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Extending Hardy fields by non-$\mathcal{C}^\infty$-germs

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Dedicated to Kasia, Kalina and Kajtek

Abstract. For a large class of Hardy fields their extensions containing non-$\mathcal{C}^\infty$-germs are constructed. Hardy fields composed of only non-$\mathcal{C}^\infty$-germs, apart from constants, are also considered.

1. Introduction. This paper concerns the problem of differentiability class of elements in a Hardy field. Recall the following (see [Bou] ([1]), [LigRob]):

Definition 1. A *Hardy field* is any subfield k of the ring of all germs at $+\infty$ of differentiable functions

$$f : (\eta, +\infty) \to \mathbb{R} \quad (\eta \in \mathbb{R}) \tag{1.1}$$

closed under differentiation.

This definition implies that each germ f of k is infinitely differentiable, but f may a priori not admit any representative of class $\mathcal{C}^\infty$. We will call such a germ f *singular*. An example of a Hardy field with singular germs was given by M. Boshernitzan [Bos1] (see also [Gok1], [Gok2]). Developing his idea we will prove the following:

Theorem 1. *Suppose a Hardy field k contains an element $h > 0$ such that for each $f \in k$ there exists $n \in \mathbb{N}$ for which $f < h^n$. Then k can be extended to a Hardy field by a singular germ.*

Definition 2. We call a Hardy field k satisfying the assumption of the theorem above a *self-bounded Hardy field*. The germ h is called a *self-bound* of k.

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([1]) It seems that the name *Hardy field* was used there for the first time.

The author's interest in singular germs in Hardy fields is partly motivated by the following open problem in o-minimal geometry (cf. example (5) below): *Does there exist an o-minimal structure on the ordered field* $\mathbb{R}$ *not admitting a* $\mathcal{C}^\infty$*-cell decomposition?*

Here are some examples of Hardy fields:

(1) $\mathbb{Q}$ is a Hardy field as the field of rational-valued constant functions germs. As it is the smallest field of characteristic 0, it is a subfield of any Hardy field.
(2) Similarly, $\mathbb{R}$ is a Hardy field. In both examples the differentiation is trivial (2). Any Hardy field can be extended to a Hardy field containing $\mathbb{R}$, because if k is a Hardy field and $a \in \mathbb{R}$ then $k(a)$ is a Hardy field as well (see example (3)).
(3) Let k be a Hardy field and let f be a $\mathcal{C}^1$-germ such that $f' \in k$. Then there exists the smallest Hardy field containing both k and f, namely $k(f)$ (see [Ros4, Theorem 2]).
(4) The field $\mathbb{R}(x)$ of germs of rational functions of one variable x over $\mathbb{R}$ is a Hardy field. By (3), any Hardy field can be extended to a Hardy field containing the germ x.
(5) Let $\mathcal{S}$ be an *o*-minimal structure on the ordered field $\mathbb{R}$ (see [Dri]). Then the germs at $+\infty$ of all functions $f : (a, +\infty) \to \mathbb{R}$ definable in $\mathcal{S}$ form a Hardy field.

We adopt the following notation. Let $\mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$ denote the ring of germs at $+\infty$ of all functions $f : (\eta, +\infty) \to \mathbb{R}$ ($\eta \in \mathbb{R}$) such that for each $n \in \mathbb{N}$ there exists $a(f, n) > 0$ such that $f_{(a(f,n),+\infty)}$ is a $\mathcal{C}^n$-function. We will call such f an *almost* $\mathcal{C}^\infty$*-germ* (3). Denote by $\mathcal{C}^\infty_{\mathbb{R},+\infty}$ the subring of $\mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$ consisting of all $\mathcal{C}^\infty$-germs, i.e. germs of all functions f which are $\mathcal{C}^\infty$ in some $(\eta, +\infty)$. Hence, by *singular germs* we mean elements of $\mathcal{C}^{(\infty)}_{\mathbb{R},+\infty} \setminus \mathcal{C}^\infty_{\mathbb{R},+\infty}$.

Definition 3 (see [Bos1]). An element $f \in \mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$, or a family $\{f_\nu\}_\nu \subset \mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$, is said to be $\mathcal{D}$*-consistent* (4) *with a Hardy field* k if there exists a Hardy field containing both k and f, respectively k and $\{f_\nu\}_\nu$. Denote by $k\langle f\rangle$, resp. $k\langle\{f_\nu\}_\nu\rangle$, the smallest such field, which obviously equals $k(f, f', \ldots)$, resp. $k(\{f_\nu\}_\nu, \{f'_\nu\}_\nu, \ldots)$.

Finally, a germ $f \in \mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$ is said to be $\mathcal{D}$*-consistent* if it is $\mathcal{D}$-consistent with $\mathbb{R}$.

(2) For a set F of germs we denote it by writing $F' = 0$.

(3) Following the terminology introduced in [Paw], where the term "almost $\mathcal{C}^\infty$-function" is used for functions with a similar differentiability property.

(4) Differentially consistent—see [Bos1].

REMARK 1. A germ $f \in \mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$ is $\mathcal{D}$-consistent with a Hardy field k if each element of the ring $k[f, f', \ldots]$ is of constant sign, in other words,

$$\forall n \in \mathbb{N}\ \forall p \in k[T_0, \ldots, T_n]\ \exists \varepsilon \in \{=, >, <\} : \quad p(f, \ldots, f^{(n)})\, \varepsilon\, 0.$$

Notice that Theorem 1 says that for any self-bounded Hardy field k there exists a singular germ which is $\mathcal{D}$-consistent with k.

Sometimes, it is more convenient to mean by a Hardy field a set of germs in right-hand neighbourhoods of 0. That is, we consider the germs at 0 of $f : (0, \eta) \to \mathbb{R}$ ($\eta > 0$) instead of germs at $+\infty$ of functions (1.1). In that case, we will use a similar notation $\mathcal{C}^{(\infty)}_{\mathbb{R},0^+}$ and $\mathcal{C}^{\infty}_{\mathbb{R},0^+}$, respectively. Via the isomorphism of rings

$$\mathcal{C}^{(\infty)}_{\mathbb{R},0^+} \ni f \mapsto f\left(\frac{1}{x}\right) \in \mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$$

we get a bijection of classes of Hardy fields ([5]) in $\mathcal{C}^{(\infty)}_{\mathbb{R},0^+}$ and in $\mathcal{C}^{(\infty)}_{\mathbb{R},+\infty}$. It follows that it does not matter which case we have in mind; all properties of Hardy fields hold equivalently in both cases. Denote by $\mathcal{C}_{\mathbb{R},0^+}$ the ring of all continuous germs, i.e. germs admitting continuous representatives. Here are further naturally arising examples of Hardy fields:

(6) Let k be a Hardy field and let $\varphi \in \mathcal{C}_{\mathbb{R},0^+}$ be an algebraic element over k. Then there exists the smallest Hardy field $k(\varphi)$ containing both k and φ (see [Ros4, Theorem 1]).

