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Gelfand transform for a Boehmian space of analytic functions

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Abstract. Let $H^\infty(\mathbb{D})$ denote the usual commutative Banach algebra of bounded analytic functions on the open unit disc $\mathbb{D}$ of the finite complex plane, under Hadamard product of power series. We construct a Boehmian space which includes the Banach algebra A where A is the commutative Banach algebra with unit containing $H^\infty(\mathbb{D})$. The Gelfand transform theory is extended to this setup along with the usual classical properties. The image is also a Boehmian space which includes the Banach algebra $C(\Delta)$ of continuous functions on the maximal ideal space Δ (where Δ is given the usual Gelfand topology). It is shown that every $F \in C(\Delta)$ is the Gelfand transform of a suitable Boehmian. It should be noted that in the classical theory the Gelfand transform from A into $C(\Delta)$ is not surjective even though it can be shown that the image is dense. Thus the context of Boehmians enables us to identify every element of $C(\Delta)$ as the Gelfand transform of a suitable convolution quotient of analytic functions. (Here the convolution is the Hadamard convolution).

1. Introduction. The theory of Gelfand transform is applicable to the Banach algebra A containing the Banach algebra $H^\infty(\mathbb{D})$ of bounded analytic functions in the open unit disc $\mathbb{D}$ with unit element added. If Δ denotes the usual maximal ideal space of this algebra (or the non-trivial complex homomorphisms on A), equipped with its Gelfand topology, then the Gelfand transform is an injective continuous mapping of A onto a dense subset of $C(\Delta)$. Since the Gelfand transform is not surjective, it is interesting to ask whether it is possible to define new objects whose Gelfand transforms exhaust $C(\Delta)$. We answer this question affirmatively by constructing a Boehmian space (which are convolution quotients of elements of $H^\infty(\mathbb{D})$) and proving that every element of $C(\Delta)$ is indeed the Gelfand transform of a suitable Boehmian. In fact we show that the Gelfand transform can be extended as a linear, bijective, bicontinuous (in the delta sense) map of one Boehmian space onto another. The required preliminaries are

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developed in Section 2 and the main theorem is proved in the final section. Indeed we shall also give a necessary and sufficient condition on a Boehmian for its Gelfand transform to represent an element of $C(\Delta)$.

2. Preliminaries. Let $\mathbb{R}$ and $\mathbb{C}$ denote the usual real line and the complex plane. For $f, g \in H^\infty(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, define

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let

$$B = \Big\{\phi : \mathbb{D} \to \mathbb{C},\ \text{analytic} \Big/ \phi(z) = \sum_{n=0}^{\infty} c_n z^n \text{ with } \sum_{n=0}^{\infty} |c_n| < \infty\Big\}.$$

For $f \in H^\infty(\mathbb{D})$ we have $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$ and hence if $\phi(z) = \sum_{n=0}^{\infty} c_n z^n \in B$ then $\|\phi\|_\infty \le \sum_{n=0}^{\infty} |c_n| < \infty$. This shows that $B \subset H^\infty(\mathbb{D})$. The following results can be found in [Re], [CD], [Sz].

Lemma 2.1. *$H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard convolution (without unit) and if $f, g \in H^\infty(\mathbb{D})$ then $f * g \in B \subset H^\infty(\mathbb{D})$.*

Theorem 2.2. *The non-trivial complex homomorphisms on $H^\infty(\mathbb{D})$ are of the form δ_n, $n = 0, 1, 2, \ldots$, where δ_n is defined by $\delta_n(f) = f^{(n)}(0)/n!$.*

Theorem 2.3. *If E is the maximal ideal space of $H^\infty(\mathbb{D})$ then E is a locally compact Hausdorff space and the maximal ideal space Δ of A is the one-point compactification of E.*

Theorem 2.4. *The Gelfand transform $\Phi : H^\infty(\mathbb{D}) \to C(E)$ maps $H^\infty(\mathbb{D})$ onto a subset of $C_0(E)$ (which in turn is a subset of $C(E)$), where $C_0(E)$ is the space of all continuous complex functions on E vanishing at ∞.*

We shall prove the following result in respect of the theory of Gelfand transform on A where A is the commutative Banach algebra containing $H^\infty(\mathbb{D})$ with unit. We first note that if Δ denotes the maximal ideal space of A with its usual Gelfand topology then the Gelfand transform $\Phi : A \to C(\Delta)$ is given by $\Phi(f) = \hat{f}$, where $\hat{f}(h) = h(f)$, $h \in \Delta$ and $f \in A$.

