ANNALES
POLONICI MATHEMATICI
101.1 (2011)

Pullback attractors for nonautonomous parabolic equations involving weighted p-Laplacian operators

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Abstract. Using the asymptotic *a priori* estimate method, we prove the existence of pullback attractors for nonautonomous quasilinear degenerate parabolic equations involving weighted p-Laplacian operators in bounded domains, without restriction on the growth order of the polynomial type nonlinearity and on the exponential growth of the external force. The results obtained improve some recent ones for nonautonomous reaction-diffusion equations. Moreover, a relationship between pullback attractors and uniform attractors is given.

1. Introduction. Nonautonomous equations appear in many applications in the natural sciences, so they are of great importance and interest. The long-time behavior of solutions of such equations have been studied extensively in the last years. The first attempt was to extend the notion of global attractors to the nonautonomous case leading to the concept of uniform attractor (see [8]). It is remarkable that the conditions ensuring the existence of the uniform attractor parallel those for the autonomous case. However, one disadvantage of the uniform attractor is that it need not be "invariant" unlike the global attractor for autonomous systems. Moreover, it is well-known that the trajectories may be unbounded for many nonautonomous systems when time tends to infinity and the uniform attractor for such systems does not exist. In order to overcome this drawback, a new concept, called a pullback attractor, has been introduced for nonautonomous equations. Several variations are then developed. On the one hand, there exists the pullback attractor of "fixed" bounded sets as the most usual option [9]. On the other hand, several authors use the concept of attraction in a universe $\mathcal{D}$ not only composed by a "fixed" set, but also moving in time, which usually appears in applications and is defined in terms of a tempered

2010 *Mathematics Subject Classification*: 35B41, 35K65, 35D05.
Key words and phrases: nonautonomous parabolic equation, weighted p-Laplacian operator, global solution, pullback attractor, uniform attractor, compactness and monotonicity methods, asymptotic *a priori* estimate method.

DOI: 10.4064/ap101-1-1

condition [6]. We refer the reader to the interesting paper [18] for a comparison of these two concepts of pullback attractors. The theory of pullback attractors has been developed for both nonautonomous and random dynamical systems and has proved useful in the understanding of the dynamics of nonautonomous dynamical systems because it allows one to consider a larger class of nonautonomous forces than the theory of uniform attractors does.

In this paper, we study the long-time behavior of solutions to the following nonautonomous quasilinear parabolic equation with a variable, nonnegative coefficient, defined on a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with boundary $\partial\Omega$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma|\nabla u|^{p-2}\nabla u) + f(u) &= g(x,t), \qquad x \in \Omega,\ t > \tau, \\ u|_{t=\tau} &= u_\tau(x), \qquad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{1.1}$$

where $2 \leq p < N$, $u_\tau \in L^2(\Omega)$ is given, the diffusion coefficient σ, the nonlinearity f, and the external force g satisfy some conditions specified later.

Problem (1.1) may be degenerate in the sense that the measurable, nonnegative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes. More precisely, we assume that the function $\sigma : \Omega \to \mathbb{R}$ satisfies

(H1) $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,p)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for all $z \in \overline{\Omega}$.

The physical motivation of assumption (H1) is related to the modeling of reaction diffusion processes in composite materials, occupying a bounded domain Ω, which at some points behave as *perfect insulators*. Following [10, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\sigma(x)$ vanishes at those points. Note that in various diffusion processes, the equation involves diffusion $\sigma(x) \sim |x|^\alpha$, $\alpha \in (0,p)$.

In the case that $\sigma(x)$ satisfies condition (H1), problem (1.1) contains some important classes of parabolic equations, such as semilinear heat equations (when $\sigma = 1, p = 2$), semilinear degenerate parabolic equations (when $p = 2$), p-Laplacian equations (when $\sigma = 1, p \neq 2$), etc. In the autonomous degenerate case, that is, the case of g independent of time t, the existence and long-time behavior of solutions to problem (1.1) when $p = 2$ have been studied in [11, 12] and recently in [1, 2]; the quasilinear case $1 < p \neq 2 < N$ has been investigated in [3, 4].

In this paper we continue the study of the long-time behavior of solutions to problem (1.1) in the case of the external force g depending on time t by using the theory of pullback attractors. To study problem (1.1) we assume the following conditions:

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying:

$$C_1|u|^q - k_1 \le f(u)u \le C_2|u|^q + k_2, \qquad q \ge 2, \tag{1.2}$$

$$f'(u) \ge -\ell, \tag{1.3}$$

where C_i, k_i and ℓ are positive constants;

(H3) $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\begin{gathered}\int\limits_{-\infty}^{0} e^{\zeta s}(\|g(s)\|^2_{L^2(\Omega)} + \|g'(s)\|^2_{L^2(\Omega)})\,ds < \infty,\\ \int\limits_{-\infty}^{0}\int\limits_{-\infty}^{s} e^{\zeta r}\|g(r)\|^2_{L^2(\Omega)}\,dr\,ds < \infty,\end{gathered} \tag{1.4}$$

where ζ is a fixed positive number;

(H4) $[1, p^*_\alpha) \cap \mathcal{I}[p', q'] \ne \emptyset$, where $p' := p/(p-1)$ is the conjugate exponent of p and

$$\mathcal{I}[p', q'] := \{(1-t)p' + tq' : 0 \le t \le 1\}, \qquad p^*_\alpha := \frac{pN}{N - p + \alpha}.$$

Let us make some comments about assumptions (H2)–(H4). The nonlinearity f is assumed to have polynomial growth and to satisfy a standard dissipative condition. Typical examples of functions satisfying condition (H2) are polynomials with odd degree and positive leading coefficient. The conditions in (H3) hold if $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ and there exist $\gamma \in (0, \zeta), \tau \in \mathbb{R}$ (we can assume $\tau < 0$) and $M_\tau > 0$ such that $\|g(t)\|^2_{L^2(\Omega)} + \|g'(t)\|^2_{L^2(\Omega)} \le M_\tau e^{-\gamma t}$ for all $t \le \tau$. In particular, (H3) holds if $\|g(t)\|^2_{L^2(\Omega)} + \|g'(t)\|^2_{L^2(\Omega)} \le M e^{\zeta|t|}$ for all $t \in \mathbb{R}$. Finally, (H4) is a technical condition, which is necessary to prove the existence of a weak solution to problem (1.1) using the compactness method (see [3] for details).

