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# Plane Jacobian conjecture for simple polynomials

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**Abstract.** A non-zero constant Jacobian polynomial map $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ has a polynomial inverse if the component $P$ is a simple polynomial, i.e. its regular extension to a morphism $p : X \to \mathbb{P}^1$ in a compactification $X$ of $\mathbb{C}^2$ has the following property: the restriction of $p$ to each irreducible component $C$ of the compactification divisor $D = X - \mathbb{C}^2$ is of degree 0 or 1.

**1.** Let $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map, $P, Q \in \mathbb{C}[x, y]$, and denote by $JF := P_x Q_y - P_y Q_x$ the Jacobian of $F$. The mysterious Jacobian conjecture (JC) (see [4] and [2]), posed first by Ott-Heinrich Keller [7] in 1939 and still open, asserts that $F$ has a polynomial inverse if the Jacobian $JF$ is a non-zero constant. In 1979 by an algebraic approach Razar [18] proved this conjecture for the simplest geometrical case when $P$ is a rational polynomial, i.e. the generic fibre of $P$ is a punctured sphere, and all fibres $P = c$, $c \in \mathbb{C}$, are irreducible. In an attempt to understand the geometrical nature of (JC), this case was also reproved by Heitmann [5] and Lê and Weber [11] using some other approaches. In fact, as observed by Neumann and Norbury in [13], every rational polynomial with all irreducible fibres is equivalent to a coordinate polynomial. Most recently, Lê in [8] and [9] presented the following observation, which was announced at the Hanoi conference, 2006, and the Kyoto conference, 2007.

Theorem 1 (Theorem 3.2 and Corollary 3.8 in [9]). *A non-zero constant Jacobian polynomial map $F = (P, Q)$ has a polynomial inverse if $P$ is a simple rational polynomial.*

Here, following [12], a polynomial map $P : \mathbb{C}^2 \to \mathbb{C}$ is *simple* if, when $P$ is extended to a morphism $p : X \to \mathbb{P}^1$ of a compactification $X$ of $\mathbb{C}^2$, the restriction of $p$ to each irreducible component $\ell$ of the compactification

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divisor $D = X - \mathbb{C}^2$ is of degree either 0 or 1. In fact, as in the proof of Theorem 1 presented in [9], if a component of a non-zero constant Jacobian map $F = (P, Q)$ is a simple rational polynomial, then this component determines a locally trivial fibration.

In this short paper we would like to present another explanation for Theorem 1 from the viewpoint of the geometry of the non-proper value set of the map $F$. In fact, we shall prove

THEOREM 2. *A non-zero constant Jacobian polynomial map $F = (P, Q)$ has a polynomial inverse if $P$ is a simple polynomial.*

The proof of this theorem will be carried out in the next sections.

**2.** Given a polynomial map $F = (P, Q)$ of $\mathbb{C}^2$. Following [6], the *non-proper value set* $A_F$ of $F$ is the set of all values $a \in \mathbb{C}^2$ such that there exists a sequence $b_i \in \mathbb{C}^2$ with $\|b_i\| \to \infty$ and $F(b_i) \to a$. This set $A_F$ is either empty or an algebraic curve in $\mathbb{C}^2$ for which every irreducible component is the image of a non-constant polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$. Our argument in the proof of Theorem 2 is based on the following facts, which were presented in [14] and can be deduced from [3] (see also [15] and [16] for other refined versions).

THEOREM 3. *Suppose $F = (P, Q)$ is a polynomial map with non-zero constant Jacobian. If $A_F \neq \emptyset$, then every irreducible component of $A_F$ can be parameterized by polynomial maps $\xi \mapsto (\varphi(\xi), \psi(\xi))$ with*

$$\deg \varphi / \deg \psi = \deg P / \deg Q.$$

This theorem together with the Abhyankar–Moh theorem [1] on embedding line into plane allows us to obtain:

THEOREM 4. *A polynomial map $F$ of $\mathbb{C}^2$ must have singularities if its non-proper value set $A_F$ has an irreducible component isomorphic to the line.*

A simple proof of Theorem 4 recently presented in [16] gives a description of the singularities in terms of Newton–Puiseux data in this situation.