Theorem 2.5. *The Gelfand transform $\Phi : A \to C(\Delta)$ is an injective continuous linear map with $\|\hat{f}\|_\infty \le \|f\|_\infty$ for $f \in H^\infty(\mathbb{D})$. Moreover, $h \in \Delta$ if and only if $h = \delta_n$ for some $n = 0, 1, 2, \ldots$ or $h = \delta_\infty$ where $\delta_\infty(f, \alpha) = \alpha$ for all $f \in H^\infty(\mathbb{D})$ and $\alpha \in \mathbb{C}$. Further $\hat{A}$ is dense in $C(\Delta)$.*

Proof. Using Theorem 2.2, the required conclusions readily follow from their analogs in the Gelfand theory of integrable functions except for the denseness, which is an easy consequence of the Stone–Weierstrass theorem (see [Ru]). ∎

We now start constructing the required Boehmian spaces. We recall that (see [M1]) a Boehmian space is obtained using a base space G which is a additive commutative semigroup and S a subsemigroup of G with a map $* : G \times S \to G$ satisfying

1. For $f, g \in S$, $f * g = g * f \in S$.
2. If $f \in G$ and $g, h \in S$ then $(f * g) * h = f * (g * h)$.
3. If $f, g \in G$ and $h \in S$ then $(f + g) * h = (f * h) + (g * h)$.

We also need χ (called a set of *delta sequences*) $\subset S^{\mathbb{N}}$ (where $\mathbb{N}$ is the set of natural numbers) satisfying

1. For $f, g \in G$, $[f * \phi_n = g * \phi_n \ (n = 1, 2, \ldots), \{\phi_n\} \in \chi] \Rightarrow f = g$.
2. $\{\phi_n\}, \{\psi_n\} \in \chi \Rightarrow \{\phi_n * \psi_n\} \in \chi$.

Consider the class $\mathcal{A}_1$ of ordered pairs of sequences defined by

$$\mathcal{A}_1 = \{(\{f_k\}, \{\phi_k\}) : f_k \in G, \ \{\phi_k\} \in \chi \, (k \in \mathbb{N})\}.$$

An element $(\{f_k\}, \{\phi_k\}) \in \mathcal{A}_1$ is said to be a *quotient of sequences* (also denoted by $\frac{f_k}{\phi_k}$) if

$$f_k * \phi_l = f_l * \phi_k \quad \forall k, l \in \mathbb{N}.$$

Let $\mathcal{A}$ denote the set of all quotients of sequences in $\mathcal{A}_1$.

Two quotients of sequences $\frac{f_k}{\phi_k}$ and $\frac{g_k}{\psi_k}$ are said to be *equivalent* (denoted by $\frac{f_k}{\phi_k} \sim \frac{g_k}{\psi_k}$) if

$$f_k * \psi_l = g_l * \phi_k \quad \forall k, l \in \mathbb{N}.$$

It can be easily shown that $\sim$ is an equivalence relation on $\mathcal{A}$. The equivalence class containing a quotient of sequence $\frac{f_k}{\phi_k}$ is called a *Boehmian* and is denoted by $\left[\frac{f_k}{\phi_k}\right]$. The set $\mathscr{B}$ of all such Boehmians is called a *Boehmian space.*

We now observe that condition 1 for delta sequences ensures that G gets embedded injectively in $\mathscr{B}$ using the map $f \mapsto \left[\frac{f * \phi_k}{\phi_k}\right]$ for $f \in G$ and any sequence $\{\phi_k\} \in \chi$.

Definition 2.6. Let G be as above with a notion of convergence of sequences. A sequence $\{x_n\}$ of Boehmians is δ-*convergent* to a Boehmian x (we write $\delta\text{-}\lim x_n = x$ or $x_n \xrightarrow{\delta} x$ as $n \to \infty$) if there exists a delta sequence $\{\delta_k\}$ such that $x_n * \delta_k, x * \delta_k \in G$ for all $k, n \in \mathbb{N}$ and $x_n * \delta_k \to x * \delta_k$ as $n \to \infty$ in G for each $k \in \mathbb{N}$.

For a complete description of the construction of Boehmian spaces and their properties we refer the reader to [M1]. There are also various other types of construction of generalized quotient spaces which are studied in detail in the literature (see [AM], [M2]). However we shall restrict ourselves to the original construction described above.

