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Pencils of irreducible rational curves and plane Jacobian conjecture

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Abstract. In certain cases the invertibility of a polynomial map $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ can be characterized by the irreducibility and the rationality of the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$.

1. Introduction. Let $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map with finite fibres. The mysterious Jacobian Conjecture (see [1] and [2] for its history and surveys), posed first by Ott-Heinrich Keller [8] in 1939 and still open, asserts that $F = (P, Q)$ is invertible if its Jacobian $\det DF$ is a non-zero constant. In 1978 Razar [15] discovered a remarkable fact that F is invertible if $\det DF \equiv c \neq 0$ and, in addition, all the fibres $P = \lambda$, $\lambda \in \mathbb{C}$, are *irreducible rational* curves, i.e. curves diffeomorphic to a sphere with a finite number of punctures. In various attempts to understand the nature of the plane Jacobian conjecture, this fact has been reproved by Heitmann [4], Lê and Weber [9], Friedland [3], and Némethi and Sigray [10] with several different approaches. In fact, Vistoli [17] and Neumann and Norbury [11] observed that every rational polynomial with all fibres irreducible is equivalent to the projection $(x, y) \mapsto x$ up to polynomial diffeomorphisms.

In this paper we note that in certain cases the invertibility of a polynomial map $F = (P, Q)$ of $\mathbb{C}^2$ with finite fibres can be characterized by the topology of the pencil of the affine curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$. Our result is

Theorem 1 (Main Theorem). *Let $F = (P, Q)$ be a polynomial map of $\mathbb{C}^2$ with finite fibres. Suppose all the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$, are irreducible and rational. Then the following are equivalent:*

(a) *$(0, 0)$ is a regular value of F;*
(b) $\det DF \equiv c \neq 0$*;*
(c) *F is invertible.*

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Theorem 1 leads to the following criterion for the invertibility of polynomial maps of $\mathbb{C}^2$.

THEOREM 2. *Let F be a polynomial map of $\mathbb{C}^2$ with finite fibres. If for generic points $q \in \mathbb{C}^2$ the inverse images $F^{-1}(l)$ of all complex lines l in $\mathbb{C}^2$ passing through q are irreducible rational curves, then F is invertible.*

Proof. Since the fibres of F are finite, we have $\det DF \not\equiv 0$. Then, by the assumptions we can assume that $(0,0)$ is a regular value of F and that for all lines l passing through $(0,0)$ the inverse image $F^{-1}(l)$ is an irreducible rational curve. Hence, by Theorem 1 the map F is invertible. ■

Note that Theorem 2 still holds in higher dimensions under the additional assumption $\det DF \equiv c \neq 0$ ([14, Corollary 1.3]).

In an attempt to understand the plane Jacobian conjecture it is worth considering the following questions:

QUESTION 1. *Does the Jacobian condition ensure the irreducibility of all of the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$?*

QUESTION 2. *Is a non-zero constant Jacobian polynomial map $F = (P, Q)$ of $\mathbb{C}^2$ invertible if all the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$, are irreducible?*

Kaliman [7] observed that to prove the plane Jacobian conjecture it is sufficient to consider non-zero constant Jacobian polynomial maps $F = (P, Q)$ all of whose fibres $P = c$, $c \in \mathbb{C}$, are irreducible. In general to Question 2 note that the irreducibility of the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$, alone does not guarantee the invertibility of the polynomial map $F = (P, Q)$. For example, the map $F(x, y) = (x, x^2 + y^3)$ is not invertible, but the curves $ax + b(x^2 + y^3) = 0$, $(a : b) \in \mathbb{P}^1$, are irreducible. Further examination of the relation between the Jacobian condition and the geometry of the pencil of curves $aP + bQ = 0$ would be useful in pursuit of the solution of the plane Jacobian problem.

The proof of Theorem 1 will be carried out in Section 3 after some necessary preparations in the next section.

2. Two lemmas on the pencil of curves $aP + bQ = 0$. From now on, $F = (P, Q)$ is a given polynomial map of $\mathbb{C}^2$ with finite fibres. In this section we are concerned with compactifications of the pencil of the curves $aP + bQ = 0$. The results obtained will be used in our proof of Theorem 1 in Section 3.