**3.** To use Theorem 4 in the situation of simple polynomials, we first need to describe the non-proper value curve $A_F$ in terms of the regular extension of $F$ in a compatification $X \supset \mathbb{C}^2$. Any polynomial $F = (P, Q)$ can be extended to a rational map $F : \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$ and we can resolve the points of indeterminacy by blowing ups to get a regular map $f = (p, q) : X \to \mathbb{P}^1 \times \mathbb{P}^1$ that coincides with $F = (P, Q)$ on $\mathbb{C}^2 \subset X$. We call the exceptional curve $D = X - \mathbb{C}^2$ the *divisor at infinity*. The divisor $D$ is a connected algebraic curve, every irreducible component of which is isomorphic to $\mathbb{P}^1$, and the dual graph of $D$ is a tree. Recall that a dual graph of the divisor $D$ is a graph in which each vertex corresponds to an irreducible component of $D$ and each edge joining two vertices $\ell$ and $\ell'$ corresponds to an intersection point of $\ell$ and $\ell'$. An irreducible component $\ell$ of $D$ is a *horizontal* component of $P$ ($Q$) if the restriction of $p$ (resp. $q$) to $\ell$ is not a constant mapping. An irreducible component $\ell$ of $D$ is a *dicritical* component of $F$ if the restriction of $f$ to $\ell$ is not a constant mapping. A dicritical component of $F$ must be a horizontal component of $P$ or $Q$. Although the compactification defined above is not unique, the horizontal components of $P$ and $Q$ as well as the dicritical components of $F$ are essentially independent of the choice of the compactification $X$ of $\mathbb{C}^2$, up to birational maps between the compactifications.

Set $D_\infty := f^{-1}((\{\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{\infty\}))$. The following description of the dual graph of the divisor $D$ is well-known (see, for example, [19], [17] and [10]).

Proposition 1.

(i) *The dual graph of the divisor $D$ is a tree.*
(ii) *The dual graph of the curve $D_\infty$ is a tree.*
(iii) *The dual graph of each connected component of the closure of $D - D_\infty$ is a linear path of the form*

$$\odot - \circ - \circ - \cdots - \circ - \circ$$

*in which the beginning vertex $\odot$ is a dicritical component of $F$ and the next possible vertices $\circ$ are those components such that the restrictions of $f$ are finite constant mappings.*

The following provides a description of the non-proper value set $A_F$ of $F$ in terms of regular extension of $F$ in a compatification $X$ of $\mathbb{C}^2$.

Proposition 2. (i) *We have*

$$A_F = \bigcup_{\text{dicritical components } \ell \text{ of } F} f(\ell) \cap \mathbb{C}^2.$$

(ii) *Let $\ell$ be a dicritical component of $F$. Then $\ell$ and the curve $D_\infty$ have a unique common point. Let $\ell^* := \ell - D_\infty$. Then the curve $\ell^*$ is isomorphic to $\mathbb{C}$ and*