We now define two Boehmian spaces as follows. $\mathscr{B}_1$ is formed using

$$G = H^\infty(\mathbb{D}), \qquad S = B,$$

$*$ being the Hadamard convolution and $\chi = \{\{\phi_k\} \subset B \,/\, \phi_k(z) = \sum_{n=0}^{\infty} c_{nk} z^n$ with $c_{nk} \neq 0$ for all k, n, $c_{nk} \to 1$ for each n as $k \to \infty$, and $\sum_{n=0}^{\infty} |c_{nk}| \leq M$, where M is independent of $k\}$. For example the sequence defined by $\phi_k(z) = \sum_{n=0}^{\infty} e^{-n^2/k} z^n$ $(k = 1, 2, \ldots)$ is a delta sequence. The second Boehmian space $\mathscr{B}_2$ is obtained by taking $G = C_0(E)$, $S = \hat{B} = \{\hat{f} : f \in B\}$, $\chi = \{\{\hat{\phi}_k\} \mid \{\phi_k\}$ is a delta sequence in $\mathscr{B}_1\}$ and taking $*$ to be pointwise multiplication. (Note that here ˆ refers to the Gelfand transform on $H^\infty(\mathbb{D})$). We also observe that $C(\Delta)$ can be identified in $\mathscr{B}_2$ using the map $f \mapsto \left[\frac{f\hat{\phi}_k}{\hat{\phi}_k}\right]$, as can be easily verified. Since the Boehmian space $\mathscr{B}_1$ already contains the Dirac delta (which is represented by $\left[\frac{\phi_k}{\phi_k}\right] \in \mathscr{B}_1$), it is not necessary for us to construct our Boehmian space using the Banach algebra A. For this reason it is easy to see that the Banach algebra A can be identified inside the Boehmian space $\mathscr{B}_1$ using the continuous map

$$(f, \alpha) \mapsto \left[\frac{f * \phi_k + \alpha\phi_k}{\phi_k}\right].$$

Here the continuity is understood in the sense that $f_n \to f$ as $n \to \infty$ in G implies $x_n = \left[\frac{f_n * \phi_k}{\phi_k}\right] \xrightarrow{\delta} x = \left[\frac{f * \phi_k}{\phi_k}\right]$ in $\mathscr{B}_1$. Further it is easy to see that $\mathscr{B}_1$ is larger than A using the following example. Consider $f(z) = \sum_{n=0}^{\infty} n! z^n$ (whose radius of convergence is 0). This can be identified with $\left[\frac{g_k}{\phi_k}\right]$ where $g_k(z) = \sum_{n=0}^{\infty} n! e^{-n^2/k} z^n$ and $\phi_k(z) = \sum_{n=0}^{\infty} e^{-n^2/k} z^n$.

Addition and scalar multiplication in the Boehmian spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ are defined canonically. Moreover we can define convolution of two Boehmians in $\mathscr{B}_1$ as follows:

$$\left[\frac{f_k}{\phi_k}\right] * \left[\frac{g_k}{\psi_k}\right] = \left[\frac{f_k * g_k}{\phi_k * \psi_k}\right].$$

In a similar way we can define the product of two Boehmians in $\mathscr{B}_2$. We observe that the Gelfand transform theory which is available on A can be extended to the Boehmian space $\mathscr{B}_1$ and we can study the extended Gelfand theory in this context. While in the classical case the Gelfand transform maps A onto a proper subalgebra of $C(\Delta)$, in our theory it is possible to show that the extended Gelfand transform is a linear, bijective and bicontinuous (in the delta sense) map of $\mathscr{B}_1$ onto $\mathscr{B}_2$. In particular every element of $C(\Delta)$ can be viewed as the Gelfand transform of a suitable Boehmian. Thus the framework of Boehmians enables us to realise the whole of $C(\Delta)$ as the Gelfand transforms of convolution quotients of $H^\infty(\mathbb{D})$. We shall give the full details in the following section.

3. Main results

DEFINITION 3.1. Let $x = \big[\frac{f_k}{\phi_k}\big] \in \mathscr{B}_1$. Then the *Gelfand transform* $\hat{x}$ of x is defined as the Boehmian $\big[\frac{\hat{f}_k}{\hat{\phi}_k}\big] \in \mathscr{B}_2$.

It is easy to see that $\hat{x} \in \mathscr{B}_2$ and is well-defined. Further if x represents $(f, \alpha) \in A$ in the sense that $x = \big[\frac{f*\phi_k+\alpha\phi_k}{\phi_k}\big]$ then $\hat{x} = \big[\frac{(\hat{f}+\alpha)\hat{\phi}_k}{\hat{\phi}_k}\big]$ represents $\hat{f}+\alpha$, which is nothing but the Gelfand transform of $(f, \alpha) \in A$, and therefore our definition is consistent with the classical definition of the Gelfand transform on A.

