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Differential analogues of the Brück conjecture

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Abstract. We give some growth properties for solutions of linear complex differential equations which are closely related to the Brück Conjecture. We also prove that the Brück Conjecture holds when certain proximity functions are relatively small.

1. Introduction and main results. In this paper a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [8, 9, 13]. In particular, we denote the order, hyperorder and lower hyperorder of growth of a meromorphic f by $\sigma(f)$, $\sigma_2(f)$ and $\mu_2(f)$, respectively. For a set $E \subset \mathbb{R}_+$, let $\lambda(E)$ be the logarithmic measure of E. The upper and lower logarithmic densities of E are defined by

$$\overline{\log \operatorname{dens}}(E) = \limsup_{r\to\infty} \frac{\lambda(E\cap[1,r])}{\log r}, \qquad \underline{\log \operatorname{dens}}\,(E) = \liminf_{r\to\infty} \frac{\lambda(E\cap[1,r])}{\log r},$$

respectively. We note that E may be different each time it occurs. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities".

In 1996, R. Brück [1] posed the following conjecture.

Brück Conjecture. Let f be an entire function such that its hyperorder $\sigma_2(f)$ is finite but not a positive integer. If f and f' share a finite value a CM, then $f' - a \equiv c(f - a)$, where c is a non-zero constant.

In 1998, Gundersen and Yang [7] verified that the conjecture is true when f is of finite order. Later on, Chen and Shon [4] proved that the conjecture holds when $\sigma_2(f) < 1/2$. Some results have also been obtained in the case when a is a small function (see Liu and Gu [10], Wang and Li [12], Zhang and Yang [17, 18]). Recently, Chang and Zhu [3] considered the case where the order of a is less than the order of f. Their main result reads as follows.

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Theorem A. *Let f and a be entire functions such that $\sigma(a)<\sigma(f)<\infty$. If f and f' share a CM, then $f'-a\equiv c(f-a)$ for a non-zero constant c.*

Suppose that f is an entire function and $a\neq 0$ is a finite value. If f and $f^{(k)}$ $(k\geq 1)$ share the value a CM, then

$$\frac{f^{(k)}-a}{f-a}=e^{Q(z)}, \tag{1.1}$$

where Q is an entire function. Set $F=f/a-1$. Then F is an entire function. By (1.1), we see that F satisfies the differential equation

$$f^{(k)}-e^{Q(z)}f=1. \tag{1.2}$$

In 1999, Yang [14] proved that every solution of the equation (1.2) is an entire function of infinite order, provided that Q is a non-constant polynomial. This raises the following questions (see [15]), which are closely related to the Brück Conjecture.

Question 1. *Let Q be a polynomial of degree $n\geq 1$, and let f be a solution of* (1.2). *Is it true that $\sigma_2(f)=n$?*

Question 2. *Let Q be a transcendental entire function, and let f be a solution of* (1.2). *Is it true that $\sigma_2(f)=+\infty$?*

It is easy to see that affirmative answers to both questions lead to a proof of the Brück Conjecture. In [16], Yang proved

Theorem B. *Let Q be an entire function and k be a positive integer. Then every solution f of equation* (1.2) *satisfies $\sigma_2(f)=\sigma(e^{Q(z)})$ with at most one exception.*

In [2], Cao proved that $\sigma_2(f)\leq n$ if Q is a polynomial of degree $n\geq 1$. Concerning Question 2, he got an affirmative answer provided that Q is a transcendental entire function with $\sigma_2(Q)\leq 1/2$.

In this paper, we obtain the following result which answers Question 1 completely.

Theorem 1.1. *Let Q be a non-constant polynomial, k be a positive integer, and let f be a solution of the equation*

$$f^{(k)}-e^{Q(z)}f=A(z), \tag{1.3}$$

where A is an entire function satisfying $\sigma(A)<\sigma(f)$. Then f satisfies $\sigma_2(f)=\mu_2(f)=\deg Q$.

