Z_u \cap S_\varepsilon$ were an accumulation point of $\Sigma_{u,\varepsilon}$. The curve selection lemma implies that we could find a definable C^1 curve

$$\gamma : [0, \eta) \to S_\varepsilon \quad \text{for which } \gamma(0) = a \text{ and } \gamma((0, \eta)) \subset \Sigma_{u,\varepsilon}.$$

Since $\frac{d}{dt}(f_u \circ \gamma)(t) = 0$, the function f_u would be constant on our curve, and thus $f_u(\gamma(t)) = f_u(\gamma(0)) = 0$ for all $t \in (0, \eta)$, which is a contradiction.

Therefore, for all $\varepsilon > 0$ small enough, we have

$$\varphi_u(\varepsilon) := \inf\{f_u(x) : x \in \Sigma_{u,\varepsilon}\} = \min\{f_u(x) : x \in \Sigma_{u,\varepsilon}\} > 0.$$

Hence and by piecewise uniform asymptotics (see e.g. [12]), there exist finitely many real numbers $\lambda_1, \ldots, \lambda_s > 0$ such that for all $u \in V$ and for all sufficiently small (depending on u) $\varepsilon > 0$ we have

$$\varphi_u(\varepsilon) = c \cdot \varepsilon^{\lambda_i} + o(\varepsilon^{\lambda_i}) \quad \text{as } \varepsilon \to 0^+$$

for some $i = 1, \ldots, s$ and $c = c(u) > 0$. Now, if we take

$$k(f) := \max\{\lambda_1, \ldots, \lambda_s\} \quad \text{and} \quad k > k(f),$$

then 0 is a regular value of $g_u|S_\varepsilon$ for all $\varepsilon > 0$ small enough, because the restrictions $f_u|S_\varepsilon$ and $g_u|S_\varepsilon$ have the same critical points on the spheres S_ε. Moreover, $f_u|S_\varepsilon$ and $g_u|S_\varepsilon$ have no critical points on $M_{u,\varepsilon} \setminus L_{u,\varepsilon}$.

Next, we are to show that the modification g_u has an isolated (if any) critical point at the origin. Otherwise, again by the curve selection lemma, we could find a non-constant definable C^1 curve

$$\gamma : [0, \eta) \to \mathbb{R}^n \quad \text{with } \gamma(0) = 0$$

which consists of only critical points of g_u. Since $\frac{d}{dt}(g_u \circ \gamma)(t) = 0$, the function g_u would be constant on our curve, and thus g_u would vanish on γ. But the points of the curve γ are critical points of the restrictions $g_u|S_\varepsilon$ as well, and consequently, 0 would be a critical value of the restrictions $g_u|S_\varepsilon$ for all $\varepsilon > 0$ small enough—contrary to what we have already shown.

Finally, through piecewise linearization of definable functions (cf. [32, Chap. II] and also [30, 31]), the set $L_{u,\varepsilon}$ is a deformation retract of its neighbourhood

$$\{x \in S_\varepsilon : f_u(x) \le c\}$$

for some small $c > 0$. Since $f_u|S_\varepsilon$ has no critical points on

$$M_{u,\varepsilon} \setminus L_{u,\varepsilon} = \{x \in S_\varepsilon : f_u(x) \le \varepsilon^k\} \setminus \{x \in S_\varepsilon : f_u(x) \le 0\},$$

we can—using integration of vector fields—homotopically deform the level surface $\{x \in S_\varepsilon : f(x) = c\}$ into $\{x \in S_\varepsilon : f(x) = \varepsilon^k\}$ by pushing it at constant vertical speed along the gradient curves of the function $f_u|S_\varepsilon$. More precisely, we push the level surface along the trajectories of the gradient field $\nabla f/|\nabla f|^2$ with respect to a fixed riemannian metric on S_ε. Notice that the trajectories of the vector fields $\nabla f/|\nabla f|^2$ and ∇f differ merely in parametrization. Consequently, $L_{u,\varepsilon} = \{x \in S_\varepsilon : f_u(x) \le 0\}$ is a deformation retract of $M_{u,\varepsilon} = \{x \in S_\varepsilon : f_u(x) \le \varepsilon^k\}$, which completes the proof of the proposition.