In order to study problem (1.1) we introduce the natural energy space $\mathcal{D}^{1,p}_0(\Omega, \sigma)$ defined as the closure of $C^\infty_0(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^{1,p}_0(\Omega,\sigma)} := \Big(\int\limits_{\Omega} \sigma(x)|\nabla u|^p\,dx\Big)^{1/p},$$

and prove some compactness results. The main aim of this paper is to prove the existence of a pullback attractor in the space $\mathcal{D}^{1,p}_0(\Omega, \sigma) \cap L^q(\Omega)$ for the process associated to problem (1.1).

Let us describe the methods used in this paper (we refer the reader to Sect. 2 for relevant concepts). First, we use the compactness and monotonicity methods [15] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of a family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ of pullback absorbing sets in $\mathcal{D}^{1,p}_0(\Omega, \sigma) \cap L^q(\Omega))$ for the process associated to problem (1.1). By the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^2(\Omega)$, the process is pullback asymptotically compact in $L^2(\Omega)$. This immediately implies the existence of a pullback attractor in $L^2(\Omega)$. When proving the existence of pullback attractors in $L^q(\Omega)$ and in $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$, to overcome the difficulty due to the lack of embbeding results, we use the asymptotic *a priori* estimate method initiated in [17] for autonomous equations and developed in [16] for nonautonomous equations. One of the main new features in our paper is that the existence of pullback attractors is proved for a class of quasilinear degenerate parabolic equations. It is also worth noticing that, when $p = 2$, $\sigma = 1$, our results improve the recent ones in [20, 13, 14] for nonautonomous Laplacian equations and, as far as we know, the results are new even for p-Laplacian equations.

The content of the paper is as follows. In Section 2, for the convenience of readers, we recall some concepts and results on function spaces and pullback attractors which we will use. In Section 3, we construct the process associated to problem (1.1) and prove the existence of pullback attractors in various spaces by using the asymptotic *a priori* estimate method. The existence of uniform attractors and a relationship between pullback attractors and uniform attractors are proved in the last section.

2. Preliminaries

2.1. Function spaces and operators. In order to study problem (1.1), we introduce the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|v\|_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)|\nabla v|^p\,dx\Big)^{1/p},$$

and denote by $\mathcal{D}^{-1,p'}(\Omega,\sigma)$ the dual space of $\mathcal{D}_0^{1,p}(\Omega,\sigma)$.

We recall some compactness results from [3].

LEMMA 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^n$, $N \geq 2$, and σ satisfies the hypothesis* (H1). *Then the following embeddings hold:*

(i) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset W_0^{1,\beta}(\Omega)$ *continuously if* $1 \leq \beta < pN/(N+\alpha)$;
(ii) $\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset L^r(\Omega)$ *compactly if* $1 \leq r < p_\alpha^*$.

Put

$$L_{p,\sigma}u := -\operatorname{div}(\sigma(x)|\nabla u|^{p-2}\nabla u), \qquad u \in \mathcal{D}_0^{1,p}(\Omega,\sigma).$$

The following lemma, whose proof is straightforward, gives some important properties of the operator $L_{p,\sigma}$.

LEMMA 2.2. *The operator $L_{p,\sigma}$ maps $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ into its dual $\mathcal{D}^{-1,p'}(\Omega,\sigma)$. Moreover,*

(1) $L_{p,\sigma}$ *is hemicontinuous, i.e., for all* $u, v, w \in \mathcal{D}_0^{1,p}(\Omega, \sigma)$*, the map* $\lambda \mapsto \langle L_{p,\sigma}(u + \lambda v), w\rangle$ *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$*.*

(2) $L_{p,\sigma}$ *is strongly monotone when* $p \geq 2$*, that is, there exists* $\delta > 0$ *such that*

$$\langle L_{p,\sigma}u - L_{p,\sigma}v, u - v\rangle \geq \delta \|u - v\|^2_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} \quad \textit{for all } u, v \in \mathcal{D}_0^{1,p}(\Omega, \sigma).$$

LEMMA 2.3. *If* $p > 2$*, then for any* $\zeta > 0$*, there is a positive number* $C = C(p, \zeta)$ *such that*

$$\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} \geq \zeta \|u\|^2_{L^2(\Omega)} - C \quad \textit{for all } u \in \mathcal{D}_0^{1,p}(\Omega, \sigma). \tag{2.1}$$

Proof. We have

$$\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} \geq \lambda_1 \|u\|^p_{L^p(\Omega)}, \tag{2.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $L_{p,\sigma}u := -\mathrm{div}(\sigma|\nabla u|^{p-2}\nabla u)$ in Ω with the homogeneous Dirichlet condition. On the other hand, we have

$$\|u\|^p_{L^p(\Omega)} \geq |\Omega|^{(2-p)/2} \|u\|^p_{L^2(\Omega)}, \tag{2.3}$$

and, by the Young inequality, for any $z > 0$ we have

$$z^p \geq \zeta |\Omega|^{(p-2)/2} \lambda_1^{-1} z^2 - C, \tag{2.4}$$

where C depends only on p and ζ. Combining (2.2)–(2.4) we obtain (2.1). ∎

2.2. Pullback attractors. Let X be a Banach space with norm $\|\cdot\|$. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X. For $A, B \subset X$, the Hausdorff semi-distance between A and B is defined by

$$\mathrm{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|.$$

Let $\{U(t, \tau) : t \geq \tau,\ t, \tau \in \mathbb{R}\}$ be a process in X, i.e., a two-parameter family of mappings $U(t, \tau) : X \to X$ such that $U(\tau, \tau) = \mathrm{Id}$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau$, $t, s, \tau \in \mathbb{R}$. The process $\{U(t, \tau)\}$ is said to be *norm-to-weak continuous* if $U(t, \tau)x_n \rightharpoonup U(t, \tau)x$ for all $t \geq \tau$ whenever $x_n \to x$ in X. The following result can be used to verify that a process is norm-to-weak continuous.