(7) Denote by $\mathcal{N}_{\mathbb{R},0^+}$ the field of germs of Nash functions. It is a Hardy field, being the algebraic closure of $\mathbb{R}(x)$ in $\mathcal{C}_{\mathbb{R},0^+}$.

(8) The field $\mathcal{M}_{\mathbb{R},0^+}$ of germs of functions meromorphic at 0 is a Hardy field extending $\mathbb{R}(x)$.

(9) The algebraic closure $\mathcal{A}_{\mathbb{R},0^+}$ of $\mathcal{M}_{\mathbb{R},0^+}$ in $\mathcal{C}_{\mathbb{R},0^+}$ is a Hardy field extending $\mathcal{N}_{\mathbb{R},0^+}$. The field $\mathcal{A}_{\mathbb{R},0^+}$ consists of germs of semianalytic functions (see [Loj]) and is isomorphic to $\mathbb{R}(\{x^*\})$, i.e. the field of all real convergent Puiseux series

$$\sum_{\nu=k}^{\infty} a_\nu x^{\nu/m},$$

where $k, m \in \mathbb{Z}$, $m > 0$, $a_\nu \in \mathbb{R}$ and there exist $M, L > 0$ such that $|a_\nu| \le ML^\nu$ for each $\nu \ge k$ (see [Rui], [Wal]).

(10) The fields $\mathbb{R}(\sin x, \cos x)$ and $\mathbb{R}(\exp x)$ are subfields of the Hardy field $\mathcal{M}_{\mathbb{R},0^+}$ not containing the germ of the identity function x.

There are germs $\mathcal{D}$-consistent with any Hardy field. By the above examples, so are for instance all constant germs, the germ x (as the germ of

([5]) Containing the germ of the identity function.

the identity function), each element of $\mathcal{N}_{\mathbb{R},0^+}$ and the germs $\exp_k(x)$ and $\log_k(x)$ for $k = 1, 2, \dots$ (6). Moreover, given a Hardy field k, the following germs are $\mathcal{D}$-consistent with k: elements of $\mathcal{C}_{\mathbb{R},0^+}$ that are algebraic over k, substitutions of any element of k into exp and log (7), germs of antiderivatives of representatives of elements k. Proofs of these properties can be found in [Bos1, Sections 5–7] and [Ros4, Chapter 1]. It is obvious that if a Hardy field k does not contain singular germs then, by the extension procedures listed above, it is not possible to produce an extension containing singular germs. What is more, all elements of the field $\mathcal{A}_{\mathbb{R},0^+}$ have analytic representatives.

2. Self-bounded Hardy fields. First, recall from [Ros4, Chapter 2] the notions of canonical valuation and comparability in a Hardy field k. The following equivalence relation is defined in $k^* := k \setminus \{0\}$. We write $f \asymp g$ and say f is *equivalent to* g iff $\lim_{x\to 0^+} f(x)/g(x) \in \mathbb{R}^*$. Denote by $v(f)$ the equivalence class of a germ f and by Γ the set of all equivalence classes.

THEOREM 2 (see [Ros4, Chapter 2]). *Γ has a structure of an ordered abelian group such that $v : k^* \to \Gamma$ satisfies the following conditions*:

1. *if $a, b \in k^*$, then $v(a \cdot b) = v(a) + v(b)$;*
2. *if $a \in k^*$, then $v(a) \geq 0$ iff $\lim_{x\to 0^+} a(x) \in \mathbb{R}$;*
3. *$v(0) := +\infty$ and if $a, b \in k$, then $v(a + b) \geq \min\{v(a), v(b)\}$ with equality if $v(a) \neq v(b)$;*
4. *if $a, b \in k^*$ and $v(a), v(b) \neq 0$, then $v(a) \geq v(b)$ iff $v(a') \geq v(b')$;*
5. *if $a, b \in k$ and $v(a) > v(b) \neq 0$, then $v(a') > v(b')$.*

DEFINITION 4. We say that $f, g \in k^*$ are *comparable* if $v(f), v(g) \neq 0$ and there exist $n, m \in \mathbb{N}$ such that $|v(f)| \leq n|v(g)|$ and $|v(g)| \leq m|v(f)|$.

Comparability of germs is an equivalence relation defined on the set of all $f \in k^*$ such that $v(f) \neq 0$. The equivalence class of a germ f is said to be the *comparability class* of f and denoted by $[f]$. Since for each $f \in k^*$ the germs f, $-f$, $1/f$, $-1/f$ have the same valuations (8), we can always choose from $[f]$ a representative that is *increasing to* $+\infty$, i.e. such that $\lim_{x\to 0^+} f(x) = +\infty$. It follows that considering comparability one can restrict oneself to germs of functions increasing to $+\infty$. For such f and g let $f \leq g$; then f and g are comparable iff there exists $n \in \mathbb{N}$ that $g \leq f^n$. We say $[f]$ is greater than $[g]$ and write $[g] < [f]$ if for any representatives f and g of these classes which are increasing to $+\infty$, the inequality $g^n < f$ holds for each $n \in \mathbb{N}$.

(6) Defined at 0^+ as follows: $\exp_0(x) := 1/x$ and $\exp_{k+1}(x) := \exp\exp_k(x)$ for $k \in \mathbb{N}$, and similarly for $\log_k$.

(7) Obviously, for log we may only consider germs with positive representatives.

(8) Up to sign.

Observe that if $k \not\subseteq \mathbb{R}$, then the Hardy field k is self-bounded iff the set of its comparability classes contains the biggest element. Hence, if k is self-bounded, then any germ increasing to $+\infty$ which is a representative of the biggest comparability class is a self-bound of k. Notice that if $k \not\subseteq \mathbb{R}$, then any self-bound of k is increasing to $+\infty$.

The following two classes of Hardy fields are worth mentioning.

EXAMPLE 1. A Hardy field k is said to be *polynomially bounded* if for each $f \in k$ there exists $n \in \mathbb{N}$ such that $f \le y^n$, where $y := x$ in the case of Hardy fields of germs at $+\infty$, and $y := 1/x$ in our case of 0^+. If k contains the germ y, then self-boundedness is obviously a generalization of polynomial boundedness. If y does not belong to k, then the self-boundedness of k follows from [Ros5, Chapter 3, Proposition 5].

EXAMPLE 2. Any Hardy field k of finite rank is self-bounded as well. The *rank of a Hardy field* k, denoted by $\operatorname{rk} k$, is the number of comparability classes of germs in k. We refer the reader to [Ros5]–[Ros7]. Here, we quote only two interesting facts: $\operatorname{rk} k = \#\{v(f'/f) : f \in k^*, v(f) \ne 0\}$ and if $k \subset K$ are Hardy fields such that $r := \operatorname{tr.deg}_k K < \infty$, then $\operatorname{rk} K \le r + \operatorname{rk} k$.