Let $D_\lambda := \{(x, y) \in \mathbb{C}^2 : aP(x, y) + bQ(x, y) = 0\}$ for $\lambda = (a : b) \in \mathbb{P}^1$ and denote by r_λ the number of irreducible components of the curve D_λ. Regarding the plane $\mathbb{C}^2$ as a subset of the projective plane $\mathbb{P}^2$, we can associate to F the rational map $G : \mathbb{P}^2 \to \mathbb{P}^1$ given by $G(x, y) = (P(x, y) : Q(x, y)) \in \mathbb{P}^1$, which is well defined outside the finite set $B := F^{-1}(0,0)$ and a possible finite subset of the line at infinity of $\mathbb{C}^2$. We can extend G to a regular morphism $g : X \to \mathbb{P}^1$ from a compactification X of $\mathbb{C}^2 \setminus B$ to $\mathbb{P}^1$. By a *horizontal component* (resp. a *constant component*) of G we mean an irreducible component l of the divisor $\mathcal{D} := X \setminus (\mathbb{C}^2 \setminus B)$ such that the restriction g_l of g to l is a non-constant mapping (resp. a constant mapping). Let us denote by h_G the number of horizontal components of G. The number h_G depends on P and Q, but not on the compactification X of $\mathbb{C}^2$.

We can construct such an extension $g : X \to \mathbb{P}^1$ by a minimal sequence of blow-ups

$$\pi : X \to \mathbb{P}^2 \tag{1}$$

that removes all the indeterminacy points of the rational map G. In such an extension g the divisor $\mathcal{D}$ is the disjoint union of the connected divisors $\mathcal{D}_\infty := \pi^{-1}(L_\infty)$ and $\mathcal{D}_b := \pi^{-1}(b)$, $b \in B$, where L_∞ indicates the line at infinity of $\mathbb{C}^2 \subset \mathbb{P}^2$. Denote by h_∞ and h_b the numbers of horizontal components of G contained in the divisors $\mathcal{D}_\infty$ and $\mathcal{D}_b$, $b \in B$, respectively. Obviously,

$$h_\infty > 0 \quad \text{and} \quad h_b > 0 \text{ for } b \in B \tag{2}$$

and

$$h_G = h_\infty + \sum_{b\in B} h_b. \tag{3}$$

LEMMA 1. *If the generic curve D_λ is irreducible and rational, then*

$$\sum_{\lambda\in\mathbb{P}^1} (r_\lambda - 1) = h_\infty + \sum_{b\in B} h_b - 2. \tag{4}$$

The equality (4) is a folklore fact, which can be deduced from the estimate on the total reducibility order of pencils of curves obtained by Vistoli in [17]. The proof presented below is quite elementary and is analogous to that of Kaliman [6] for the total reducibility order of polynomials in two variables.

Proof of Lemma 1. Fix a regular morphism g which is a blow-up version of G. Let C_λ be the fibre $g = \lambda$, $\lambda \in \mathbb{P}^1$, and let C be a generic fibre of g. We will use Suzuki's formula [16]

$$\sum_{\lambda\in\mathbb{P}^1} (\chi(C_\lambda) - \chi(C)) = \chi(X) - 2\chi(C). \tag{5}$$

Here, $\chi(V)$ indicates the Euler–Poincaré characteristic of V.

Let us denote by m the number of irreducible components of the divisor $\mathcal{D}$ and by m_λ the number of irreducible components of C_λ contained in $\mathcal{D}$. Then $\chi(X) = m + 2$ and

$$m = h_\infty + \sum_{b \in B} h_b + \sum_{\lambda \in \mathbb{P}^1} m_\lambda.$$

Since the generic curves D_λ are irreducible and rational, the generic fibre C is a copy of $\mathbb{P}^1$ and the fibres C_λ are connected rational curves with simple normal crossings. Therefore, $\chi(C) = 2$ and $\chi(C_\lambda) = r_\lambda + m_\lambda + 1$.