LEMMA 2.4 ([21]). *Let* X, Y *be two Banach spaces, and* X^*, Y^* *their respective dual spaces. Assume that* X *is dense in* Y*, the injection* $i : X \to Y$ *is continuous, its adjoint* $i^* : Y^* \to X^*$ *is dense, and* $\{U(t, \tau)\}$ *is a continuous (or weak continuous) process on* Y*, that is,* $U(t, \tau)x_n \to U(t, \tau)x$ *in* Y *as* $x_n \to x$ *in* Y *(or* $U(t, \tau)x_n \rightharpoonup U(t, \tau)x$ *in* Y *as* $x_n \rightharpoonup x$ *in* Y*), for all* $t \geq \tau, \tau \in \mathbb{R}$*. Then* $\{U(t, \tau)\}$ *is norm-to-weak continuous on* X *iff for all* $t \geq \tau$*,* $U(t, \tau)$ *maps compact subset of* X *to bounded subsets of* X*.*

DEFINITION 2.5 ([14]). The process $\{U(t, \tau)\}$ is said to be *pullback asymptotically compact* if for any $t \in \mathbb{R}$ and $D \in \mathcal{B}(X)$, and any sequences

$\tau_n \to -\infty$ and $x_n \in D$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 2.6. A process $\{U(t,\tau)\}$ is called *pullback ω-limit compact* if for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $D \in \mathcal{B}(X)$, there exists a $\tau_0 = \tau_0(D,\varepsilon,t) \le t$ such that

$$\alpha\Big(\bigcup_{\tau\le\tau_0} U(t,\tau)D\Big) \le \varepsilon,$$

where α is the Kuratowski measure of noncompactness of $B \in \mathcal{B}(X)$,

$$\alpha(B) = \inf\{\delta > 0 : B \text{ has a finite open cover by sets of diameter } \le \delta\}.$$

Lemma 2.7 ([14]). *A process $\{U(t,\tau)\}$ is pullback asymptotically compact if it is ω-limit compact.*

Definition 2.8. A family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ of bounded sets is called *pullback absorbing* for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$ and $D \in \mathcal{B}(X)$, there exists $\tau_0 = \tau_0(D,t) \le t$ such that

$$\bigcup_{\tau\le\tau_0} U(t,\tau)D \subset B(t).$$

Definition 2.9. The family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a *pullback attractor* for $\{U(t,\tau)\}$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;
(2) $\hat{\mathcal{A}}$ is invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t) \quad \text{for } t \ge \tau;$$

(3) $\hat{\mathcal{A}}$ is pullback attracting, i.e.,

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D, A(t)) = 0 \quad \text{for all } D \in \mathcal{B}(X), \text{ and } t \in \mathbb{R};$$

(4) if $\{C(t) : t \in \mathbb{R}\}$ is any family of closed pullback attracting sets then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.10 ([14]). *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback asymptotically compact. If there exists a family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ of pullback absorbing sets, then $\{U(t,\tau)\}$ has a unique pullback attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)B(\tau)}.$$

3. Existence of pullback attractors. Denote

$$V = L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)) \cap L^q(\tau,T;L^q(\Omega)),$$
$$V^* = L^{p'}(\tau,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)) + L^{q'}(\tau,T;L^{q'}(\Omega)),$$

where q' is the conjugate of q. From now on, for brevity, we denote by $|\cdot|_2, (\cdot,\cdot)$ the norm and scalar product of $L^2(\Omega)$, and by $|\cdot|_q$, $\|\cdot\|$ the norms in the spaces $L^q(\Omega)$ and $\mathcal{D}_0^{1,p}(\Omega,\sigma)$, respectively.

DEFINITION 3.1. A function $u(x,t)$ is called a *weak solution* of (1.1) on (τ,T) if

$$u \in V, \quad \frac{\partial u}{\partial t} \in V^*,$$
$$u|_{t=\tau} = u_\tau \quad \text{a.e. in } \Omega,$$

and
$$\int_\tau^T \int_\Omega \left(\frac{\partial u}{\partial t}\varphi + \sigma|\nabla u|^{p-2}\nabla u \nabla\varphi + f(u)\varphi \right) dx\, dt = \int_\tau^T \int_\Omega g\varphi\, dx\, dt$$

for all test functions $\varphi \in V$.

It follows from Theorem 1.8 in [8, p. 33] that if $u \in V$ and $du/dt \in V^*$, then $u \in C([\tau,T]; L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful.

THEOREM 3.2. *Under conditions* (H1)–(H4), *for any $T > \tau$ and $u_\tau \in L^2(\Omega)$, problem* (1.1) *has a unique weak solution u on (τ,T). Moreover,*

$$|u(t)|_2^2 \le C\Big(e^{-\zeta(t-\tau)}|u_\tau|_2^2 + 1 + e^{-\zeta t}\int_{-\infty}^t e^{\zeta s}|g(s)|_2^2\, ds\Big). \tag{3.1}$$

Proof. Under conditions (H1)–(H4), one can prove the existence of a weak solution of problem (1.1) using the compactness and monotonicity methods [15, Chapters 1–2]. The proof is similar to the one in the autonomous case (see [3, 4]), so it is omitted here. We only prove (3.1). From (1.1) we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \|u\|^p + \int_\Omega f(u)u\, dx = \int_\Omega g(t)u\, dx.$$

By (1.2) and the Cauchy inequality, we obtain

$$\frac{d}{dt}|u|_2^2 + 2\|u\|^p + 2C_1|u|_q^q \le 2k_1|\Omega| + \frac{1}{\eta}|g(t)|_2^2 + \eta|u|_2^2. \tag{3.2}$$

Using Lemma 2.3 we get

$$\frac{d}{dt}|u|_2^2 + \zeta|u|_2^2 \le \frac{1}{\eta}|g(t)|_2^2 + C. \tag{3.3}$$

Applying the Gronwall lemma to (3.3), we get the desired inequality (3.1). ■

By Theorem 3.2, problem (1.1) defines a process:

$$U(t,\tau) : L^2(\Omega) \to \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega),$$

where $U(t,\tau)u_\tau$ is the unique weak solution of (1.1) with initial datum u_τ.

LEMMA 3.3. *Under conditions* (H1)–(H4), *the weak solution u of* (1.1) *satisfies the following inequality for all $t > \tau$:*

(3.4)
$$|u|_2^2 + \|u\|^p + |u|_q^q \le c\Big(\Big(1 + (t-\tau) + \frac{1}{t-\tau}\Big)e^{-\zeta(t-\tau)}|u_\tau|_2^2 + \Big(1 + \frac{1}{t-\tau}\Big) + \Big(1 + \frac{1}{t-\tau}\Big)e^{-\zeta t}\int_{-\infty}^{t} e^{\zeta s}|g(s)|_2^2\,ds + \Big(1 + \frac{1}{t-\tau}\Big)e^{-\zeta t}\int_{-\infty}^{t}\int_{-\infty}^{s} e^{\zeta r}|g(r)|_2^2\,dr\,ds\Big).$$

Hence there exists a family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ of pullback absorbing sets in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ for $\{U(t,\tau)\}$.