There are self-bounded Hardy fields of infinite rank; for instance, the field $\mathbb{R}(\{\log_k\}_{k=0,1,\ldots})$ (6) (see [Har2, Section III, Theorem 13]), which has infinitely many comparability classes represented by $\{\log_k\}_{k=0,1,\ldots}$. The biggest one of them is represented by $\log_0$. Of course, in general, Hardy fields of infinite rank are not self-bounded (9); for instance, the Boshernitzan class E which is defined as the intersection of all maximal Hardy fields (see [Bos1], [Bos2]).

3. Differentially transcendental power series. In this section we show a theorem [Bos1, Proposition 7.4] saying that any real analytic function of one variable such that the coefficients of its expansion at 0 are algebraically independent over $\mathbb{Q}$, does not satisfy any polynomial differential equation with coefficients in $\mathbb{R}$. In our construction of a field containing singular germs we will base on a version given in Theorem 3 below, which is a slight generalization of [Gok1, Proposition 0.1].

(9) Although Hardy fields of countable rank are bounded in the usual sense, i.e. for any such field K there exists a continuous germ g such that $f < g$ for each $f \in k$. Indeed, denote by $\{[\cdot]_\nu\}_{\nu=0}^\infty$ the set of all comparability classes in K. For any $\nu = 0, 1, \ldots$ let f_ν be a positive representative of $[\cdot]_\nu$, increasing to $+\infty$. Let $g_n : (\alpha_n, +\infty) \to (0, +\infty)$ $(n = 0, 1, \ldots)$ be the sequence of all powers f_ν^k, where $\nu = 0, 1, \ldots,\ k = 1, 2, \ldots.$ One can assume that $\{\alpha_n\}$ is increasing to $+\infty$. Let $h_n : \mathbb{R} \to [0, 1]$ be a continuous function with $\operatorname{supp} h_n \subset (\alpha_n, +\infty)$ and $h_n(x) = 1$ if $x \ge \alpha_{n+1}$. It is easy to see that K is bounded by $\varphi := \sum_{n=0}^\infty h_n g_n$.

Let $K \subset F$ be fields such that F is a differential field.

DEFINITION 5. An element $f \in F$ is said to be *differentially algebraic* over K if it satisfies a nontrivial polynomial differential equation with coefficients from K, i.e. there exist $n \in \mathbb{N}$ and $p \in K[T_0, \ldots, T_n] \setminus \{0\}$ such that $p(f, \ldots, f^{(n)}) = 0$. We say that F is differentially algebraic over K, or equivalently that the extension $K \subset F$ is differentially algebraic, if each element of F is differentially algebraic over K. Finally, an element $f \in F$ is said to be *differentially transcendental* over K if it is not differentially algebraic over K.

PROPOSITION 1. *$f \in F$ is differentially algebraic over K iff the transcendence degree of the extension $K\langle f\rangle$ over K is finite, i.e.* $\operatorname{tr.deg}_K K\langle f\rangle < \infty$.

Indeed, let $f \in F$ be differentially algebraic over K. Take $n \in \mathbb{N}$ as in Definition 5 to be the smallest possible and choose $p \in K[T_0, \ldots, T_n] \setminus \{0\}$ of minimal degree such that $p(f, \ldots, f^{(n)}) = 0$. Differentiating this equality, we find that $f^{(n+1)}$ is algebraically dependent on $f, \ldots, f^{(n)}$ over K. Repeating differentiating we see that $f^{(i)}$ for $i = n + 2, \ldots$ are algebraic over $K(f, \ldots, f^{(n)})$. On the other hand, if $\operatorname{tr.deg}_K K\langle f\rangle < \infty$ then of course there exists a derivative $f^{(n)}$ algebraically dependent on $K(f, f', \ldots, f^{(n-1)})$ (cf. [MarMesPil, Chapter II, Lemma 1.9]). As a consequence, differential algebraicity of field extensions is a transitive property (a fact hard to find in the literature) ([10]):

PROPOSITION 2. *Let $F_1 \subset F_2 \subset F_3$ be fields and assume that F_2 and F_3 are differential fields such that F_2 is differentially algebraic over F_1. Then $f \in F_3$ is differentially algebraic over F_2 if and only if it is differentially algebraic over F_1.*

Proof. Let $f \in F_3$ be differentially algebraic over F_2. Consider $n \in \mathbb{N}$ and a polynomial $p \in F_2[T_0, \ldots, T_n] \setminus \{0\}$ such that $p(T) = \sum_\alpha b_\alpha T^\alpha$ and $p(f, \ldots, f^{(n)}) = 0$. Take the smallest possible $n \in \mathbb{N}$ and p of minimal degree. By Proposition 1, $\{f^{(i)}\}_{i=0}^\infty$ are algebraic over

$$L := F_1\Big(\bigcup_\alpha \{b_\alpha^{(i)}\}_{i=0}^\infty, \{f^{(i)}\}_{i=0}^n\Big).$$

Hence, $\operatorname{tr.deg}_L L(\{f^{(i)}\}_{i=0}^\infty)$ is finite. Since $\operatorname{tr.deg}_{F_1} F_1(\bigcup_\alpha \{b_\alpha^{(i)}\}_{i=0}^\infty)$ is finite by assumption and $F_1 \subset L$, we conclude that $\operatorname{tr.deg}_{F_1} F_1(\{f^{(i)}\}_{i=0}^\infty)$ is finite. ∎

Consequently, any $f \in F_3$ is differentially transcendental over F_2 if and only if it is differentially transcendental over F_1.

([10]) All the properties of differentially algebraic extensions of fields considered here can be found in [MijMal, Section 2].

THEOREM 3. *Let F be a field of characteristic 0 and let $\{a_i\}_{i=0}^\infty \subset F$ be a sequence algebraically independent over $\mathbb{Q}$. Then the series*

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \in F[[x]], \tag{3.1}$$

with usual differentiation ([11]) *with respect to x, is differentially transcendental over $F(x)$.*

Proof. Since $\mathbb{Q} \subset F$ and $F \subset F(x)$ are differentially algebraic extensions ([12]), by Propositions 1 and 2, f is differentially transcendental over $F(x)$ iff it is differentially transcendental over $\mathbb{Q}$. Suppose the contrary, i.e. f is differentially algebraic over $\mathbb{Q}$. There exist $k \in \mathbb{N}$ and $q \in \mathbb{Q}[T_0,\dots,T_k]\setminus\{0\}$ such that $q(f(x),\dots,f^{(k)}(x)) = 0$. It follows that $q(f(0),\dots,f^{(k)}(0)) = q(a_0,\dots,k!a_k) = 0$, a contradiction. ■

COROLLARY 1. *If $\{a_i\}_{i=0}^\infty \subset \mathbb{R}$ is a sequence algebraically independent over $\mathbb{Q}$ and k is a Hardy field, then f is differentially transcendental over $k(x)$* ([13]).

This follows easily from the fact that k may be extended to a Hardy field K containing $\mathbb{R}$. Theorem 3 holds for $F = K$ and then we use the remark following Proposition 2.