DEFINITION 3.2. A bijective map $\varPhi$ from a Boehmian space $\mathscr{B}_1$ onto a Boehmian space $\mathscr{B}_2$ is said to be *bicontinuous* if $x_n \xrightarrow{\delta} x$ in $\mathscr{B}_1$ as $n \to \infty$ implies $y_n = \varPhi(x_n) \xrightarrow{\delta} y = \varPhi(x)$ in $\mathscr{B}_2$ as $n \to \infty$ and vice versa.

THEOREM 3.3. *The Gelfand transform $\varPhi : \mathscr{B}_1 \to \mathscr{B}_2$ given by $\varPhi(x) = \hat{x}$ is linear, bijective and bicontinuous (in the delta sense). Moreover $(x*y)^\wedge = \hat{x}\hat{y}$.*

Proof. It is obvious that $\varPhi$ is linear and we can also easily show that the Gelfand transform is injective and that $(x*y)^\wedge = \hat{x}\hat{y}$ for $x, y \in \mathscr{B}_1$. We now show that the Gelfand transform on $\mathscr{B}_1$ is surjective. Let $y = \big[\frac{f_k}{\hat{\phi}_k}\big] \in \mathscr{B}_2$. Fix $\{\psi_k\} \in \chi$ (with $\psi_k(z) = \sum_{n=0}^{\infty} d_{nk} z^n$) and consider $g_k(z) = \sum_{n=0}^{\infty} b_{nk} z^n$, where $b_{nk} = f_k(\delta_n)\hat{\psi}_k(\delta_n) = f_k(\delta_n) d_{nk}$. We have $|b_{nk}| \le M_k |d_{nk}|$ (where $M_k = \sup_{h\in E} |f_k(h)| < \infty$). It is now easy to see that $g_k(z) \in B$ for every $k = 1, 2, \ldots$ and that

$$\hat{g}_k(\delta_n) = b_{nk} = f_k(\delta_n)\hat{\psi}_k(\delta_n).$$

(Note that in view of Theorem 2.3 every element of E is of the form δ_n, $n = 0, 1, 2, \ldots$). Hence

$$y = \left[\frac{f_k}{\hat{\phi}_k}\right] = \left[\frac{f_k\hat{\psi}_k}{\hat{\phi}_k\hat{\psi}_k}\right] = \left[\frac{\hat{g}_k}{\hat{\phi}_k\hat{\psi}_k}\right] = \left[\frac{g_k}{\phi_k * \psi_k}\right]^\wedge = \hat{x}$$

with $x = \big[\frac{g_k}{\phi_k*\psi_k}\big] \in \mathscr{B}_1$. This completes the proof of the fact that $\varPhi$ is surjective.

We next claim that $\varPhi$ is continuous in the sense that $x_m \xrightarrow{\delta} x$ in $\mathscr{B}_1$ as $m \to \infty$ implies $\hat{x}_m \xrightarrow{\delta} \hat{x}$ in $\mathscr{B}_2$ as $m \to \infty$ (see Definition 3.2). Indeed we can assume $x_m = \big[\frac{f_{km}}{\phi_k}\big]$, $x = \big[\frac{f_k}{\phi_k}\big]$ and $f_{km} \to f_k$ in $H^\infty(\mathbb{D})$ as $m \to \infty$ for each $k = 1, 2, \ldots$. Using the fact that the Gelfand transform is continuous on A it follows that $\hat{f}_{km} \to \hat{f}_k$ in $C_0(E)$ as $m \to \infty$ for each $k = 1, 2, \ldots$. Since $\hat{x}_m = \big[\frac{\hat{f}_{km}}{\hat{\phi}_k}\big]$, $\hat{x} = \big[\frac{\hat{f}_k}{\hat{\phi}_k}\big]$ we see that $\hat{x}_m \xrightarrow{\delta} \hat{x}$ in $\mathscr{B}_2$ as $m \to \infty$. Thus $\varPhi$ is continuous. For the continuity of $\varPhi^{-1}$ we shall assume $y_m \to y$ as $m \to \infty$, where $y_m = \left[\frac{\hat{f}_{km}}{\hat{\phi}_k}\right]$, $y = \left[\frac{\hat{f}_k}{\hat{\phi}_k}\right] \in \mathscr{B}_2$ (note that Φ is bijective), and show that in $\mathscr{B}_1$, $x_m \xrightarrow{\delta} x$ as $m \to \infty$.