Proof. Multiplying (3.1) by $e^{\zeta t}$ and integrating from τ to t, we get

$$\int_\tau^t e^{\zeta s}|u|_2^2\,ds \le C\Big((t-\tau)e^{\zeta\tau}|u_\tau|_2^2 + e^{\zeta t} + \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\zeta r}|g(r)|^2\,dr\,ds\Big). \tag{3.5}$$

From (3.2) and (2.1), we have

$$\frac{d}{dt}|u|_2^2 + \|u\|^p + 2C_1|u|_q^q \le C + \frac{1}{\eta}|g(t)|_2^2. \tag{3.6}$$

Thus,

$$\frac{d}{dt}(e^{\zeta t}|u|_2^2) + e^{\zeta t}(\|u\|^p + 2C_1|u|_q^q) \le \zeta e^{\zeta t}|u|_2^2 + Ce^{\zeta t} + \frac{e^{\zeta t}}{\eta}|g(t)|_2^2. \tag{3.7}$$

Integrating this inequality and using (3.1), we have

$$\int_\tau^t e^{\zeta s}(\|u\|^p + 2C_1|u|_q^q)\,ds \le C\Big((1 + \zeta(t-\tau))e^{\zeta\tau}|u_\tau|_2^2 + e^{\zeta t} + \int_{-\infty}^{t} e^{\zeta s}|g(s)|_2^2\,ds + \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\zeta r}|g(r)|^2\,dr\,ds\Big). \tag{3.8}$$

Combining (3.5) and (3.8), we get

$$\int_\tau^t e^{\zeta s}(\|u\|^p + 2C_1|u|_q^q + |u|_2^2)\,ds \le C\Big((1 + t-\tau)e^{\zeta\tau}|u_\tau|_2^2 + e^{\zeta t} + \int_{-\infty}^{t} e^{\zeta s}|g(s)|_2^2\,ds + \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\zeta r}|g(r)|_2^2\,dr\,ds\Big). \tag{3.9}$$

From (1.2) and (3.6), we obtain

$$\frac{d}{dt}|u|_2^2 + \|u\|^p + C_5 \int_\Omega F(u)\,dx \le \frac{1}{\eta}|g(t)|_2^2 + C_7, \tag{3.10}$$

where $F(u) = \int_0^u f(s)\,ds$.

We now give some formal calculations; a rigorous proof uses Galerkin approximations and Lemma 11.2 in [19]. Multiplying $(1.1)_1$ by u_t and integrating over Ω, we have

$$|u_t|_2^2 + \frac{1}{2}\frac{d}{dt}\Big(\|u\|^p + 2\int_\Omega F(u)\,dx\Big) = \int_\Omega g(t)u_t\,dx \le \frac{1}{2}|g(t)|_2^2 + \frac{1}{2}|u_t|_2^2.$$

Thus,

$$\frac{d}{dt}\Big(\|u\|^p + 2\int_\Omega F(u)\,dx\Big) \le |g(t)|_2^2. \tag{3.11}$$

Combining (3.10), (3.11) and using (2.1), we have

$$\frac{d}{dt}G(u) + C_8 G(u) \le C_9|g(t)|_2^2 + C,$$

where $G(u) = |u|_2^2 + \|u\|^p + 2\int_\Omega F(u)\,dx$. This implies that

$$\frac{d}{dt}((t-\tau)e^{\zeta t}G(u)) \le (1+(\zeta - C_8)(t-\tau))G(u)e^{\zeta t} + (C + C_9|g(t)|_2^2)(t-\tau)e^{\zeta t}.$$

Integrating this inequality from τ to t, we obtain

$$\begin{aligned}(t-\tau)G(u) \le (1 + C_{11}(t-\tau))\int_\tau^t G(u)e^{\zeta s}\,ds \\ + C_{10}(t-\tau)e^{\zeta t} + C_9(t-\tau)\int_\tau^t e^{\zeta s}|g(s)|_2^2\,ds.\end{aligned}$$

Using (3.9) we get the required inequality (3.4). Put

$$r_0^2(t) = 2c\Big(1 + e^{-\zeta t}\int_{-\infty}^t e^{\zeta s}|g(s)|_2^2\,ds + e^{-\zeta t}\int_{-\infty}^t\int_{-\infty}^s e^{\zeta r}|g(r)|_2^2\,dr\,ds\Big). \tag{3.12}$$

Then for any $D \in \mathcal{B}(L^2(\Omega))$ and any $t \in \mathbb{R}$, by (3.4), there exists $\tau_0(D,t) \le t$ such that

$$|u|_2^2 + \|u\|^p + |u|_q^q \le r_0^2(t) \quad \text{for all } \tau \le \tau_0,\ u_\tau \in D,$$

i.e., there exists a family $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ of bounded pullback absorbing sets in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ of $\{U(t,\tau)\}$. ∎

Remark 3.4. Let $\mathcal{R}$ be the set of all functions $r : \mathbb{R} \to (0,\infty)$ such that

$$\lim_{t\to-\infty} e^{\zeta t}r^2(t) = 0,$$

and denote by $\mathcal{D}$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega))$ such that $D(t) \subset \overline{B}(r(t))$ for some $r \in \mathcal{R}$, where $\overline{B}(r(t))$ denotes the closed ball in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ with radius $r(t)$. From the proof above, we see that there exists a family of pullback $\mathcal{D}$-absorbing sets in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ for the process $\{U(t,\tau)\}$.

From Lemma 3.3 we see that the process $\{U(t,\tau)\}$ maps each compact set in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ to a bounded set in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$ for any $t \geq \tau$, and thus by Lemma 2.4, the process $\{U(t,\tau)\}$ is norm-to-weak continuous in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$. Since $\{U(t,\tau)\}$ has a family of pullback absorbing sets in $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$, in order to prove the existence of pullback attractors, we need only check that $\{U(t,\tau)\}$ is pullback asymptotically compact.

3.1. Pullback attractor in $L^2(\Omega)$. Because $\mathcal{D}_0^{1,p}(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ compactly, the process $\{U(t,\tau)\}$ is pullback asymptotically compact in $L^2(\Omega)$. Thus, we immediately get the following result.

THEOREM 3.5. *Assume conditions* (H1)–(H4) *hold. Then the process* $\{U(t,\tau)\}$ *associated to problem* (1.1) *has a pullback attractor* $\hat{\mathcal{A}}_2$ *in* $L^2(\Omega)$.

3.2. Pullback attractor in $L^q(\Omega)$. From now on, for the sake of brevity, we will use the notation

$$\Omega(\Phi) = \{x \in \Omega : \Phi \text{ is true}\},$$

where Φ is a logical condition.