COROLLARY 2. *If $\{a_i\}_{i=0}^\infty \subset \mathbb{R}$ is a sequence algebraically independent over $\mathbb{Q}$ and f is convergent, then f is differentially transcendental over any subfield L of $\mathbb{R}(\{x^*\})$ which is differentially algebraic over $\mathbb{Q}$.*

Taking $F = \mathbb{R}$ in the assumptions of Theorem 3 we find that f is differentially algebraic over $\mathbb{R}(x)$, hence over $\mathbb{Q}$. Since $f \in \mathbb{R}(\{x^*\})$, the remark following Proposition 2, with $F_1 = \mathbb{Q}$ and $F_2 = L$, shows that f is differentially transcendental over L.

4. A simplified version of Boshernitzan's example. Now we will give a simplified version of Boshernitzan's construction of a singular germ, which is suitable for extending the field $\mathcal{N}_{\mathbb{R},0^+}$ of Nash germs.

Let us start with the following example of a singular germ:

EXAMPLE 3. Let $g : (0,1) \to \mathbb{R}$ be the function given by the formula

$$g(x) := \left(x - \frac{1}{n}\right)^{n+1} \cdot \left(\frac{1}{n-1} - x\right)^{n+1}, \quad x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), \ n = 2,3,\dots \tag{g}$$

Then g has the following properties:

([11]) Hence, $F' = 0$.

([12]) The first because of trivial differentiation in F, and the second because x is differentially algebraic over $\mathbb{Q}$.

([13]) The independent variable of the germs in k is different from x.

$(g.1)$ $g \in \mathcal{C}^n\big(0, \frac{1}{n-1}\big) \setminus \mathcal{C}^{n+1}\big(0, \frac{1}{n-1}\big)$ for $n = 2, 3, \ldots$,
$(g.2)$ $\lim_{x\to 0^+} g^{(k)}(x)/x^s = 0$ for each k and $s \in \mathbb{N}$.

We are going to apply Corollary 2 with $L = \mathcal{N}_{x,0^+}$ ([14]). We have the following construction of a singular germ ([15]) $\mathcal{D}$-consistent with $\mathcal{N}_{x,0^+}$. Let h be the germ of $f + g$, where f is as in Corollary 2 and g is the function given in Example 3. Obviously, h is a singular germ. If $n \in \mathbb{N}$ and $p \in \mathcal{N}_{x,0^+}[T_0, \ldots, T_n] \setminus \{0\}$ we have the formula

$$p(h, \ldots, h^{(n)}) = p(f, \ldots, f^{(n)}) + \sum_{\substack{\sigma\in\mathbb{N}^{n+1}\\ 0<|\sigma|\le\deg p}} p_\sigma \cdot g^{\sigma_0}(g')^{\sigma_1}\cdots(g^{(n)})^{\sigma_n},$$

with p_σ being polynomials with respect to $f, \ldots, f^{(n)}$ over $\mathcal{N}_{x,0^+}$. Since we have $p(f, \ldots, f^{(n)}) \neq 0$, there exists $k \in \mathbb{N}$ such that $x^k < |p(f, \ldots, f^{(n)})|$ and at the same time $|p_\sigma/p(f, \ldots, f^{(n)})| < 1/x^k$ for each σ. Therefore, by $(g.2)$,

$$\lim_{x\to 0^+} \sum_{\substack{\sigma\in\mathbb{N}^{n+1}\\ 0<|\sigma|\le\deg p}} \frac{p_\sigma}{p(f, \ldots, f^{(n)})}\, g^{\sigma_0}(g')^{\sigma_1}\cdots(g^{(n)})^{\sigma_n} = 0.$$

This means $p(h, \ldots, h^{(n)})$ is of constant sign, and Remark 1 finishes the construction.

Remark 2. The field $\mathcal{N}_{x,0^+}$ cannot be replaced by $\mathcal{M}_{x,0^+}$ in the example above, though it is also polynomially bounded. The reason is that $p(f, \ldots, f^{(n)}) \in \mathcal{M}_{x,0^+}$ and so $g = (f+g) - f$ would be in $\mathcal{M}_{x,0^+}\langle h\rangle$ but g obviously does not belong to any Hardy field.

In order to obtain singular germs which are $\mathcal{D}$-consistent with any fixed self-bounded Hardy field we will modify the above construction.

5. Proof of main theorem. First we will prove the following:

Theorem 4. *Let $K \not\subseteq \mathbb{R}$ be a Hardy field self-bounded by h. Let ψ be a real function analytic at $0 \in \mathbb{R}$ and whose Taylor coefficients at 0 are algebraically independent over $\mathbb{Q}$. Then $\varphi := \psi(\exp(-h)) \in \mathcal{C}^{(\infty)}_{x,0^+}$ is differentially transcendental* ([16]) *over K and $\mathcal{D}$-consistent with K.*

Proof. Let $n \in \mathbb{N}$, $T = (T_0, \ldots, T_n)$ and let $p \in K[T]$ be of degree d. Note that φ is differentially transcendental over K iff for any such polynomial p the equality $p(\varphi(x), \varphi'(x), \ldots, \varphi^{(n)}(x)) = 0$ implies $p = 0$.

([14]) To emphasize the independent variable of germs we write $\mathcal{N}_{x,0^+}$ instead of $\mathcal{N}_{\mathbb{R},0^+}$.

([15]) Basing on [Bos1, Section 7] and [Gok2, Theorem 0.1].

([16]) With differentiation with respect to x, the independent variable of the germ φ, which is also the independent variable of germs in K.

Before proceeding with the proof let us make some remarks and introduce the notation that will be useful later.

(1) Set $y := \exp(-h)$ and observe that y is $\mathcal{D}$-consistent with K. For any $f \in K$ we have $yf \to 0$ as $x \to 0^+$ since the comparability class of y is greater ([17]) than that of any f.

(2) If Y is an independent variable (later, we will substitute y for Y) and $\psi(Y) = \sum_{q=0}^{\infty} a_q Y^q \in \mathbb{R}\{Y\}$ is an expansion of ψ at 0, then by Corollary 1 the series $\psi(Y)$ is differentially transcendental over $K(Y)$; i.e. for each $n \in \mathbb{N}$ and for each $p \in K(Y)[T_0, \ldots, T_n]$, if $p(\psi(Y), \ldots, \psi^{(n)}(Y)) = 0$, then $p = 0$.

(3) We have the following relations between derivatives of φ and ψ:

$$\varphi^{(k)}(x) = \sum_{i=1}^{k} \psi^{(i)}(y) y^i h_{ki}, \tag{5.1}$$

where $k = 1, 2, \ldots$ and $\{h_{ki}\}$ are of the following polynomial form:

$$h_{ki} = \sum_{\substack{\mu \in \mathbb{N}^k \\ i \le |\mu|,\, \|\mu\| = k}} d_\mu^{ki} (h'(x))^{\mu_1} \cdots (h^{(k)}(x))^{\mu_k}$$

with integers d_μ^{ki} and $|\mu| := \mu_1 + \cdots + \mu_k$, $\|\mu\| := 1 \cdot \mu_1 + \cdots + k \cdot \mu_k$. Obviously $\{h_{ki}\} \subset K$ and observe that $h_{kk} = (-h'(x))^k$.