Our claim is that $f_{km} * \phi_k \to f_k * \phi_k$ in $H^\infty(\mathbb{D})$ as $m \to \infty$ for each $k = 1, 2, \ldots$. Indeed, by hypothesis, we have $|a_{nkm} - a_{nk}| < \epsilon$ uniformly for all n as $m \to \infty$ for each fixed k, where $f_k(z) = \sum_{n=0}^\infty a_{nk} z^n$, $f_{km}(z) = \sum_{n=0}^\infty a_{nkm} z^n$ and $\phi_k(z) = \sum_{n=0}^\infty c_{nk} z^n$. Hence

$$\sup_{z\in\mathbb{D}} |(f_{km} * \phi_k)(z) - (f_k * \phi_k)(z)| \le \sum_{n=0}^\infty |a_{nkm} - a_{nk}|\,|c_{nk}| \le \epsilon \sum_{n=0}^\infty |c_{nk}| \le \epsilon M \quad \text{(say)}.$$

This shows that $(x_m * \phi_k) * \phi_k \to (x * \phi_k) * \phi_k$ as $m \to \infty$ or equivalently $x_m \xrightarrow{\delta} x$ in $\mathscr{B}_1$ as $m \to \infty$. Thus Φ^{-1} is continuous. The proof of the theorem is now complete. ■

The above theorem in particular implies that given $f \in C(\Delta)$ there exists $x = \left[\frac{f_k}{\phi_k}\right] \in \mathscr{B}_1$ such that $\hat{x} = \left[\frac{f\hat{\phi}_k}{\hat{\phi}_k}\right]$. However it is also possible to characterise the Boehmians whose Gelfand transforms represent functions in $C(\Delta)$. We have the following.

THEOREM 3.4. *Let* $x = \left[\frac{f_k}{\phi_k}\right] \in \mathscr{B}_1$ *with* $f_k(z) = \sum_{n=0}^\infty a_{nk} z^n \in H^\infty(\mathbb{D})$. *Then* $\hat{x} = \left[\frac{\hat{f}_k}{\hat{\phi}_k}\right]$ *represents* $f \in C(\Delta)$ (*in the sense that* $f\hat{\phi}_k = \hat{f}_k$) *if and only if* $\lim_{k\to\infty} a_{nk} = b_n$ *exists for each* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} b_n = \alpha$ *exists for some* $\alpha \in \mathbb{C}$.

Proof. Let $\hat{f}_k = f\hat{\phi}_k$ in $C_0(E)$ and note that

$$\begin{aligned} \hat{f}_k(\delta_n) &= f(\delta_n)\hat{\phi}_k(\delta_n), \\ a_{nk} &= f(\delta_n) c_{nk} \quad (\hat{\phi}_k(\delta_n) = c_{nk}), \\ \lim_{k\to\infty} a_{nk} &= f(\delta_n) \lim_{k\to\infty} c_{nk} = f(\delta_n), \\ \lim_{n\to\infty}\lim_{k\to\infty} a_{nk} &= \lim_{n\to\infty} f(\delta_n) = f(\delta_\infty) \text{ exists.} \end{aligned}$$

(Note that f is continuous on Δ and its Gelfand topology implies $\delta_n \to \delta_\infty$ as $n \to \infty$).

Conversely assume that $\lim_{n\to\infty}\lim_{k\to\infty} a_{nk} = \alpha$ exists. Define

$$f(\delta_n) = \frac{\hat{f}_k(\delta_n)}{\hat{\phi}_k(\delta_n)} \quad \text{for } \delta_n \ne \delta_\infty \quad \text{and} \quad f(\delta_\infty) = \alpha.$$

Thus f is defined for all $h \in \Delta$. Since $c_{nk} \ne 0$ for all k, n, we have $\hat{\phi}_k(\delta_n) \ne 0$, so $f(\delta_n)$ is well defined on E. Being the quotient of two continuous functions,

f is continuous on E. Using the quotient property of $\left[\frac{f_k}{\phi_k}\right]$, it also follows that $f(\delta_n)$ is independent of k. In order to prove that f is continuous on $\varDelta$ we have to show that $f(\delta_n) \to f(\delta_\infty)$ as $n \to \infty$. Indeed,

$$\begin{aligned} f(\delta_n) &= \lim_{k\to\infty} \hat{f}_k(\delta_n)/\lim_{k\to\infty} \hat{\phi}_k(\delta_n) \\ &= \lim_{k\to\infty} a_{nk} \quad \text{(since } c_{nk} \to 1 \text{ as } k \to \infty \text{ for each } n). \end{aligned}$$

Now

$$\lim_{n\to\infty} f(\delta_n) = \lim_{n\to\infty} \lim_{k\to\infty} a_{nk} = \alpha = f(\delta_\infty).$$

Thus $f \in C(\varDelta)$. This completes the proof of our theorem. ∎

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