To prove that $\{U(t,\tau)\}$ is pullback asymptotically compact in $L^q(\Omega)$, we need the following lemma.

LEMMA 3.6. *Let* $\{U(t,\tau)\}$ *be a norm-to-weak continuous process in* $L^q(\Omega)$ *and* $L^2(\Omega)$, *and let* $\{U(t,\tau)\}$ *satisfy the following two conditions:*

(i) $\{U(t,\tau)\}$ *is pullback asymptotically compact in* $L^2(\Omega)$;

(ii) *for any* $\varepsilon > 0$, $t \in \mathbb{R}$, *and* $D \in \mathcal{B}(L^2(\Omega))$, *there exist constants* $M = M(\varepsilon, D)$ *and* $\tau_0 = \tau_0(\varepsilon, D) \leq t$ *such that*

$$\Big(\int_{\Omega(|U(t,\tau)u_\tau| \geq M)} |U(t,\tau)u_\tau|^q \, dx\Big)^{1/q} < \varepsilon \quad \text{for any } \tau \leq \tau_0 \text{ and } u_\tau \in D.$$

Then $\{U(t,\tau)\}$ *is pullback asymptotically compact in* $L^q(\Omega)$.

Proof. For any fixed $\varepsilon > 0$ and $D \in \mathcal{B}(L^2(\Omega))$, it follows from condition (i) and Lemma 2.7 that there exists $\tau_1 = \tau_1(D, \varepsilon) \leq \tau_0$ such that

$$\alpha\Big(\bigcup_{\tau \leq \tau_1} U(t,\tau)D\Big) \leq (3M)^{(2-q)/2}(\varepsilon/2)^{q/2} \quad \text{in } L^2(\Omega),$$

i.e., $\bigcup_{\tau \leq \tau_1} U(t,\tau)D$ has a finite $(3M)^{(2-q)/2}(\varepsilon/2)^{q/2}$-net in $L^2(\Omega)$. From condition (ii) and Lemma 5.3 in [21] we deduce that $\bigcup_{\tau \leq \tau_1} U(t,\tau)D$ has a finite

ε-net in $L^q(\Omega)$. By the definition of the measure of noncompactness, we obtain

$$\alpha\Big(\bigcup_{\tau\le\tau_1} U(t,\tau)D\Big) \le \varepsilon \quad \text{in } L^q(\Omega),$$

i.e., $\{U(t,\tau)\}$ is pullback ω-limit compact in $L^q(\Omega)$. Using Lemma 2.7 once again, $\{U(t,\tau)\}$ is pullback asymptotically compact in $L^q(\Omega)$. ∎

THEOREM 3.7. *Assume conditions* (H1)–(H4) *hold. Then the process* $\{U(t,\tau)\}$ *associated to problem* (1.1) *has a pullback attractor* $\hat{\mathcal{A}}_q$ *in* $L^q(\Omega)$.

Proof. It is sufficient to show that the process $\{U(t,\tau)\}$ satisfies the condition (ii) in Lemma 3.6.

Take M large enough such that $\tilde{C}_1|u|^{q-1} \le f(u)$ in

$$\Omega_1 = \Omega(u(t)\ge M) = \{x\in\Omega : u(x,t)\ge M\},$$

and denote

$$(u-M)_+ = \begin{cases} u-M, & u\ge M,\\ 0, & u\le M.\end{cases}$$

In Ω_1 we have

$$\begin{aligned}(3.13)\qquad g(t)((u-M)_+)^{q-1} &\le \frac{\tilde{C}_1}{2}((u-M)_+)^{2q-2} + \frac{1}{2\tilde{C}_1}|g(t)|^2\\ &\le \frac{\tilde{C}_1}{2}((u-M)_+)^{q-1}|u|^{q-1} + \frac{1}{2\tilde{C}_1}|g(t)|^2,\end{aligned}$$

and

(3.14)
$$\begin{aligned} f(u)((u-M)_+)^{q-1} &\ge \tilde{C}_1|u|^{q-1}((u-M)_+)^{q-1}\\ &\ge \frac{\tilde{C}_1}{2}((u-M)_+)^{q-1}|u|^{q-1} + \frac{\tilde{C}_1 M^{q-2}}{2}((u-M)_+)^q.\end{aligned}$$

Multiplying equation $(1.1)_1$ by $|(u-M)_+|^{q-1}$ and using (3.13) and (3.14), we deduce that

$$\begin{aligned}\frac{2}{q}\frac{d}{dt}|(u-M)_+|_q^q + (q-1)\int_{\Omega_1}\sigma(x)|\nabla(u-M)_+|^p|(u-M)_+|^{q-2}\,dx\\ + \tilde{C}_1 M^{q-2}\int_{\Omega_1}|(u-M)_+|^q \le \int_{\Omega_1}\frac{1}{\tilde{C}_1}|g(t)|^2\,dx.\end{aligned}$$

Therefore

$$\frac{d}{dt}|(u-M)_+|_q^q + CM^{q-2}|(u-M)_+|_q^q \le C|g(t)|_2^2,$$

which implies that

$$\frac{d}{dt}\big((t-\tau)e^{\rho t}|(u-M)_+|_q^q\big) \le e^{\rho t}|(u-M)_+|_q^q + C(t-\tau)e^{\rho t}|g(t)|_2^2, \tag{3.15}$$

where $\rho = CM^{q-2}$. Integrating (3.15) we get

$$\begin{aligned}(t-\tau)e^{\rho t}|(u-M)_+|_q^q &\le \int_\tau^t e^{\rho s}|(u-M)_+|_q^q\,ds + C(t-\tau)\int_\tau^t e^{\rho s}|g(s)|_2^2\,ds \\ &\le e^{(\rho-\zeta)t}\int_\tau^t e^{\zeta s}|u|_q^q\,ds + \frac{C(t-\tau)e^{(\rho-\gamma)t}}{\rho-\gamma},\end{aligned}$$

and then

$$|(u-M)_+|_q^q \le \frac{1}{t-\tau}e^{-\zeta t}\int_\tau^t e^{\zeta s}|u|_q^q\,ds + \frac{Ce^{-\gamma t}}{\rho-\gamma}. \tag{3.16}$$