(4) Let $\zeta_{ki}(Y) = Y^i h_{ki}$ and $\varphi^{(k)}(x) = \sum_{i=1}^{k} \psi^{(i)}(y) \zeta_{ki}(y)$. If we also define

$$\zeta_{ki}(Y) = \begin{cases} 1 & \text{for } k = 0,\ i = 0, \\ 0 & \text{for } k \neq 0,\ i = 0 \text{ or } 0 \le k < i, \end{cases}$$

then the matrix

$$A(Y) := [\zeta_{ki}(Y)]_{\substack{k=0,\ldots,n \\ i=0,\ldots,n}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & Yh_{11} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & Yh_{n1} & \cdots & Y^n h_{nn} \end{bmatrix},$$

is invertible in $K(Y)$, due to (3). Hence, for $T = (T_0, \ldots, T_n)$, the homomorphism $\ell : K(Y)[T] \ni p(T) \mapsto p(A(Y)T) \in K(Y)[T]$ is a bijection and $p(A(Y)T) = 0$ implies $p(T) = 0$. If $\Phi(x) := (\varphi(x), \varphi'(x), \ldots \ldots, \varphi^{(n)}(x))$ and $\Psi(y) := (\psi(y), \psi'(y), \ldots, \psi^{(n)}(y))$ then (5.1) can be written as $\Phi(x) = A(y)\Psi(y)$.

Suppose now $p(T) = \sum_{\alpha \in \mathbb{N}^{n+1}} c_\alpha T^\alpha$, where $|\alpha| \le d$ and $\{c_\alpha\} \subset K$, is such that $p(\Phi(x)) = p(A(y)\Psi(y)) = 0$. One can assume that all germs h_{ki} and c_α have limit 0 as $x \to 0^+$, since otherwise we replace p by $h^{-\lambda} p$ with $\lambda \in \mathbb{N}$

([17]) In fact, y is a bound of K in the usual sense.

sufficiently large. Hence, $p(\Phi)$ is a substitution of germs $\{h_{ki}\}$, $\{c_\alpha\}$ and y for variables H_{ki}, C_α and Y in a convergent power series $F(\{H_{ki}\},\{C_\alpha\},Y)$ with coefficients in $\mathbb{R}$. Notice that

$$F(\{H_{ki}\},\{C_\alpha\},Y)=\sum_{\mu=0}^{\infty}\Lambda_\mu(\{H_{ki}\},\{C_\alpha\})Y^\mu, \tag{5.2}$$

where the Λ_μ are polynomials in H_{ki}, C_α with coefficients in $\mathbb{R}$.

We claim that $\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})=0$ for each μ. Suppose the contrary and let $\mu_0:=\min\{\mu\in\mathbb{N}:\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})\neq 0\}$. Then

$$-\Lambda_{\mu_0}(\{h_{ki}\},\{c_\alpha\})=y\sum_{\mu=\mu_0+1}^{\infty}\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})y^{\mu-\mu_0-1}.$$

Since $\Lambda_{\mu_0}(\{h_{ki}\},\{c_\alpha\})\in K$, there exists $\varrho\in\mathbb{N}$ with $h^{-\varrho}\le|\Lambda_{\mu_0}(\{h_{ki}\},\{c_\alpha\})|$ and at the same time

$$y\Big|\sum_{\mu=\mu_0+1}^{\infty}\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})y^{\mu-\mu_0-1}\Big|<yC$$

for some positive constant C ([18]). Thus $h^{-\varrho}\le yC$, a contradiction, and therefore $F(\{h_{ki}\},\{c_\alpha\},Y)=0$.

Now, having $p(A(Y)\Psi(Y))=0$ and applying (2) to $p(A(Y)T)$ as an element of $K(Y)[T]$ we get $p(A(Y)T)=0$. Hence $p=0$ by (4), proving that φ is differentially transcendental over K.

It remains to prove that φ is $\mathcal{D}$-consistent with K. By Remark 1 we need to show that for each $p(T)\in K[T]\setminus\{0\}$ the nonzero germ $p(\Phi(x))$ is of constant sign. Taking an appropriate λ as in the first part of the proof, we get

$$h^{-\lambda}p(\Phi(x))=F(\{h_{ki}\},\{c_\alpha\},y),$$

where F is a convergent power series as in (5.2). Therefore there are $M,L>0$ such that $|\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})|<ML^\mu$ for each μ. If μ_0 is the smallest number for which $\Lambda_\mu:=\Lambda_\mu(\{h_{ki}\},\{c_\alpha\})$ is a nonzero germ in K, then

$$\frac{h^{-\lambda}p(\Phi(x))}{\Lambda_{\mu_0}y^{\mu_0}}=1+\sum_{\mu=\mu_0+1}^{\infty}\frac{\Lambda_\mu}{\Lambda_{\mu_0}}\,y^{\mu-\mu_0}.$$

Since h is a self-bound of K there exists $\varrho_0\in\mathbb{N}$ such that $h^{-\varrho_0}\le|\Lambda_{\mu_0}|$, and hence for some $C>0$ we get

$$\Big|\sum_{\mu=\mu_0+1}^{\infty}\frac{\Lambda_\mu}{\Lambda_{\mu_0}}\,y^{\mu-\mu_0}\Big|\le yh^{\varrho_0}ML^{\mu_0+1}\sum_{\mu=\mu_0+1}^{\infty}(Ly)^{\mu-\mu_0-1}<yh^{\varrho_0}C.$$

([18]) By convergence of (5.2).

Since $yh^{\varrho_0}C \to 0$ as $x \to 0^+$, we obtain

$$\lim_{x\to 0^+} \frac{h^{-\lambda}p(\Phi(x))}{\Lambda_{\mu_0}y^{\mu_0}} = 1. \tag{5.3}$$

The proof is complete. ■

For $f,\, g \in k^*$ we put

$$f \sim g \quad \text{iff} \quad \lim_{x\to 0^+} \frac{f(x)}{g(x)} = 1.$$

In the remainder of this section we will consider the situation described in Theorem 4. Notice that by (5.3) we have the following:

REMARK 3. For each $n \in \mathbb{N}$ and $p \in K[T_0,\dots,T_n] \setminus \{0\}$ there exist $\mu_p \in \mathbb{N}$ and $\Lambda_p \in K \setminus \{0\}$ such that $p(\Phi(x)) \sim \Lambda_p y^{\mu_p}$.

Let $g : (0,1) \to \mathbb{R}$ be the function given in Example 3 and suppose the assumptions of Theorem 4 hold.