Now, by the above we have

$$\chi(X) - 2\chi(C) = h_\infty + \sum_{b \in B} h_b + \sum_{\lambda \in \mathbb{P}^1} m_\lambda - 2 \tag{6}$$

and

$$\sum_{\lambda \in \mathbb{P}^1} (\chi(C_\lambda) - \chi(C)) = \sum_{\lambda \in \mathbb{P}^1} (r_\lambda - 1) + \sum_{\lambda \in \mathbb{P}^1} m_\lambda. \tag{7}$$

Putting (6) and (7) into (5) we get the desired equality (4). ∎

Regarding polynomials P and Q as rational maps from $\mathbb{P}^2$ into $\mathbb{P}^1$, the blow-up $X \xrightarrow{\pi} \mathbb{P}^2$ in (1) also provides natural extensions $p, q : X \to \mathbb{P}^1$ of P and Q, which may have some indeterminacy points. If necessary, we can replace X by its convenient blow-up version so that p and q are regular morphisms and $f = (p, q) : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a regular extension of F. The restrictions of p and q to each irreducible component $l \subset \mathcal{D}$ then determine holomorphic maps from l to $\mathbb{P}^1$, denoted by p_l and q_l respectively. We can divide the horizontal components l of G into the following types:

I. $l \subset \mathcal{D}_b$. Then $(p_l, q_l) \equiv (0, 0)$.

II. $l \subset \mathcal{D}_\infty$. Then either
  - (a) $(p_l, q_l) \equiv (\infty, \infty)$,
  - (b) $(p_l, q_l) \equiv (0, 0)$, or
  - (c) (p_l, q_l) is a non-constant mapping with $(p_l : q_l) \neq \text{const}$.

Obviously, in Type (IIc), $(p_l, q_l)(l) \cap \mathbb{C}^2 \neq \emptyset$.

By a *dicritical component* of F we mean an irreducible component $l \subset \mathcal{D}_\infty$ such that (p_l, q_l) is a non-constant mapping. Recall from [5] that the *non-proper value set* A_F of F is the set of all $a \in \mathbb{C}^2$ such that a is a limit point of $F(v_k)$ for a sequence $v_k \in \mathbb{C}^2$ tending to ∞. The set A_F is a plane curve composed of the images of some polynomial maps from $\mathbb{C}$ into $\mathbb{C}^2$ [5]. When F has finite fibres, by definitions

$$v \notin A_F \;\Leftrightarrow \sum_{w \in F^{-1}(v)} \deg_w F = \deg_{\text{geo}} F, \tag{8}$$

where $\deg_w F$ is the multiplicity of F at w and $\deg_{\text{geo}} F$ is the number of solutions of the equation $F(x, y) = v$ for generic points $v \in \mathbb{C}^2$. Obviously, by the definitions

$$A_F = \bigcup_{l \text{ a dicritical component of } F} (f(l) \cap \mathbb{C}^2).$$

In particular, F is a proper map of $\mathbb{C}^2$ if and only if F does not have dicritical components.

LEMMA 2.

(a) *G has at least one horizontal component of Type* (IIa).
(b) *If $A_F \neq \emptyset$, then G has at least one horizontal component of Types* (IIb) *or Type* (IIc). *If $(0,0) \in A_F$, then G has at least one horizontal component of Type* (IIb).
(c) *If l is a dicritical component of F, then either l is a horizontal component of G or $f(l) \cap \mathbb{C}^2$ is a line passing through* $(0,0)$.

Proof. (a) Note that each generic fibre C_λ is the union of D_λ and a finite number of points lying in horizontal components of G, at which the rational map (p,q) is well defined. If G did not have horizontal components of Type (IIa), the map (p,q) would have finite values on $C_\lambda \cap \mathcal{D}$, and hence P and Q would be constant on each connected component of D_λ. This is impossible, since the fibres of F are finite.