By (3.16) and (3.9), we have

$$\begin{aligned}|(u-M)_+|_q^q \le C\Bigg(\Big(1+\frac{1}{t-\tau}\Big)e^{-\zeta(t-\tau)}|u_\tau|_2^2 + \frac{1}{t-\tau} + \frac{e^{-\zeta t}}{t-\tau}\int_{-\infty}^t e^{\zeta s}|g(s)|_2^2\,ds \\ + \frac{e^{-\zeta t}}{t-\tau}\int_{-\infty}^t\int_{-\infty}^s e^{\zeta r}|g(r)|_2^2\,dr\,ds\Bigg) + \frac{Ce^{-\gamma t}}{\rho-\gamma}.\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exist $M_1 > 0$ and $\tau_1 < t$ such that for any $\tau < \tau_1$ and any $M \ge M_1$, we have

$$\int_{\Omega(u(t)\ge M)} |(u-M)_+|^q\,dx \le \varepsilon. \tag{3.17}$$

Repeating the same reasoning with $(u+M)_-$ instead of $(u-M)_+$, we deduce that there exist $M_2 > 0$ and $\tau_2 < t$ such that for any $\tau < \tau_2$ and any $M \ge M_2$, we have

$$\int_{\Omega(u(t)\le -M)} |(u+M)_-|^q\,dx \le \varepsilon, \tag{3.18}$$

where

$$(u+M)_- = \begin{cases} u+M, & u \le -M, \\ 0, & u \ge -M. \end{cases}$$

Letting $M_0 = \max\{M_1, M_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$, we obtain

$$\int_{\Omega(|u|\ge M)} (|u|-M)^q\,dx \le \varepsilon \quad \text{for } \tau \le \tau_0 \text{ and } M \ge M_0.$$

Using (3.17) and (3.18), we have

$$\begin{aligned}
(3.19)\qquad \int\limits_{\Omega(|u|\geq 2M)} |u|^q\,dx &= \int\limits_{\Omega(|u|\geq 2M)} ((|u|-M)+M)^q\,dx \\
&\leq 2^{q-1}\Big(\int\limits_{\Omega(|u|\geq 2M)} (|u|-M)^q\,dx + \int\limits_{\Omega(|u|\geq 2M)} M^q\,dx\Big) \\
&\leq 2^{q-1}\Big(\int\limits_{\Omega(|u|\geq M)} (|u|-M)^q\,dx + \int\limits_{\Omega(|u|\geq M)} (|u|-M)^q\,dx\Big) \\
&\leq 2^q\varepsilon.
\end{aligned}$$

This completes the proof. ∎

3.3. Pullback attractor in $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$. First, we prove the following lemma.

LEMMA 3.8. *Assume conditions* (H1)–(H4) *hold. Then for any $t\in\mathbb{R}$ and any bounded subset $B\subset L^2(\Omega)$, there exists a positive constant $T=T(B,t)\leq t$ such that*

$$(3.20)\qquad |u_t(t)|_2^2 \leq C\Big(1+e^{-\zeta t}\int\limits_{-\infty}^{t} e^{\zeta s}(|g(s)|_2^2+|g'(s)|_2^2)\,ds\Big),$$

for all $\tau\leq T$ and all $u_\tau\in B$, where $C>0$ is independent of t and B.

Proof. We give some formal calculations; a rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [19]. By differentiating $(1.1)_1$ in time t and setting $v=u_t$, we get

$$\begin{aligned}
v_t &- \mathrm{div}(\sigma(x)|\nabla u|^{p-2}\nabla v) \\
&\quad - (p-2)\mathrm{div}(\sigma(x)|\nabla u|^{p-4}(\nabla u\cdot\nabla v)\nabla u) + f'(u)v = g'(r).
\end{aligned}$$

Multiplying the above equality by $e^{\zeta r}v$ and then integrating over Ω, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dr}(e^{\zeta r}|v|_2^2) &+ e^{\zeta r}\int\limits_{\Omega}\sigma(x)|\nabla u|^{p-2}|\nabla v|^2\,dx \\
&+ (p-2)e^{\zeta r}\int\limits_{\Omega}\sigma(x)|\nabla u|^{p-4}(\nabla u\cdot\nabla v)^2\,dx + e^{\zeta r}(f'(u)v,v) \\
&= \frac{\zeta}{2}e^{\zeta r}|v|_2^2 + \frac{1}{2}e^{\zeta r}(g'(r),v).
\end{aligned}$$

Using (1.3), the Cauchy inequality, and noting that $p\geq 2$, we obtain

$$(3.21)\qquad \frac{d}{dr}(e^{\zeta r}|v|_2^2) \leq C(e^{\zeta r}|g'(r)|_2^2 + e^{\zeta r}|v|_2^2).$$

We set $\tau \le r \le t-1$ and $F(s) = \int_0^s f(\xi)\,d\xi$; then by (1.2), we deduce that

$$\tilde{C}_1\|u\|^q_{L^q(\Omega)} - \tilde{k}_1|\Omega| \le \int_\Omega F(u)\,dx \le \tilde{C}_2\|u\|^q_{L^q(\Omega)} + \tilde{k}_2|\Omega|. \tag{3.22}$$

Multiplying $(1.1)_1$ by u, then using (1.3) and the Cauchy inequality, we get

$$\begin{aligned}\frac{d}{dr}(e^{\zeta r}|u|_2^2) &= \zeta e^{\zeta r}|u|_2^2 + e^{\zeta r}\frac{d}{dr}|u|_2^2 \\ &\le 2\zeta e^{\zeta r}|u|_2^2 - 2e^{\zeta r}\|u\|^p - 2\tilde{C}_1 e^{\zeta r}|u|_q^q + \frac{1}{\zeta}e^{\zeta r}|g(r)|_2^2 + 2e^{\zeta r}k_1|\Omega| \\ &\le -e^{\zeta r}\|u\|^p - 2\tilde{C}_1 e^{\zeta r}|u|_q^q + C(e^{\zeta r}|g(r)|_2^2 + e^{\zeta r}),\end{aligned} \tag{3.23}$$

where we have used the fact that $\|u\|^p \ge 2\zeta|u|_2^2 - \tilde{C}$ (see Lemma 2.3). Integrating the last inequality over the interval $[\tau, t]$, we obtain

$$e^{\zeta t}|u|_2^2 \le e^{\zeta\tau}|u_\tau|^2 + C\Big(\int_{-\infty}^t e^{\zeta s}|g(s)|^2\,ds + e^{\zeta t}\Big). \tag{3.24}$$