Corollary 1.2. *Let f and a be entire functions such that $\sigma(a)<\sigma(f)<\infty$, and let k be a positive integer. If f and $f^{(k)}$ share a CM, then $f^{(k)}-a\equiv c(f-a)$ for a non-zero constant c.*

Clearly, Corollary 1.2 generalizes Theorem A. However, our proof, based on Theorem 1.1, is different from that in [3].

It is known that the Brück Conjecture holds when $a = 0$, and that if f satisfies

$$N(r, 1/f') = S(r, f), \tag{1.4}$$

then

$$\frac{f' - 1}{f - 1} = c \tag{1.5}$$

for a non-zero constant c (see [1]). That is, the Brück Conjecture holds when $N(r, 1/f')$ is small. Zhang [19] proved that if (1.4) is relaxed to

$$\overline{N}(r, 1/f') < (\lambda + o(1))T(r, f),$$

where $\lambda \in (0, 1/4)$, then (1.5) also holds.

Next we consider the case where either $m(r, 1/f^{(j)})$ or $m(r, 1/(f - a))$ is small. Here j is an integer and a is a finite value.

Theorem 1.3. *Let f be an entire function. Then the Brück Conjecture holds, provided that one of the following assumptions is satisfied:*

(1) *There exists a positive integer j such that*

$$m(r, 1/f^{(j)}) \le \log\{rT(r, f)\}, \qquad r \ge r_0.$$

(2) $m(r, 1/(f - a)) \le \log\{rT(r, f)\}$, $r \ge r_0$.

Remark. By calculating carefully, we can prove that Theorem 1.3 is still valid upon replacing "$\le \log\{rT(r, f)\}$" with "$= O(\log\{rT(r, f)\})$".

2. Some lemmas

Lemma 2.1 ([5, Lemma 2]). *Let f be an entire function of infinite order, and let $\nu(r, f)$ be the central index of f. Then*

$$\limsup_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r} = \sigma_2(f) \quad \text{and} \quad \liminf_{r\to\infty} \frac{\log\log\nu(r, f)}{\log r} = \mu_2(f).$$

Lemma 2.2 ([8, Satz 21.3]). *Let f be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then, for all r outside a set E of finite logarithmic measure,*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^k (1 + o(1)),$$

where k is a positive integer.

Lemma 2.3. *Let f and g be entire functions such that $\sigma(g) < \sigma(f)$. Then there exists a set E with $\overline{\log \mathrm{dens}}(E) > 0$ such that*

$$\frac{|g(z)|}{M(r, f)} = o(1)$$

for all z such that $|z| = r \in E$ is sufficiently large.

Proof. We divide the proof into two cases.

CASE 1: $\sigma(g) < \sigma(f) < \infty$. The proof in this case is a modification of a similar reasoning in [11, Lemma 2.5]. Therefore, we omit it here.

CASE 2: $\sigma(g) < \sigma(f) = \infty$. For any $\varepsilon > 0$, we have

$$|g(z)| \le e^{r^{\sigma(g)+\varepsilon}}. \tag{2.1}$$

Let r_n^* be a sequence tending to infinity such that

$$\sigma(f) = \lim_{n\to\infty} \frac{\log\log M(r_n^*, f)}{\log r_n^*}.$$

Defining $E_t = \bigcup_{n=1}^{\infty} [r_n^*, r_n^{*1+2t}]$ for $t > 0$, we have

$$\begin{aligned}\overline{\log\operatorname{dens}}(E_t) &\ge \limsup_{n\to\infty} \frac{\lambda(E_t \cap [1, r_n^{*1+2t}])}{(1+2t)\log r_n^*} \\ &\ge \limsup_{n\to\infty} \frac{\lambda(E_t \cap [r_n^*, r_n^{*1+2t}])}{(1+2t)\log r_n^*} = \frac{2t}{1+2t} > 0.\end{aligned}$$