THEOREM 5. *$f := (\psi + g) \circ \exp(-h) \in \mathcal{C}^{(\infty)}_{x,0^+}$ is $\mathcal{D}$-consistent with K.*

Proof. If $\gamma := g(y)$ and $\varphi = \psi(y)$ is as before, then $f = \varphi + \gamma$. For any $n \in \mathbb{N}$ and $p \in K[T_0,\dots,T_n] \setminus \{0\}$ we have

$$p(f,\dots,f^{(n)}) = p(\Phi(x)) + \sum_{\substack{\sigma\in\mathbb{N}^{n+1}\\ 0<|\sigma|\le \deg p}} p_\sigma(\Phi(x)) \cdot \gamma^{\sigma_0}(\gamma')^{\sigma_1}\cdots(\gamma^{(n)})^{\sigma_n},$$

with p_σ being polynomials with respect to $\varphi,\dots,\varphi^{(n)}$ over K. By Remark 3, $p(\Phi(x)) \sim \Lambda_p y^{\mu_p}$ for some nonzero $\Lambda_p \in K$ and $\mu_p \in \mathbb{N}$. Proceeding in a similar way for all nonzero p_σ, for each σ we have

$$\lim_{x\to 0^+} \frac{p_\sigma(\Phi(x))\gamma^{\sigma_0}(\gamma')^{\sigma_1}\cdots(\gamma^{(n)})^{\sigma_n}}{\Lambda_p y^{\mu_p}} = \lim_{x\to 0^+} \frac{\gamma^{\sigma_0}(\gamma')^{\sigma_1}\cdots(\gamma^{(n)})^{\sigma_n}}{\Lambda_p\Lambda_{p_\sigma}^{-1}y^{\mu_p-\mu_{p_\sigma}}},$$

and so, since $y < |\Lambda_p\Lambda_{p_\sigma}^{-1}|$, we obtain

$$\left|\frac{\gamma^{\sigma_0}(\gamma')^{\sigma_1}\cdots(\gamma^{(n)})^{\sigma_n}}{\Lambda_p\Lambda_{p_\sigma}^{-1}y^{\mu_p-\mu_{p_\sigma}}}\right| < \left|\frac{\gamma^{\sigma_0}(\gamma')^{\sigma_1}\cdots(\gamma^{(n)})^{\sigma_n}}{y^{\mu_p-\mu_{p_\sigma}+1}}\right|.$$

Therefore, due to (g.2), we have $p(f,\dots,f^{(n)}) \sim \Lambda_p y^{\mu_p}$. ■

Notice that if K does not contain singular germs, then its extension by the singular germ f given in Theorem 5 contains singular germs. Thus the proof of Theorem 1 is finished. Moreover, if the assumption $K \not\subseteq \mathbb{R}$ in Theorem 4 is not satisfied, Theorems 4 and 5 can be used for $K(x)$. Next, $\mathcal{D}$-consistency of f with K follows obviously from that with $K(x)$.

In the proof of Theorem 5, it is shown in fact that for each $n \in \mathbb{N}$ and $p \in K[T_0,\dots,T_n] \setminus \{0\}$ there exist $\mu_p \in \mathbb{N}$ and $\Lambda_p \in K \setminus \{0\}$ for which

$$p(f,\dots,f^n) \sim \Lambda_p y^{\mu_p}, \quad \text{where} \quad y = \exp(-h).$$

As a consequence we have the following:

THEOREM 6. *$K\langle f\rangle$ has the following properties*:

(1) *its valuation group equals $\mathbb{Z}\oplus\Gamma$ equipped with the lexicographical order with $\mathbb{Z}$ naturally ordered and with Γ the valuation group of K*;
(2) *if* $\operatorname{rk} K$ *is finite*, *then* $\operatorname{rk} K\langle f\rangle = \operatorname{rk} K + 1$;
(3) *$K\langle f\rangle$ is a self-bounded Hardy field and its self-bound is given by any germ u increasing to $+\infty$ such that $v(u) = -v(y)$.*

Proof. First, we note that the valuation group of $K\langle f\rangle$ is contained in $\mathbb{Z}\oplus\Gamma$. This follows from the fact that for each $u\in K\langle f\rangle^*$ there exist $\nu_u\in\mathbb{Z}$ and $\Lambda_u\in K^*$ such that $u\sim \Lambda_u y^{\nu_u}$. Thus

$$v(u) = v(\Lambda_u y^{\nu_u}) = v(\Lambda_u) + \nu_u v(y),$$

where obviously $v(\Lambda_u) < v(y)$ for each $\Lambda_u\in K^*$. To show the opposite inclusion, let $(\alpha,\beta)\in\mathbb{Z}\oplus\Gamma$. We have $v(f-a_0) = v(y)$ with a nonzero $a_0 := \psi(0)$. Hence, for each $\widetilde{u}\in K$ satisfying $v(\widetilde{u}) = \beta$, if $u := \widetilde{u}\cdot(f-a_0)^\alpha$ then $v(u) = (\alpha,\beta)$.

It remains to check that the lexicographical order in $\mathbb{Z}\oplus\Gamma$ is the same as that in the valuation group of $K\langle f\rangle$. Take $(\alpha_i,\beta_i)\in\mathbb{Z}\oplus\Gamma$ for $i=1,2$ and $u_i\in K\langle f\rangle^*$ such that $v(u_i) = (\alpha_i,\beta_i)$ and notice that there exist $\Lambda_i\in K^*$ for which $u_i\sim\Lambda_i y^{\alpha_i}$. It is sufficient to show that $v(u_1)<v(u_2)$ iff either $\alpha_1<\alpha_2$, or $\alpha_1=\alpha_2$ and $\beta_1<\beta_2$. This is a consequence of the equality

$$v\left(\frac{u_2}{u_1}\right) = v\left(\frac{\Lambda_2 y^{\alpha_2}}{\Lambda_1 y^{\alpha_1}}\right) = v\left(\frac{\Lambda_2}{\Lambda_1}\,y^{\alpha_2-\alpha_1}\right). \tag{5.4}$$

Indeed, $v(u_1)<v(u_2)$ iff $\lim_{x\to 0^+}(\Lambda_2/\Lambda_1)y^{\alpha_2-\alpha_1} = 0$ and since $v(\Lambda)<v(y)$ for each $\Lambda\in K^*$, by (5.4) we see that one of two cases holds:

- $\alpha_1=\alpha_2$; then $0<v(\Lambda_2/\Lambda_1)$ and hence $\beta_1<\beta_2$;
- $\alpha_1\neq\alpha_2$; then $0<\alpha_2-\alpha_1$ or $0<\alpha_1-\alpha_2$. If $0<\alpha_1-\alpha_2$ we obtain

$$v\left(\frac{\Lambda_2}{\Lambda_1}\right) < v(y) < v(y^{\alpha_1-\alpha_2}) + v\left(\frac{\Lambda_2}{\Lambda_1}\,y^{\alpha_2-\alpha_1}\right) = v\left(\frac{\Lambda_2}{\Lambda_1}\right),$$

  a contradiction.

The proof of (1) is finished.