By (1.3) and (3.22), we infer from (3.23) that

$$\frac{d}{ds}(e^{\zeta s}|u|_2^2) + C\Big(e^{\zeta s}\|u\|^p + 2e^{\zeta s}\int_\Omega F(u)\,dx\Big) \le C(e^{\zeta s}|g(s)|_2^2 + e^{\zeta s}). \tag{3.25}$$

Integrating this inequality from r to $r+1$ and using (3.24), we obtain

$$\begin{aligned}&\int_r^{r+1}\Big(e^{\zeta s}\|u\|^p + 2e^{\zeta s}\int_\Omega F(u)\,dx\Big)\,ds \\ &\le C\Big(e^{\zeta r}|u(r)|_2^2 + \int_r^{r+1}(e^{\zeta s}|g(s)|^2 + e^{\zeta s})\,ds\Big) \\ &\le C\Big(e^{\zeta\tau}|u_\tau|_2^2 + \int_{-\infty}^t e^{\zeta s}|g(s)|^2\,ds + e^{\zeta t}\Big) < \infty \quad \text{for any } r \in [\tau, t-1].\end{aligned} \tag{3.26}$$

Now multiplying $(1.1)_1$ by $e^{\zeta r}u_t = e^{\zeta r}v$, we have

$$\begin{aligned}&e^{\zeta r}|v|_2^2 + \frac{d}{dr}\Big(\frac{2}{p}e^{\zeta r}\|u\|^p + 2e^{\zeta r}\int_\Omega F(u)\,dx\Big) \\ &\qquad\le \zeta\Big(\frac{2}{p}e^{\zeta r}\|u\|^2 + 2e^{\zeta r}\int_\Omega F(u)\,dx\Big) + e^{\zeta r}|g(r)|_2^2.\end{aligned} \tag{3.27}$$

By (3.26), (3.27), and the uniform Gronwall inequality, we obtain

$$e^{\zeta r}\|u(r)\|^2 + e^{\zeta r}\int_\Omega F(u)\,dx \le C\Big(e^{\zeta\tau}|u_\tau|_2^2 + \int_{-\infty}^t e^{\zeta s}|g(s)|_2^2\,ds + e^{\zeta t}\Big). \tag{3.28}$$

On the other hand, integrating (3.27) from r to $r+1$, by (3.23), (3.26) and (3.28), we have

$$\int_r^{r+1} e^{\zeta s}|v|_2^2\,ds \le C\Big(e^{\zeta\tau}|u_\tau|_2^2 + \int_{-\infty}^{t} e^{\zeta s}|g(s)|_2^2\,ds + e^{\zeta t}\Big).$$

Then, by (3.21), using the uniform Gronwall lemma once again, we get

$$e^{\zeta t}|v|_2^2 \le C\Big(e^{\zeta\tau}|u_\tau|_2^2 + \int_{-\infty}^{t} e^{\zeta s}(|g(s)|_2^2 + |g'(s)|_2^2)\,ds + e^{\zeta t}\Big),$$

that is,

$$|v(t)|_2^2 \le C\Big(e^{-\zeta(t-\tau)}|u_\tau|_2^2 + e^{-\zeta t}\int_{-\infty}^{t} e^{\zeta s}(|g(s)|_2^2 + |g'(s)|_2^2)\,ds + 1\Big).$$

This completes the proof. ∎

We are now in a position to prove the main theorem.

THEOREM 3.9. *Assume conditions* (H1)–(H4) *hold. Then the process* $\{U(t,\tau)\}$ *associated to problem* (1.1) *has a pullback attractor* $\hat{\mathcal{A}}$ *in* $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega)$.

Proof. By Lemma 3.3, $\{U(t,\tau)\}$ has a family of bounded pullback absorbing sets in $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$. It remains to show that $\{U(t,\tau)\}$ is pullback asymptotically compact in $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$, i.e., for any $t\in\mathbb{R}$, any bounded set $D\in\mathcal{B}(\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega))$, and any sequences $\tau_n\to-\infty$ and $u_{\tau_n}\in D$, the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ is precompact in $\mathcal{D}_0^1(\Omega,\sigma)\cap L^q(\Omega)$. Thanks to Theorem 3.7, we only need to show that the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,\sigma)$.

Denote $u_n(t)=U(t,\tau_n)u_{\tau_n}$. By Theorem 3.5, we can assume that $\{u_n(t)\}$ is a Cauchy sequence in $L^2(\Omega)$. Since $L_{p,\sigma}$ is strongly monotone when $p\ge 2$, we have

$$\begin{aligned}
&\delta\|u_n(t)-u_m(t)\|^2\\
&\le \langle L_{p,\sigma}u_n(t)-L_{p,\sigma}u_m(t), u_n(t)-u_m(t)\rangle\\
&= -\Big\langle \frac{du_n}{dt}(t)-\frac{du_m}{dt}(t), u_n(t)-u_m(t)\rangle - \langle f(u_n(t))-f(u_m(t)), u_n(t)-u_m(t)\Big\rangle\\
&\le \Big|\frac{d}{dt}u_n(t)-\frac{d}{dt}u_m(t)\Big|_2^2 |u_n(t)-u_m(t)|_2^2 + \ell|u_n(t)-u_m(t)|_2^2,
\end{aligned}$$

where we have used condition (1.3). Because $\{u_n(t)\}$ is a Cauchy sequence in $L^2(\Omega)$ and by Lemma 3.8, one gets

$$\|u_n(t)-u_m(t)\|\to 0 \quad \text{as } m,n\to\infty.$$

The proof is complete. ∎

Remark 3.10. The pullback attractor in Theorems 3.5, 3.7 and 3.9 is the same object. Using the universe $\mathcal{D}$ in Remark 3.4, one may establish the existence of a pullback $\mathcal{D}$-attractor; as a corollary of the results in [18], this attractor works in the norms of $L^2(\Omega), L^q(\Omega)$ and $\mathcal{D}_0^{1,p}(\Omega,\sigma)\cap L^q(\Omega)$ (the same in the three frameworks) and contains the attractor obtained in Theorems 3.5, 3.7 and 3.9.

4. A relationship between pullback attractors and uniform attractors

4.1. Existence and structure of a uniform attractor. First, we recall the concept of kernel sections. The *kernel* $\mathcal{K}$ of the process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot)\,|\,U(t,\tau)u(\tau) = u(t), \operatorname{dist}(u(t),u(0)) \le C_u, \forall t \ge \tau,\ t,\tau \in \mathbb{R}\}.$$

The set $\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathcal{K}\}$ is said to be the *kernel section* at time $s \in \mathbb{R}$.