$$f(\ell^*) = f(\ell) \cap \mathbb{C}^2.$$

(iii) *We have*

$$A_F = \bigcup_{\text{dicritical components } \ell \text{ of } F} f(\ell^*).$$

*Proof.* (i) Note that the closure of $\mathbb{C}^2$ in the compactification $X$ coincides with $X$. If $\ell \subset D$ is a dicritical component of $F$ and $b \in \ell$ such that $f(b) \in \mathbb{C}^2$, we can take a sequence $b_i \in \mathbb{C}^2$ tending to $b$. Then the sequence $F(b_i) = f(b_i)$ will tend to $f(b)$. Hence, by definition $f(b) \in A_F$. So, we get $f(\ell) \cap \mathbb{C}^2 \subset A_F$. Conversely, if $a \in A_F$ and $a = \lim_{i\to\infty} F(b_i)$ for a sequence $b_i \in \mathbb{C}^2$, then in view of Proposition 1(iii) we can assume that $b_i$ tends to a point $b$ lying in an irreducible component $L$ of a connected component $C$ of the closure of $D - D_\infty$. Let $\ell$ be the unique dicritical component in $C$. If $\ell \equiv L$, we have $a \in f(\ell)$. Otherwise, the restrictions of $f$ to $L$ as well as to other irreducible components of $C$ differing from $\ell$ are constant mappings with value $a$. Then, by the structure of the curve $C$ (Proposition 1(iii)), we can take another sequence $b'_i \in \mathbb{C}^2$ tending to a point $b' \in \ell$ such that $f(b') = a$. Thus, the value $a$ always belongs to the image $f(\ell)$ for a dicritical component $\ell$ of $F$.

(ii) Let $\ell$ be a dicritical component of $F$. By Proposition 1 the dual graphs of the divisors $D$ and $D_\infty$ are trees and the component $\ell$ is the beginning vertex of the dual graph of a connected component of the closure of $D - D_\infty$. This ensures that $\ell$ intersects $D_\infty$ in a unique point, the curve $\ell^* := \ell - D_\infty$ is isomorphic to $\mathbb{C}$ and $f(\ell^*) = f(\ell) \cap \mathbb{C}^2$.

(iii) Results from (i) and (ii). ∎

**4.** Now, we consider the situation when the restriction of $p$ to a dicritical component $\ell$ of $F$ is of degree 1.

LEMMA 1. *Let $\ell$ be a dicritical component of $F$. If the restriction of $p$ to $\ell$ is of degree* 1, *then the image $f(\ell^*)$ is isomorphic to the line $\mathbb{C}$.*

*Proof.* Suppose $\ell$ is a dicritical component of $F$ and the degree of the restriction $p_{|\ell}$ equals 1. Then $p_{|\ell} : \ell \to \mathbb{P}^1$ is injective, and hence bijective, since $\ell$ is isomorphic to $\mathbb{P}^1$. This ensures that the curve $f(\ell^*)$ intersects each line $\{(u, v) \in \mathbb{C}^2 : u = c\}$, $c \in \mathbb{C}$, in a unique point. Then the polynomial $H(u, v)$ defining the curve $f(\ell^*) \subset \mathbb{C}^2$ can be chosen of the form $v + h(u)$, $h \in \mathbb{C}[u]$. So, the automorphism $A(u, v) := (u, v - h(u))$ maps isomorphically the curve $f(\ell^*)$ onto the line $v = 0$. ∎

*Proof of Theorem 2.* Suppose $F = (P, Q)$ with $JF \equiv c \neq 0$ and $P$ is a simple polynomial. Note that each dicritical component of $F$ must be a horizontal component of $P$ or $Q$. Since $JF \equiv c \neq 0$ and $P$ is simple, in view of Theorem 4 and Lemma 1, a horizontal component of $P$ cannot be a dicritical component of $F$. So, if $\ell$ is a dicritical component of $F$, then $\ell$ must be a horizontal component of $Q$ and the restriction $p_{|\ell}$ maps $\ell$ to a finite constant. Thus, for such $\ell$ the image $f(\ell^*)$ is a line $u = \text{const}$. This is impossible again by Theorem 4 as $JF \equiv c \neq 0$. Hence, $F$ has no dicritical component. Thus, $A_F = \emptyset$ by Proposition 2 and $F$ is a proper map by the definition of $A_F$. Therefore, by simple connectedness of $\mathbb{C}^2$, the locally diffeomorphic map $F$ must be bijective. Thus, $F$ is an automorphism of $\mathbb{C}^2$.

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