Corollary. *If $k > k(f)$, then*

$$\chi(\mathrm{lk}(0; W_u)) = 1 - \deg_0(\nabla g_u) \quad \textit{for each } u \in V.$$

This follows immediately from Khimshiashvili's formula. Here we set $\deg_0(\nabla g_u) = 0$ if 0 is not a critical point of g_u, i.e. if $\nabla g_u(0) \ne 0$.

Hence, and by the Corollary to the Theorem from Section 2, we obtain

Theorem 1. *The Euler characteristic of each link, $\chi(\mathrm{lk}(0; W_u))$ for $u \in U$, is the sum of the signs of a finite number of smooth definable functions $\sigma_1, \ldots, \sigma_r \in \mathcal{D}(U)$:*

$$\chi(\mathrm{lk}(0; W_u)) = \operatorname{sgn} \sigma_1(u) + \cdots + \operatorname{sgn} \sigma_r(u) \quad \textit{for } u \in U.$$

Proof. This is straightforward, because $\deg_0(\nabla g_u)$ is the sum of the signs of a finite number of functions from $\mathcal{D}(U)$ if $u \in V$, and if $u \in U \setminus V$ we have

$$\chi(\mathrm{lk}(0; W_u)) = \begin{cases} 0 & \text{for } n \text{ even}, \\ 1 - \operatorname{sgn} f_u(0) & \text{for } n \text{ odd}. \end{cases}$$

Remarks. (1) We shall apply Theorem 1 to a modification g of the definable family f of the form

$$g_u(x) = f_u(x) - c(x_1^2 + \cdots + x_n^2)^k, \quad c \in \mathbb{R};$$

we may regard g as a definable family with parameter set $U \times \mathbb{R}$. Thus the Euler characteristic

$$\chi(\mathrm{lk}(0; \{x \in \mathbb{R}^n : f_u(x) \le c(x_1^2 + \cdots + x_n^2)^k\}))$$

is a finite sum of the signs of smooth definable functions $\sigma_i(u, c) \in \mathcal{D}(U \times \mathbb{R})$, $i = 1, \ldots, s$.

(2) Working with links, we actually deal with closed definable subsets of the spheres S_ε, which are triangulable (see e.g. [32, Chap. II]). Therefore we have at our disposal such tools of algebraic topology as exact Mayer–Vietoris sequence (because every triad under consideration is excisive) or Alexander duality (for singular homology and cohomology); see e.g. [33].

(3) Clearly, the foregoing proposition holds true for modifications g of the definable family f of the form

$$g_u(x) = f_u(x) - c\,(x_1^2 + \cdots + x_n^2)^k \quad \text{with } c > 0,$$

provided that the integer k satisfies $k > k(f)$. Thus, for any $c > 0$, we get

$$\chi(\mathrm{lk}(0; \{x \in \mathbb{R}^n : f_u(x) \le c\,(x_1^2 + \cdots + x_n^2)^k\})) = \chi(\mathrm{lk}(0; W_u)).$$

(4) By Alexander duality, we have, for any $c > 0$, the equality

$$\begin{aligned}\chi(\{x \in S_\varepsilon : f_u(x) \le -c\,(x_1^2 + \cdots + x_n^2)^k\})\\ = \chi(S_\varepsilon) - \chi(\{x \in S_\varepsilon : f_u(x) > -c\,(x_1^2 + \cdots + x_n^2)^k\}).\end{aligned}$$

Since, similarly to what we saw before, the set

$$W_u' := \{x \in S_\varepsilon : f_u(x) \ge 0\}$$

is a deformation retract of the set

$$\{x \in S_\varepsilon : f_u(x) > -c\,(x_1^2 + \cdots + x_n^2)^k\}$$

for all $\varepsilon > 0$ small enough, we get the equality

$$\chi(\mathrm{lk}(0; \{x \in \mathbb{R}^n : f_u(x) \le -c\,(x_1^2 + \cdots + x_n^2)^k\})) = \chi(S_\varepsilon) - \chi(\mathrm{lk}(0; W_u'))$$

whenever $c > 0$.