In this section, to get the existence of a uniform attractor in $\mathcal{D}_0^{1,p}(\Omega)\cap L^q(\Omega)$, instead of assumption (H3), we assume the external force g satisfies the following condition:

(H3bis) $g \in W^{1,2}_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ and

$$\sup_{t\in\mathbb{R}} \|g(t,\cdot)\|^2_{L^2(\Omega)} \le K, \qquad g' \in L^2_b(\mathbb{R}; L^2(\Omega)),$$

where $L^2_b(\mathbb{R}; L^2(\Omega))$ is the set of translation bounded functions (see [7, 8]).

Denote by $\mathcal{H}_w(g)$ the closure of $\{g(\cdot+h) : h \in \mathbb{R}\}$ in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ with the weak topology. It is known (see e.g. [7, 8]) that $\mathcal{H}_w(g)$ is weakly compact in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$. By Theorem 3.2, for each external force $\sigma \in \mathcal{H}_w(g)$, problem (1.1) has a unique weak solution $U_\sigma(t,\tau)u_\tau$ subject to the initial datum u_τ. Thus, we get a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem (1.1). The following results are proved in [5] (the structure of the uniform attractor follows from Theorem 3.9 in [7]).

Theorem 4.1. *Assume conditions* (H1), (H2), (H3bis) *and* (H4) *hold. Then the family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ *has a uniform attractor* $\mathcal{A}_{\mathcal{H}_w(g)}$ *in* $\mathcal{D}_0^{1,p}(\Omega)\cap L^q(\Omega)$. *Moreover,*

$$\mathcal{A}_{\mathcal{H}_w(g)} = \bigcup_{\sigma\in\mathcal{H}_w(g)} \mathcal{K}_\sigma(s), \qquad \forall s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ *is the kernel section at* s *of the kernel* $\mathcal{K}_\sigma$ *of the process* $\{U_\sigma(t,\tau)\}$ *with symbol* $\sigma \in \mathcal{H}_w(g)$.

4.2. The relationship between pullback attractors, uniform attractors and kernel sections. We first recall some abstract results. A set

Y is said to be *uniformly* (w.r.t. $\tau \in \mathbb{R}$) *attracting* for a process $\{U(t,\tau)\}$ if

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}(U(t+\tau,\tau)B, Y) \to 0 \quad \text{ as } t \to \infty$$

for any bounded set B. In particular, a closed set $\mathcal{A}_0$ is said to be a *uniform (w.r.t.* $\tau \in \mathbb{R}$) *attractor* for $\{U(t,\tau)\}$ if it is contained in any closed uniformly attracting set. Given a symbol σ_0, let $\Sigma_0 = \{\sigma_0(\cdot + h) : h \in \mathbb{R}\}$ be a subset of some Banach space. If the process $\{U_{\sigma_0}(t,\tau)\}$ satisfies the following translation identity:

$$U_{\sigma_0}(t+h,\tau+h) = U_{T(h)\sigma_0}(t,\tau), \quad \forall t \geq \tau,\ t,\tau \in \mathbb{R},\ h \geq 0, \tag{4.1}$$

then obviously, being uniformly (w.r.t. $\tau \in \mathbb{R}$) attracting for $\{U_{\sigma_0}(t,\tau)\}$ is equivalent to being uniformly (w.r.t $\sigma \in \Sigma_0$) attracting for $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma_0}$. It is easy to see that the uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathcal{A}_0$ of $\{U_{\sigma_0}(t,\tau)\}$ coincides with the uniform (w.r.t. $\sigma \in \Sigma_0$) attractor $\mathcal{A}_{\Sigma_0}$ of $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma_0}$.

Now we return to problem (1.1). Obviously, (H3bis) implies (H3). For problem (1.1), it is proved in Theorem 3.9 that for any $g_0 \in \mathcal{H}_w(g)$, the process $\{U_{g_0}(t,\tau)\}$ has a pullback attractor $\hat{\mathcal{A}}_{g_0} = \{A_{g_0}(t) : t \in \mathbb{R}\}$ in $\mathcal{D}_0^{1,p}(\Omega) \cap L^q(\Omega)$. Moreover, we have

THEOREM 4.2. *Assume conditions* (H1), (H2), (H3bis) *and* (H4) *hold. Then for any* $g_0 \in \mathcal{H}_w(g)$, *the process* $\{U_{g_0}(t,\tau)\}$ *has a pullback attractor* $\hat{\mathcal{A}}_{g_0} = \{A_{g_0}(t) : t \in \mathbb{R}\}$ *in* $\mathcal{D}_0^{1,p}(\Omega) \cap L^q(\Omega)$, *and*

$$A_{g_0}(s) = \mathcal{K}_{g_0}(s), \qquad \bigcup_{g_0\in\mathcal{H}_w(g)} A_{g_0}(s) = \mathcal{A}_{\mathcal{H}_w(g)}, \quad \forall s \in \mathbb{R},$$

where $\mathcal{A}_{\mathcal{H}_w(g)}$ *is the uniform attractor of problem* (1.1), *and* $\mathcal{K}_{g_0}$ *is the kernel of the process* $\{U_{g_0}(t,\tau)\}$.

Proof. Since $\hat{\mathcal{A}}_{g_0}$ is pullback attracting and $A_{g_0}(s)$ is compact, we have

$$\mathcal{K}_{g_0}(s) \subset A_{g_0}(s) \quad \text{ for any } s \in \mathbb{R}.$$

On the other hand, by the definition of $\mathcal{K}_{g_0}(s)$ and the invariance of $\hat{\mathcal{A}}_{g_0}$,

$$A_{g_0}(s) \subset \mathcal{K}_{g_0}(s) \quad \text{ for any } s \in \mathbb{R}.$$

So, we have

$$A_{g_0}(s) = \mathcal{K}_{g_0}(s) \quad \text{ for any } s \in \mathbb{R}. \tag{4.2}$$

Next, by (4.2) and Theorem 4.1,

$$\mathcal{A}_{\mathcal{H}_w(g)} = \bigcup_{g_0\in\mathcal{H}_w(g)} \mathcal{K}_{g_0}(s) = \bigcup_{g_0\in\mathcal{H}_w(g)} A_{g_0}(s), \quad \forall s \in \mathbb{R}.$$

The proof is complete. ∎

Acknowledgments. This work was supported by Vietnam's National Foundation for Science and Technology Development (NAFOSTED), Project 101.01-2010.05.

The authors would like to thank the referee for the helpful comments and suggestions which improved the presentation of the paper.

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Received 18.2.2010
and in final form 19.12.2010 (2173)