Let $K_{[\cdot]}$ be the set of all comparability classes of germs in K. Recall Definition 4, where the comparability relation in K is defined via the comparability of nonzero elements in the valuation group of K. Basing on the proof of (1) we show that $K\langle f\rangle_{[\cdot]}$ equals $\{[u]\}\cup K_{[\cdot]}$, where u is an element of $K\langle f\rangle^*\setminus K$ such that its valuation in $\mathbb{Z}\oplus\Gamma$ is nonzero and comparable with $(1,0)$. Following the proof of (1), let $u_i\in K\langle f\rangle^*$ $(i=1,2)$ be such that appropriate valuations $(\alpha_i,\beta_i)\in\mathbb{Z}\oplus\Gamma$ are nonzero. By Definition 4,

$u_1, u_2 \in K\langle f\rangle^*$ are comparable iff there exist $n, m \in \mathbb{N}$ such that

$$|(\alpha_1,\beta_1)| < m|(\alpha_2,\beta_2)| < mn|(\alpha_1,\beta_1)|.$$

Since for each u in a Hardy field all germs u, $-u$, u^{-1}, $-u^{-1}$ are comparable, without loss of generality we may assume that $u_i \to 0$ as $x \to 0^+$, thus $0 < (\alpha_i, \beta_i)$. Consider two cases:

- $\alpha_1 = 0$, and then $|(0,\beta_1)| < m|(\alpha_2,\beta_2)| < mn|(0,\beta_1)|$. If $\alpha_2 \neq 0$, then we would have $0 < m\alpha_2 < 0$. Hence $\alpha_2 = 0$. Therefore, there exist comparable $\widetilde{u}_i \in K^*$ such that $(0,\beta_i) = v(\widetilde{u}_i)$ and $[u_i] = [\widetilde{u}_i] \in K_{[\cdot]}$.
- $\alpha_1 \neq 0$, and so obviously $\alpha_2 \neq 0$. By our choice we have $\alpha_i > 0$ and $(\alpha_i,\beta_i) < (\alpha_i+1)\cdot(1,0) < 3\cdot(\alpha_i,\beta_i)$. Thus, (α_i,β_i) are comparable ([19]) with $(1,0)$ in $\mathbb{Z}\oplus\Gamma$.

We have just shown that if a germ from $K\langle f\rangle^*$ is not comparable with an element of K^*, then its valuation as an element of $\mathbb{Z}\oplus\Gamma$ is comparable with $(1,0)$. Assuming the rank of $K\langle f\rangle$ is finite, we obtain $\operatorname{rk} K\langle f\rangle = \operatorname{rk} K + 1$.

Property (3) is an easy consequence of (2). In the extension $K \subset K\langle f\rangle$ exactly one new comparability class appears. That class is greater than all those in K. Therefore, it is the biggest comparability class and every representative of it increasing to $+\infty$ is a self-bound of $K\langle f\rangle$. ∎

6. Purely singular Hardy fields. We say that a Hardy field is *purely singular* if each of its nonconstant germs is singular. The aim of this section is to show that such Hardy fields exist. We use some natural facts concerning implicit functions. In particular, recall the following identity principle:

Remark 4. Let Ω be an open connected subset of $\mathbb{R}^n$ and consider a function $\Phi : \Omega\times\mathbb{R} \ni (u,w) \mapsto \Phi(u,w) \in \mathbb{R}$ of class $\mathcal{C}^1$. Suppose $\xi_i : \Omega \to \mathbb{R}$ for $i = 1,2$ are functions such that $\Phi(u,\xi_i(u)) = 0$ and $\frac{\partial\Phi}{\partial w}(u,\xi_i(u)) \neq 0$ for each $u \in \Omega$. If there exists $u_0 \in \Omega$ that $\xi_1(u_0) = \xi_2(u_0)$, then $\xi_1(u) = \xi_2(u)$ for each $u \in \Omega$.

Proof. This follows immediately from the fact that $\{u \in \Omega : \xi_1(u) = \xi_2(u)\}$ is an open and closed subset of Ω. ∎

As a consequence of Remark 4 we will obtain the following version of the implicit function theorem over a continuous curve:

Theorem 7. *Let* $\Phi : (\alpha,\beta)\times\mathbb{R}^{k+1} \ni (x,y_0,\ldots,y_k) \mapsto \Phi(x,y_0,\ldots,y_k) \in \mathbb{R}$, *where* $(\alpha,\beta) \subset \mathbb{R}$, *be a function of class* $\mathcal{C}^\infty$. *Assume that for some* $\gamma \in (\alpha,\beta)$ *we have a continuous* $\lambda : (\alpha,\gamma] \ni x \mapsto (x,\lambda_0(x),\ldots,\lambda_k(x)) \in \mathbb{R}^{k+2}$ *such that* $\lambda((\alpha,\gamma]) \subset \Phi^{-1}(0)$ *and* $\frac{\partial\Phi}{\partial y_k}(x,\lambda_0(x),\ldots,\lambda_k(x)) \neq 0$ *for each* $x \in (\alpha,\gamma]$. *Put*

([19]) In fact, $[u_i] = [y]$ in $(K\langle f\rangle)(y)$. As a representative of the class in $K\langle f\rangle$ we may take $f - a_0$.

$\widetilde{\lambda} : (\alpha, \gamma] \ni x \mapsto (x, \lambda_0(x), \dots, \lambda_{k-1}(x)) \in \mathbb{R}^{k+1}$. *Then there exists an open* $U \subset \mathbb{R}^{k+1}$ *such that* $\widetilde{\lambda}((\alpha, \gamma]) \subset U$ *and there exists a function* $\xi : U \to \mathbb{R}$ *of class* $\mathcal{C}^\infty$ *for which* $(\mathrm{id}_U, \xi)(U) \subset \Phi^{-1}(0)$ *and* $\lambda_k(x) = \xi(\widetilde{\lambda}(x))$ *for each* $x \in (\alpha, \gamma]$.

Proof. For $a \in \mathbb{R}^n$ and $\varrho > 0$ put $K(a, \varrho) := \{u \in \mathbb{R}^n : |a - u| < \varrho\}$. Notice that the continuity of λ and the implicit function theorem imply that for each $x \in (\alpha, \gamma]$ there exist $\varepsilon_x, r_x > 0$ such that for some open cylinder in $\mathbb{R}^{k+1}$ of the form

$$\Omega_x := K(x, \varepsilon_x) \times K((\lambda_0(x), \dots, \lambda_{k-1}(x)), r_x),$$

the following conditions are satisfied:

(1) $\Omega_x \subset (\alpha, \beta) \times \mathbb{R}^k$ and $\widetilde{\lambda}(K(x, \varepsilon_x) \cap (\alpha, \gamma]) \subset \Omega_x$;

(2) there exists an implicit function $\xi_x : \Omega_x \to \mathbb{R}$ of class $\mathcal{C}^\infty$ for which $(\mathrm{id}_{\Omega_x}, \xi_x)(\Omega_x) \subset \Phi^{-1}(0)$ and $\lambda_k(t) = \xi_x(\widetilde{\lambda}(t))$ for each $t \in K(x, \varepsilon_x) \cap (\alpha, \gamma]$.