Concluding, we see that

$$\chi(\mathrm{lk}(0; W_u)) = \sum_i \lim_{c \to 0^+} \operatorname{sgn} \sigma_i(u, c),$$
$$\chi(\mathrm{lk}(0; W_u')) = \chi(S_\varepsilon) - \sum_i \lim_{c \to 0^+} \operatorname{sgn} \sigma_i(u, -c).$$

At this stage we can generalize Theorem 1 as follows.

THEOREM 2. *There exist a finite number of smooth definable functions* $\xi_1, \ldots, \xi_s \in \mathcal{D}(U)$ *such that*

$$\frac{1}{2}(\chi(\mathrm{lk}(0; W_u)) \pm \chi(\mathrm{lk}(0; W_u'))) = \operatorname{sgn} \xi_1(u) + \cdots + \operatorname{sgn} \xi_s(u) \quad \textit{for every } u \in U.$$

Proof. It suffices to show that, for every smooth definable function $\sigma(u, c) \in \mathcal{D}(U \times \mathbb{R})$, the function

$$\frac{1}{2}(\lim_{c \to 0^+} \operatorname{sgn} \sigma(u, c) + \lim_{c \to 0^+} \operatorname{sgn} \sigma(u, -c))$$

is such a finite sum of signs. Put $F_0 := U$, and for $k \geq 1$,

$$F_k := \left\{ u \in U : \sigma(u,0) = \partial \frac{\sigma}{\partial c}(u,0) = \cdots = \frac{\partial^{k-1}}{\partial c^{k-1}(u,0)} = 0 \right\}.$$

The decreasing sequence (F_k) of Zariski closed subsets of U stabilizes (see the appendix), i.e. $F_K = F_{K+1} = F_{K+2} = \cdots$ for an integer $K > 0$. Then

$$\frac{1}{2}\big(\lim_{c\to 0^+} \operatorname{sgn} \sigma(u,c) + \lim_{c\to 0^+} \operatorname{sgn} \sigma(u,-c)\big) = \begin{cases} 0 & \text{if } u \in (F_0 \setminus F_1) \cup (F_2 \setminus F_3) \cup (F_4 \setminus F_5) \cup \cdots, \\ 1 & \text{if } u \in (F_1 \setminus F_2) \cup (F_3 \setminus F_4) \cup (F_5 \setminus F_6) \cup \cdots, \end{cases}$$

and the latter function can be expressed as a desired finite sum of signs; the detailed verification is left to the reader.

Corollary. *Half of the Euler characteristic $\chi(\mathrm{lk}(0; Z_u))$ is the sum of the signs of some smooth definable functions $\zeta_1, \ldots, \zeta_s \in \mathcal{D}(U)$:*

$$\frac{1}{2}\chi(\mathrm{lk}(0; Z_u)) = \operatorname{sgn} \zeta_1(u) + \cdots + \operatorname{sgn} \zeta_s(u) \quad \textit{for } u \in U.$$

Proof. Since $Z_u = W_u \cap W'_u$, this follows immediately from the Mayer–Vietoris sequence applied to the triad $(S_\varepsilon, W_u \cap S_\varepsilon, W'_u \cap S_\varepsilon)$.

Now we shall consider one smooth definable function $\varphi \in \mathcal{D}(U)$ on an open set $U \subset \mathbb{R}^n$ and the two sets determined by φ:

$$W := \{u \in U : \varphi(u) \geq 0\}, \qquad Z := \{u \in U : \varphi(u) = 0\}.$$

The function f induces the following definable family $f = (f_u)_{u\in U}$ of smooth function germs:

$$f_u : (\mathbb{R}^n, 0) \to \mathbb{R}^n, \qquad f_u(x) := \varphi(u + x).$$

Clearly, the germs at $u \in U$ of the sets W and Z are the translations by the vector u of the germs W_u and Z_u at the origin (determined by the family f). We may therefore summarize the foregoing theorems as follows.