Clearly, $\sigma(g) + 1 < \sigma(f) = \infty$. Taking $2(\sigma(g)+1)t = \varepsilon$ and using a similar argument to Case 1, we have (2.1) and

$$\frac{\log\log M(r,f)}{\log r} \ge \frac{\sigma(f)}{1+2t} > \frac{\sigma(g)+1}{1+2t} \ge (\sigma(g)+1)(1-2t) = (\sigma(g)+1) - \varepsilon.$$

This gives

$$M(r,f) > e^{r^{\sigma(g)+1-\varepsilon}}, \quad r \in E_t. \tag{2.2}$$

The assertion follows by combining (2.1) and (2.2). ■

LEMMA 2.4 ([6, Lemma 5]). *Let F and G be non-decreasing functions on $(0,\infty)$. If $F(r) \le G(r)$ for $r \notin E \cup [0,1]$, where the set $E \subset (1,\infty)$ has finite logarithmic measure, then, for any constant $\alpha > 1$, there exists a value $r_0 > 0$ such that $F(r) \le G(\alpha r)$ for $r > r_0$.*

LEMMA 2.5 ([9, Lemma 2.3]). *Let f be a meromorphic function in $|z| \le R$. Then, for $1 \le r < R < \infty$,*

$$m\left(r, \frac{f'}{f}\right) \le C\left(\log^+ T(R,f) + \log^+ \frac{1}{R-r} + \log^+ R + \log^+ \frac{1}{r} + 1\right),$$

where C is a positive constant depending on f only.

3. Proof of Theorem 1.1. From the growth properties of both sides of (1.3), we easily get $\sigma(f) \ge 1$. Since Q is a non-constant polynomial, we write

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \qquad a_n \ne 0. \tag{3.1}$$

It follows that $|Q(z)| > |a_n z^n|/2$ when $|z| = r$ is large enough, and from (1.3) we get

$$\begin{aligned}(3.2)\qquad n\log r + O(1) &\le \log|\log e^{Q(z)}| \le |\log\log e^{Q(z)}| \\ &= \left|\log\log\left(\frac{f^{(k)}(z)}{f(z)} - \frac{A(z)}{f(z)}\right)\right|, \qquad r\to\infty.\end{aligned}$$

Since $\sigma(A) < \sigma(f)$, by Lemma 2.3 we have

$$(3.3)\qquad \frac{|A(z)|}{M(r,f)} = o(1), \qquad r\in E_1,\ r\to\infty,$$

where E_1 is a set with $\overline{\log\operatorname{dens}}(E_1) > 0$.

By Lemma 2.2, we know that there exists a subset $E_2 \subset (1,\infty)$ of finite logarithmic measure such that for some point $z = re^{i\theta(r)}$ ($\theta(r)\in[0,2\pi)$) satisfying $|z| = r \notin E_2$ and $|f(z)| = M(r,f) = \max_{|z|=r}|f(z)|$, we have

$$(3.4)\qquad \frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1+o(1)), \qquad r\to\infty.$$

We first prove that $\sigma(f) = \infty$. If $\sigma(f) < \infty$, then from (1.3), (3.3) and (3.4), we have

$$\begin{aligned}|Q(z)| = |\log e^{Q(z)}| &= |k(\log\nu(r,f) - \log re^{i\theta(r)})(1+o(1))| \\ &= |k(\log\nu(r,f) - \log r - i\theta(r))(1+o(1))| \\ &\le O(\log r), \quad r\to\infty,\end{aligned}$$

which is a contradiction. From (3.2)–(3.4), we have

$$\begin{aligned}(3.5)\qquad n\log r &\le \left|\log\log\left(\left(\frac{\nu(r,f)}{z}\right)^k (1+o(1))\right)\right| \\ &= |\log\{k(\log\nu(r,f) - \log re^{i\theta(r)})\}(1+o(1))| \\ &= |\log\{k(\log\nu(r,f) - \log r)\}(1+o(1))| \\ &\le (\log\log\nu(r,f) + \log\log r)(1+o(1)),\end{aligned}$$

where $r\in E_1\setminus E_2$, $r\to\infty$. We deduce from (3.5) and Lemma 2.1 that

$$(3.6)\qquad n \le \liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r} = \mu_2(f).$$

