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Normal pseudoholomorphic curves

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Abstract. First, we give some characterizations of J-hyperbolic points for almost complex manifolds. We apply these characterizations to show that the hyperbolic embeddedness of an almost complex submanifold follows from relative compactness of certain spaces of continuous extensions of pseudoholomorphic curves defined on the punctured unit disc. Next, we define uniformly normal families of pseudoholomorphic curves. We prove extension-convergence theorems for these families similar to those obtained by Kobayashi, Kiernan and Joseph–Kwack in the standard complex case.

1. Introduction. It is known that every bounded domain in $\mathbb{C}^n$ equipped with the standard complex structure is hyperbolic. A statement of this kind is however false for a general almost complex domain where some extra condition such as tameness or standard integrability is not imposed (see Example 2 in [17]).

Motivated by studying the geometry of unbounded domains in $\mathbb{C}^n$ in almost complex settings, the investigation of hyperbolic embeddedness of not necessarily relatively compact almost complex submanifolds of an almost complex manifold is of interest. In this paper, we focus on such studies and we use the results obtained to extend pseudoholomorphic curves.

First, we introduce and give some characterizations of J-hyperbolic points. This allows a characterization of the hyperbolic embeddedness of an almost complex submanifold (M, J) (not necessarily relatively compact) in an almost complex manifold (N, J), and extensions of pseudoholomorphic curves. We recall that several characterizations of hyperbolic embeddedness were obtained in [7] under the assumption of relative compactness. Mainly, we proved that a relatively compact almost complex submanifold (M, J) is hyperbolically embedded in (N, J) if and only if $\mathcal{O}_J(\Delta, M)$ is relatively compact in $\mathcal{O}_J(\Delta, N)$.

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As far as we know, almost all studies of hyperbolic embeddedness in the almost complex case were carried out under the assumption of the relative compactness of M in N (see [7], [13]). In the complex case, such studies have been carried out by Kobayashi [16], Kiernan [14, 15], and Joseph–Kwack [10, 11]. Their proofs are based on the Riemann extension theorem and winding numbers arguments. So, to extend such results to the almost complex setting, we need substantial modifications.

Next, we extend the notion of the uniformly normal family to almost complex curves. We prove that such curves satisfy the big Picard theorem and Noguchi-type extension convergence theorems. We recall that Joseph and Kwack [12] and Funahashi [4] extended to several complex variables the notion of Hayman's uniformly normal family defined by Lehto–Virtanen [19] in one complex variable.

Throughout the paper, we assume, by default, that all relevant objects (manifolds, structures, etc.) are smooth of class C^∞.

We denote by $M^+ = M \cup \{\infty\}$ the Aleksandrov (1-point) compactification of M, and $\mathcal{C}(X, Y)$ is the space of continuous maps from a topological space X to a topological space Y. Let X_0 and Y_0 be subspaces of X, Y respectively, such that X_0 is dense in X. If $Q \subset \mathcal{C}(X_0, Y_0)$, we denote by $\mathcal{C}[X, Y; Q]$ the set of $\tilde{f} \in \mathcal{C}(X, Y)$ such that $\tilde{f}$ is the unique extension of $f \in Q$.

For the definition of hyperbolic almost complex manifold and related material see [8, 17].

2. Hyperbolic embedding and extension of pseudoholomorphic discs. Hyperbolic points are introduced and investigated in [10]; this notion can be naturally extended to the almost complex case.

Definition 2.1. Let (M, J) be an almost complex submanifold in an almost complex manifold (N, J) with a length function G.

1. A point $p \in \overline{M}$ is called a *J-hyperbolic point* for M if there is a neighborhood U of p in N and a positive constant c such that $K^J_M \geq cG$ on $U \cap M$.
2. (M, J) is *hyperbolically embedded* in (N, J) if for every pair (p, q) of different points in $\overline{M}$, there exist neighborhoods U and V of p and q such that $d^J_M(M \cap U, M \cap V) > 0$, where $\overline{M}$ is the closure of M in N and d^J_M is the Kobayashi pseudo-distance on M.

We will denote the set of J-hyperbolic points of M by $R^J_M(N)$. It is clear from the definition that $R^J_M(N)$ is an open subset of $\overline{M}$.

Definition 2.2. We call $p \in \overline{M}$ a *degeneracy point* of $d^J_{\overline{M}}$ if there exists a point $q \in \overline{M} \setminus \{p\}$ such that $d^J_{\overline{M}}(p, q) = 0$. We denote by $S^J_M(N)$ the set of degeneracy points of $d^J_{\overline{M}}$.

Here, $d^J_{\overline{M}}$ is the extension to $\overline{M}$ of the Kobayashi pseudodistance defined as follows:

$$\text{for } p,q \in \overline{M}, \qquad d^J_{\overline{M}}(p,q) = \liminf_{p'\Rightarrow p,\, q'\Rightarrow q} d^J_M(p',q'), \qquad p',q' \in M.$$

The following proposition shows that the set of J-hyperbolic points of M is the complement in $\overline{M}$ of the set of degeneracy points.