Theorem 3. *The Euler characteristic $\chi(\mathrm{lk}(u; W))$ is the sum of the signs of some smooth definable functions $\sigma_1, \ldots, \sigma_r \in \mathcal{D}(U)$:*

$$\chi(\mathrm{lk}(u; W)) = \operatorname{sgn} \sigma_1(u) + \cdots + \operatorname{sgn} \sigma_r(u) \quad \textit{for } u \in U;$$

half of the Euler characteristic $\chi(\mathrm{lk}(u; Z))$ is the sum of the signs of some smooth definable functions $\zeta_1, \ldots, \zeta_s \in \mathcal{D}(U)$:

$$\frac{1}{2}\chi(\mathrm{lk}(u; Z)) = \operatorname{sgn} \zeta_1(u) + \cdots + \operatorname{sgn} \zeta_s(u) \quad \textit{for } u \in U.$$

Theorems 1, 2 and 3 are o-minimal analogues of Nowel's results [26] about analytic sets and functions. Finally, we state a parametric version of these formulae:

Let U and T be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Suppose $\varphi(u,t)$ is a smooth definable function on $U \times T$. For $t \in T$ put

$$W_t := \{u \in U : \varphi(u,t) \geq 0\}, \qquad Z_t := \{u \in U : \varphi(u,t) = 0\}.$$

Then the Euler characteristic $\chi(\mathrm{lk}(u; W_t))$ is the sum of the signs of some smooth definable functions $\sigma_1, \ldots, \sigma_r \in \mathcal{D}(U \times T)$:

$$\chi(\mathrm{lk}(u; W_t)) = \operatorname{sgn} \sigma_1(u,t) + \cdots + \operatorname{sgn} \sigma_r(u,t) \quad \text{for } u \in U,\ t \in T;$$

half of the Euler characteristic $\chi(\mathrm{lk}(u; Z_t))$ is the sum of the signs of some smooth definable functions $\zeta_1, \ldots, \zeta_s \in \mathcal{D}(U \times T)$:

$$\frac{1}{2}\,\chi(\mathrm{lk}(u; Z_t)) = \operatorname{sgn} \zeta_1(u,t) + \cdots + \operatorname{sgn} \zeta_s(u,t) \quad \text{for } u \in U,\ t \in T.$$

4. Appendix: smoothly definable Zariski topology is noetherian. As previously, we fix a polynomially bounded, o-minimal structure $\mathcal{R}$ on the field $\mathbb{R}$ of reals. A *definable leaf* in $\mathbb{R}^n$ is a definable, connected, locally closed subset of $\mathbb{R}^n$ that is a smooth submanifold in $\mathbb{R}^n$. We begin by decomposing the set Z into finitely many definable leaves.

THEOREM. *Let M be a definable, locally closed, smooth submanifold in an affine space $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ a smooth definable function. Then the set $Z = Z(f) := \{x \in M : f(x) = 0\}$ is a finite union of definable leaves.*

Proof. We proceed by induction on the dimension m of the ambient manifold M. The case $m = 1$ is obvious, so suppose $m > 1$. Consider the linear projections

$$\pi = \pi_{l_1,\ldots,l_m} : M \to \mathbb{R}^m, \qquad \pi(x) = \pi_{l_1,\ldots,l_m}(x) := (x_{l_1}, \ldots, x_{l_m}).$$

Then the sets

$$U = U_{l_1,\ldots,l_m} \subset M, \qquad 1 \leq l_1 < \cdots < l_m \leq n,$$

of those points at which the mappings π are local diffeomorphisms onto the image are definable open subsets of M. Since the sets U cover M, it suffices to decompose every set $Z \cap U$ into definable leaves. For simplicity of notation, we shall assume that U is a definable open subset of $\mathbb{R}^m$; we may also assume that U is connected and $f \not\equiv 0$. Put

$$Z_1 := \left\{ x \in U : \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } i = 1, \ldots, m \right\};$$

then Z_1 is a definable closed subset of U.