On the other hand, from (1.3), we get

$$(3.7)\qquad \left|\frac{f^{(k)}}{f}\right| \le \frac{|A(z)|}{|f(z)|} + |e^Q|.$$

Thus, by using (3.3), (3.4) and (3.7), we obtain

$$\left(\frac{\nu(r,f)}{r}\right)^k \le Kr^p M(r,e^Q), \qquad r\in E_1\setminus E_2,\ r\to\infty,$$

where K is a positive constant and p is a positive integer. We have

$$(\nu(r,f))^k \le K r^{p+k} M(r, e^Q), \qquad r \in E_1 \setminus E_2,\ r \to \infty. \tag{3.8}$$

It follows from Lemma 2.1 and (3.8) that

$$\sigma_2(f) \le \sigma(e^Q) = n. \tag{3.9}$$

From (3.6) and (3.9), we have

$$\sigma_2(f) = \mu_2(f) = n.$$

Theorem 1.1 is thus proved.

4. Proof of Theorem 1.3. We assume that f and f' share the finite value a CM. It follows that

$$\frac{f'-a}{f-a} = e^{\phi} \tag{4.1}$$

for an entire function ϕ. We suppose, contrary to the assertion, that ϕ is non-constant. By Lemma 2.2, we have

$$\frac{f'}{f-a} = \left(\frac{\nu(r, f-a)}{z}\right)(1+o(1)), \tag{4.2}$$

where $|z| = r$, $|f(z) - a| = M(r, f-a)$, $r \notin E$ which has a finite logarithmic measure, and $\nu(r, f-a)$ is the central index of $f-a$. From (4.1) and (4.2), we obtain

$$\frac{\nu(r, f-a)}{|z|} = |e^{\phi} + o(1)|\,|1+o(1)| \le 2M(r, e^{\phi}), \qquad r \ge r_1,\ r \notin E,$$

where r_1 is a real number. By Lemma 2.4, we have

$$\nu(r, f-a) \le 2rM(2r, e^{\phi}), \qquad r \ge r_2, \tag{4.3}$$

for a real number r_2. From Lemma 2.1 and (4.3), we have

$$\sigma_2(f-a) = \sigma_2(f) \le \sigma(e^{\phi}). \tag{4.4}$$

On the other hand, we deduce from (4.1) that

$$T(r, e^{\phi}) \le m\left(r, \frac{f'}{f-a}\right) + m\left(r, \frac{f^{(j)}}{f-a}\right) + m\left(r, \frac{1}{f^{(j)}}\right) + O(1). \tag{4.5}$$

If the assumption (1) is satisfied, then by Lemma 2.5 and (4.5), we get

$$\begin{aligned} T(r, e^{\phi}) &\le C\left(\log^+ T(2r, f) + \log^+ \frac{1}{r} + \log\{rT(r,f)\} + 1\right) \\ &\le C\left(2\log^+ T(2r, f) + \log^+ \frac{1}{r} + \log r + 1\right), \end{aligned} \tag{4.6}$$

where C is a positive constant depending only on j and f. Thus

$$\sigma(e^{\phi}) = \limsup_{r\to\infty} \frac{\log T(r, e^{\phi})}{\log r} \le \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \sigma_2(f).$$

Combining this with (4.4), we have

$$\sigma_2(f) = \sigma(e^{\phi}). \tag{4.7}$$

Since e^{ϕ} is an entire function of regular growth, it follows that $\sigma(e^{\phi})$ must be infinite or an integer. This contradicts the assumptions of the Brück Conjecture. Hence ϕ is a constant, and so the Brück Conjecture holds in this case. If the assumption (2) is satisfied, a similar argument yields Theorem 1.3.

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