(b) By definitions the non-proper value set A_F can be expressed as $A_F = f(\mathcal{D}_\infty) \cap \mathbb{C}^2$. Assume $A_F \neq \emptyset$. Let V be an irreducible component of A_F. Then the inverse $f^{-1}(V)$ must contain a component l of $\mathcal{D}_\infty$ such that $V \subset f(l)$. Obviously, $g(l) = \mathbb{P}^1$ or $g_l \equiv \text{const}$. Therefore, l is a horizontal component of Type (IIc) of G, unless $(0,0) \in A_F$ and V is a line passing through $(0,0)$. In the case $(0,0) \in A_F$, the intersection $D := f^{-1}(0,0) \cap \mathcal{D}_\infty$ is not empty. Then f maps each neighbourhood U of D onto a neighbourhood of $(0,0)$, and hence g maps each such U onto $\mathbb{P}^1$. It follows that D must contain a horizontal component of Type (IIb) of G. The conclusions are now clear.

(c) Let l be a dicritical component of F, $(p_l : q_l) \neq \text{const}$. By definitions, l is either a horizontal component of G (if $(p_l : q_l) \neq \text{const}$), or a component of a fibre of g. Obviously, in the latter case $f(l) \cap \mathbb{C}^2$ is a line passing through $(0,0)$. ∎

3. Proof of Main Theorem. In the proof we will use the following fact on the non-proper value sets of non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$.

THEOREM 3 ([12], [13]). *Let $F = (P,Q)$ be a non-zero constant Jacobian polynomial map. Then the irreducible components of A_F, if any, can be parameterized by polynomial maps $t \mapsto (\varphi(t), \psi(t))$, $\varphi, \psi \in \mathbb{C}[t]$, satisfying*

$$\frac{\deg \varphi}{\deg \psi} = \frac{\deg P}{\deg Q}.$$

In particular, A_F never contains components isomorphic to the line $\mathbb{C}$.

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let $F = (P, Q)$ be a given polynomial map $\mathbb{C}^2$ with finite fibres such that all the curves $aP + bQ = 0$, $(a : b) \in \mathbb{P}^1$, are irreducible and rational. The implication (c)⇒(a;b) is trivial. We need to prove only (a)⇒(c) and (b)⇒(c). We will use the same constructions and notations presented for $F = (P, Q)$ in the previous sections.

First, by the assumptions we can apply Lemma 1 to see that G has exactly two horizontal components,

$$h_G = h_\infty + \sum_{b \in B} h_b = 2. \tag{9}$$

Since $h_\infty > 0$ and $h_b > 0$ for $b \in B$ by Lemma 2, from (9) it follows that either

(i) $h_\infty = 2$ and $B = \emptyset$, or
(ii) $h_\infty = 1$, B consists of a unique point, say $B = \{b\}$, and $h_b = 1$.

(a)⇒(c). Assume that $(0, 0)$ is a regular value of F, i.e. $F^{-1}(0, 0)$ is non-empty and does not contain singular points of F. So, we are in case (ii): $h_\infty = 1$, $B = \{b\}$ and $h_b = 1$. Then, by Lemma 2(a) the unique horizontal component of G in $\mathcal{D}_\infty$ must be of Type (IIa). It follows that F does not have dicritical components, $A_F = \emptyset$. This means that F is a proper map of $\mathbb{C}^2$. Then, by (8) the geometric degree $\deg_{\text{geo}} F$ is equal to the number of solutions of the equation $F(x, y) = (0, 0)$, counted with multiplicity. But this equation admits b as a unique solution and b is not a singular point of F. Thus, $\deg_{\text{geo}} F = 1$ and hence F is injective. Then, by the well-known fact (see [2]) that polynomial injections of $\mathbb{C}^n$ are automorphisms, the map F must be invertible.

(b)⇒(c). Assume $\det DF \equiv \text{const} \neq 0$. If $F^{-1}(0, 0) \neq \emptyset$, then $(0, 0)$ is a regular value of F and we are done by the previous part. Assume now that $F^{-1}(0, 0) = \emptyset$. Then we are in case (i): $h_\infty = 2$ and $B = \emptyset$. In this case, by the definitions $(0, 0)$ is a non-proper value of F, $(0, 0) \in A_F$. Therefore, by Lemma 2(a)&(b), G has exactly two horizontal components, one of Type (IIa) and one of Type (IIb). In particular, none of such horizontal components can be a dicritical component of F. Hence, by Lemma 2(c), A_F must be composed of some lines passing through $(0, 0)$. This contradicts Theorem 3. Thus, F is invertible. ■

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