Proposition 2.3. *Let (M,J) be an almost complex submanifold in an almost complex manifold (N,J). For a point $p \in \overline{M}$, the following are equivalent:*

(1) $p \in R^J_M(N)$.
(2) *For every sequence (p_n) converging to p and for every sequence (q_n) with no subsequence converging to p, we have* $\liminf d^J_M(p_n,q_n) > 0$.

It is easily seen from Proposition 2.3 that (M,J) is hyperbolically embedded in (N,J) if and only if $\overline{M} = R^J_M(N)$.

Our main result in this section is the following

Theorem 2.4. *Let (M,J) be an almost complex submanifold in an almost complex manifold (N,J) with a length function G. The following are equivalent for a point $p \in \overline{M}$:*

(1) $p \in R^J_M(N)$.
(2) *If (f_n) is a sequence in $\mathcal{O}_J(\Delta^*,M)$ and (z_n) is a sequence in Δ^* such that $z_n \to 0$ and $f_n(z_n) \to p$, then for each U open about p, there is an $r \in]0,1[$ such that $f_n(\Delta^*_r) \subset U$ eventually.*
(3) *If (f_n) is a sequence in $\mathcal{O}_J(\Delta^*,M)$ and (z_n) is a sequence in Δ^* such that $z_n \to 0$ and $f_n(z_n) \to p$, then:*
   (a) *Each f_n extends to $\tilde{f}_n \in \mathcal{O}_J(\Delta,N)$.*
   (b) *There is an $r \in]0,1[$ and a subsequence $(\tilde{f}_{n_k})$ of $(\tilde{f}_n)$ such that*
$$\tilde{f}_{n_k} \to f \in \mathcal{O}_J(\Delta_r,N) \qquad \text{on } \Delta_r.$$
(4) *There is a neighborhood U of p such that*
$$\sup\{|f'(0)|_G : f \in \mathcal{O}_J(\Delta,M),\ f(0) \in U\} < \infty.$$

The proof follows the line of Joseph–Kwack's proof (see [10, Theorem 1]), although much of the key steps including the Riemann removable singularity theorem and winding numbers arguments due to Grauert–Reckziegel [6] are not available in the almost complex case.

To prove the theorem, we need the following lemmas.

Lemma 2.5. *Let (M,J) be an almost complex submanifold of an almost complex manifold (N,J) and let $p \in R^J_M(N)$. Let (z_n) and (f_n) be sequences in Δ^* and $\mathcal{O}_J(\Delta^*,M)$ respectively such that $z_n \to 0$ and $f_n(z_n) \to p$. If $\sigma_n = \{z \in \Delta : |z| = |z_n|\}$, then $f_n(\sigma_n) \to p$.*

Proof. For every sequence $\alpha_n \in \sigma_n$, we have

$$\begin{aligned}\liminf_{n\to\infty} d^J_{\overline{M}}(f_n(\alpha_n), p) &\le \liminf_{n\to\infty} d^J_M(f_n(\alpha_n), f_n(z_n)) \\ &\le \liminf_{n\to\infty} d_{\Delta^*}(\alpha_n, z_n).\end{aligned}$$

Since $d_{\Delta^*}(\alpha_n, z_n) = O(1/\log|z_n|)$ and $p \in R^J_M(N)$, we have $f_n(\alpha_n) \rightrightarrows p$.

The following statement is essentially due to E. Chirka (see [5, 9]).

LEMMA 2.6. *Let (M, J) be an almost complex manifold and let p be a point of M. Then there exist relatively compact local coordinate neighborhoods U, W of p with $W \subset\subset U$ and positive constants A and B such that the function $\psi_q : z \mapsto \log|z-q|^2 + A|z-q| + B|z|^2$ is J-plurisubharmonic on U for every $q \in \overline{W}$.*

Proof of Theorem 2.4. (1)⇒(2). Assume (1) holds and (2) does not hold. Then there exist relatively compact local coordinate neighborhoods U, W of p as in Lemma 2.6 such that $U \cap \overline{M} \subset R^J_M(N)$, and a sequence (z'_n) in Δ^* such that $|z'_n| < |z_n|$ and $f_n(z'_n) \notin U$, which implies the existence of a sequence (w_n) in Δ^* such that $|z'_n| < |w_n| < |z_n|$ and $f_n(w_n) \in \partial W$ for each n. By taking a subsequence, we may assume that $f_n(w_n) \to q \in \partial W$. Since $p \in R^J_M(N)$, by Lemma 2.5 we have $f_n(\sigma_n) \to p$, where $\sigma_n = \{z \in \Delta : |z| = |z_n|\}$. This implies that $f_n(\sigma_n) \subset W$ for sufficiently large n. Let $\mathcal{R}_n$ be the largest open annulus containing σ_n with $f_n(\mathcal{R}_n) \subset W$. Since $f_n(w_n) \rightrightarrows q \in \partial W$, there exist $a_n \ge |w_n|$ and $b_n > |z_n|$ such that $\mathcal{R}_n = \{z \in \mathbb{C} : a_n < |z| < b_n\}$.