Clearly, $Z \cap (U \setminus Z_1)$ is a smooth definable submanifold of codimension 1 in $U \setminus Z_1$, and thus it decomposes into finitely many definable leaves. So we have to decompose $Z \cap Z_1$. Put

$$Z_2 := \left\{ x \in U : \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x) = 0 \text{ for all } i_1, i_2 = 1, \dots, m \right\};$$

then

$$U \setminus Z_2 := \left\{ x \in U : \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x) \neq 0 \text{ for some } i_1, i_2 = 1, \dots, m \right\},$$

and thus $Z_1 \setminus Z_2$ is contained in a finite union of smooth definable submanifolds of codimension 1, closed in the open subsets

$$\left\{ x \in U : \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x) \neq 0 \right\}.$$

By induction hypothesis, $Z \cap Z_1 \cap (U \setminus Z_2)$ decomposes into finitely many definable leaves. So we have to decompose $Z \cap Z_1 \cap Z_2$. We now repeat this process. The proof will be finished once we show that

$$Z \cap Z_1 \cap \dots \cap Z_k = \emptyset$$

for a sufficiently large k, where

$$Z_k := \left\{ x \in U : \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(x) = 0 \text{ for all } i_1, \dots, i_k = 1, \dots, m \right\}.$$

But the above intersection is the set of those points $x \in U$ at which the function f is k-flat, and therefore our assertion follows from the fact that the structure $\mathcal{R}$ is polynomially bounded (cf. [25]).

We shall associate with any decomposition of Z into finitely many definable leaves the multi-index $\mu = (\mu_m, \mu_{m-1}, \dots, \mu_0) \in \mathbb{N}^{m+1}$, where μ_i is the number of leaves of dimension i.

Denote by $\mu(Z)$ the smallest (with respect to the lexicographical ordering) among the multi-indices of such decompositions. Now we are in a position to achieve the main goal of the Appendix.

Corollary. *The smoothly definable Zariski topology on a definable open subset U of $\mathbb{R}^m$ is noetherian.*

Proof. It suffices to show that every descending sequence of Zariski closed sets of the form

$$Z_n = Z(f_n) := \{x \in U : f_n(x) = 0\}, \qquad f_n \in \mathcal{D}(U),\ n \in \mathbb{N},$$

stabilizes. Since the algebra $\mathcal{D}(U)$ of smooth definable functions on U is quasianalytic, we can easily deduce that $\mu(Z(f)) < \mu(Z(g))$ for any two functions $f, g \in \mathcal{D}(U)$ such that $Z(f) \subset Z(g)$ and $Z(f) \neq Z(g)$. Hence our assertion follows immediately.

Remarks. (1) The same proofs remain valid for the case of analytic definable submanifolds and functions, even though we drop the assumption that the structure $\mathcal{R}$ is polynomially bounded. (The condition of polynomial boundedness plays an essential role only in the fact that a smooth definable function which is infinitely flat at a point must vanish.) Consequently, the analytic Zariski topology on the set U with respect to the $\mathbb{R}$-algebra $\mathcal{A}(U)$ of analytic definable functions remains noetherian for arbitrary o-minimal structures $\mathcal{R}$ (see also [38] for some analytical background to topological noetherianity). If the o-minimal structure $\mathcal{R}$ is not polynomially bounded, the function exp is definable (the dichotomy principle; cf. [23, 12]). Then the set of Łojasiewicz exponents for a definable family $f = (f_u)_{u \in U}$ of analytic function germs at zero is a definable subset of $\mathbb{R}$ contained in $\mathbb{Q}$, and thus it is a finite set as well. Taking the above into account, we see that the theorems and proofs of this paper still hold for the case of analytic definable functions in arbitrary o-minimal structures.

(2) The $\mathbb{R}$-algebras $\mathcal{D}(U)$ under consideration may contain much more than only real-analytic functions. In particular, $\mathcal{D}(U)$ may embrace power functions with real exponents (cf. [24]) or quasianalytic Denjoy–Carleman classes (cf. [29]).

(3) Finally, it is worth pointing out that research concerning topological invariants of algebraic and analytic sets has been conducted by many mathematicians, for instance in [1, 2, 4, 8–10, 21, 22, 26–28, 34–36].

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