In this way, we obtain a covering $\{\Omega_x\}_{x \in (\alpha, \gamma]}$ of $\widetilde{\lambda}((\alpha, \gamma])$ consisting of open subsets of $\mathbb{R}^{k+1}$ which, due to (1), satisfy the condition

$$\Omega_{x_1} \cap \Omega_{x_2} \neq \emptyset \;\Rightarrow\; \Omega_{x_1} \cap \Omega_{x_2} \cap \widetilde{\lambda}((\alpha, \gamma]) \neq \emptyset. \tag{6.1}$$

Hence, for any $x_1, x_2 \in (\alpha, \gamma]$, if $\Omega_{x_1} \cap \Omega_{x_2} \neq \emptyset$, then $\xi_{x_1} = \xi_{x_2}$ on the convex set $\Omega_{x_1} \cap \Omega_{x_2}$. Indeed, this follows by Remark 4 from (6.1) and (2). Finally, $U := \bigcup_{x \in (\alpha, \gamma]} \Omega_x$ and $\xi := \bigcup_{x \in (\alpha, \gamma]} \xi_x : U \to \mathbb{R}$ satisfy the conclusion. ∎

Theorem 8. *If* $\xi : (\alpha, \beta) \times \mathbb{R}^k \to \mathbb{R}$, *where* $(\alpha, \beta) \subset \mathbb{R}$, *is a function of class* $\mathcal{C}^\infty$ *and* $y : (\alpha, \gamma) \to \mathbb{R}$, *where* $\alpha < \gamma < \beta$, *is a solution of class* $\mathcal{C}^k$ *of* $y^{(k)} = \xi(x, y, y', \dots, y^{(k-1)})$, *then* y *is a* $\mathcal{C}^\infty$*-function.*

Proof. We prove by induction that y is of class $\mathcal{C}^p$ for each $p \geq k$. Suppose y is a $\mathcal{C}^p$-function on (α, γ) for some $p \geq k$. Then $y^{(k-1)}$ is of class $\mathcal{C}^{p-k+1}$ on (α, γ) and therefore also $\xi(x, y, y', \dots, y^{(k-1)})$ is a function of that class. Hence $y^{(k)}$ is of class $\mathcal{C}^{p-k+1}$ on (α, γ), i.e. y is of class $\mathcal{C}^{p+1}$. ∎

Theorem 9. *Each singular germ which is* $\mathcal{D}$*-consistent with a Hardy field* $k \subset \mathcal{C}^\infty_{\mathbb{R},0^+}$ *is differentially transcendental over* k.

Proof. Suppose f is differentially algebraic over k. Let

$$W(Y_0, \dots, Y_l) = \sum_\alpha a_\alpha(x) Y_0^{\alpha_0} \cdots Y_l^{\alpha_l} \in k[Y_0, \dots, Y_l] \setminus \{0\}$$

be of a minimal degree such that $W(f, \dots, f^{(l)}) = 0$ with $l \in \mathbb{N}$ the smallest possible. Since l and the degree of W are minimal and f is $\mathcal{D}$-consistent with k, $\frac{\partial W}{\partial Y_l}(f(x), f'(x), \dots, f^{(l)}(x)) \neq 0$ on some open interval of the form $(0, \varepsilon)$. Put now $\Phi(x, y_0, \dots, y_l) := W(y_0, \dots, y_l)$ and note that Φ satisfies the assumptions of Theorem 7. By Theorem 8, f is of class $\mathcal{C}^\infty$. ∎

Hence, we can prove the following theorem:

THEOREM 10. *For any singular $\mathcal{D}$-consistent germ f, $\mathbb{R}\langle f\rangle$ is a purely singular Hardy field.*

Proof. Define $k := \mathbb{R}\langle f\rangle$. Each element h of k is of the form

$$\frac{p(f, f', \ldots, f^{(l)})}{q(f, f', \ldots, f^{(l)})}$$

for some $l = 0, 1, \ldots$ and $p, q \in \mathbb{R}[Y_0, \ldots, Y_l]$, where $q(f, f', \ldots, f^{(l)}) \neq 0$. By Theorem 9, the last condition is equivalent to $q \neq 0$.

Suppose the theorem is not true and let l be the smallest possible and the sum of degrees of p and q is minimal for which a nonconstant germ h admits a representative

$$h(x) = \frac{p(f(x), f'(x), \ldots, f^{(l)}(x))}{q(f(x), f'(x), \ldots, f^{(l)}(x))}$$

of class $\mathcal{C}^\infty$ on some open interval $(0, \varepsilon)$. Consider the function

$$\Phi(x, y_0, \ldots, y_l) := h(x)q(y_0, \ldots, y_l) - p(y_0, \ldots, y_l)$$

defined on $(0, \varepsilon) \times \mathbb{R}^{l+1}$ and of class $\mathcal{C}^\infty$. We check that Φ satisfies the assumptions of Theorem 7. Of course, $\Phi(x, f(x), \ldots, f^{(l)}(x)) = 0$ for each $x \in (0, \varepsilon)$. It remains to show that $\frac{\partial \Phi}{\partial y_l}(x, f(x), \ldots, f^{(l)}(x)) \neq 0$ for each $x \in (0, \varepsilon_1)$ with some $\varepsilon_1 > 0$. Since f is $\mathcal{D}$-consistent, it is sufficient to show that $\frac{\partial \Phi}{\partial y_l}(x, f(x), \ldots, f^{(l)}(x)) = 0$ does not hold on $(0, \varepsilon_1)$, for any $\varepsilon_1 > 0$. Suppose the contrary and consider the following two cases:

- $\frac{\partial p}{\partial y_l}(y_0, \ldots, y_l) \neq 0$ and $\frac{\partial q}{\partial y_l}(y_0, \ldots, y_l) \neq 0$. Then we get

$$h(x) = \frac{\frac{\partial p}{\partial y_l}(f(x), f'(x), \ldots, f^{(l)}(x))}{\frac{\partial q}{\partial y_l}(f(x), f'(x), \ldots, f^{(l)}(x))}$$

  on some $(0, \varepsilon_1)$. This contradicts the minimality of the sum of the degrees of p and q chosen as representatives of h.
- $\frac{\partial p}{\partial y_l}(y_0, \ldots, y_l) = 0$ or $\frac{\partial q}{\partial y_l}(y_0, \ldots, y_l) = 0$. Since l is supposed to be the smallest possible, these equalities cannot both be satisfied at the same time. Hence, we get

$$h(x)\,\frac{\partial q}{\partial y_l}(f(x), f'(x), \ldots, f^{(l)}(x)) = 0$$

  or

$$\frac{\partial p}{\partial y_l}(f(x), f'(x), \ldots, f^{(l)}(x)) = 0,$$

  respectively. Each of these holds on some $(0, \varepsilon_1)$ and thus f is differentially algebraic over $\mathbb{R}$. This contradicts Theorem 9.

Since the assumptions of Theorem 7 are satisfied, f is of class $\mathcal{C}^\infty$ by Theorem 8, a contradiction. ■

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