We may assume that $a_n = |w_n|$. Otherwise, there is a sequence (w'_n) in Δ^* such that $|w'_n| = a_n$ and $f_n(w'_n) \to q' \in \partial W$. Let

$$\tilde{\mathcal{R}}_n = \{z \in \mathbb{C} : |w_n| \le |z| \le |z_n|\} \quad \text{and} \quad \gamma_n = \{z \in \Delta^* : |z| = |w_n|\}.$$

We have $f_n(w_n) \rightrightarrows q$, so by Lemma 2.5, we get $f_n(\gamma_n) \rightrightarrows q \in \partial W$. Let $\rho = \psi_p + \psi_q$, where ψ_p and ψ_q are Chirka's functions defined in Lemma 2.6. Clearly, ρ is a J-plurisubharmonic function on U satisfying $\rho^{-1}\{-\infty\} = \{p, q\}$. Therefore, for $K > 0$, we have $f_n(\partial\tilde{\mathcal{R}}_n) \subset \{z \in U : \rho(z) \le -K\}$ for n sufficiently large. By the maximum principle, for n sufficiently large we obtain

$$f_n(\tilde{\mathcal{R}}_n) \subset \{z \in U : \rho(z) \le -K\}.$$

We get a contradiction, since for K sufficiently large, the two points p and q are not in the same connected component.

(2)⇒(3). Let (f_n) be a sequence in $\mathcal{O}_J(\Delta^*, M)$ and (z_n) be a sequence in Δ^* such that $z_n \to 0$ and $f_n(z_n) \to p$. In order to prove that f_n extends to a J-holomorphic curve in Δ, it is sufficient to verify that f_n has a finite energy.

Let U and V be hyperbolic neighborhoods of p such that $\overline{U} \subset V$. There exists a positive constant c such that $K_U^J \geq cG$ where G is a length function on N. By hypothesis, there exist $r \in]0,1[$ and $n_0 \in \mathbb{N}$ such that $f_n(\Delta_r^*) \subset U$ for $n \geq n_0$. We conclude that

$$|f_n'(z)|_G \leq \frac{1}{c} K_{\Delta_r^*}(z) \quad \text{for every } z \in \Delta_r^*.$$

Hence, $E(f_{n|\Delta^*_{r/2}})$, the energy of $f_{n|\Delta^*_{r/2}}$, satisfies

$$E(f_{n|\Delta^*_{r/2}}) = \frac{1}{2} \int_{\Delta^*_{r/2}} |f_n'(z)|_G^2 \leq \frac{1}{2c^2} \int_{\Delta^*_{r/2}} K^2_{\Delta_r^*}(z) < \infty.$$

Consequently, each f_n extends to Δ; we denote the extension by $\tilde{f}_n$. We have $\tilde{f}_n(\Delta_r) \subset V$ for every $n \geq n_0$. If we choose V sufficiently small, there exists a positive constant α such that $|\tilde{f}_n'(z)| \leq \alpha$ in Δ_r. By compactness, there exists a subsequence $(\tilde{f}_{\varphi(n)})$ which converges uniformly to a J-holomorphic curve f.

(3)⇒(4). If (4) is not satisfied, there is a sequence (f_n) in $\mathcal{O}_J(\Delta, M)$ such that $f_n(0) \to p$ and $|f_n'(0)|_G \to \infty$. By continuity, there is a sequence (z_n) in Δ^* such that $z_n \to 0$ and $f_n(z_n) \to p$. Using condition (b), we conclude that there is a subsequence (f_{n_k}) of (f_n) which converges uniformly in some neighborhood of 0, contradicting $|f_n'(0)|_G \to \infty$.

(4)⇒(1). By hypothesis, there exists a positive constant C such that $\sup\{|f'(0)|_G : f \in \mathcal{O}_J(\Delta, M),\ f(0) \in U\} \leq C$. This implies $K_M^J(\xi) \geq C^{-1}|\xi|_G$, therefore p is a J-hyperbolic point.

Corollary 2.7. *Let (M,J) be a hyperbolic embedded almost complex submanifold of an almost complex manifold (N,J). Then each $f \in \mathcal{O}_J(\Delta^*, M)$ extends to $\tilde{f} \in \mathcal{C}(\Delta, N^+)$. If M is relatively compact in N, then $\tilde{f}$ is J-holomorphic.*

Proof. If there is a sequence (z_n) in Δ^* such that $z_n \to 0$ and $f(z_n) \to p \in N$, then $p \in R_M^J(N)$ and by Theorem 2.4(3)(a), f extends to $\tilde{f} \in \mathcal{O}_J(\Delta, N)$. Otherwise, for all sequences (z_n) in Δ^* such that $z_n \to 0$, we have $f(z_n) \to \infty$ and the conclusion follows.

Next, we prove our second main result.

Theorem 2.8. *Let (M,J) be an almost complex submanifold of an almost complex manifold (N,J). The following are equivalent:*

(1) *(M,J) is hyperbolically embedded in (N,J).*
(2) *$\mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$ is relatively compact in $\mathcal{C}(\Delta, N^+)$.*
(3) *$\mathcal{O}_J[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$ is relatively compact in $\mathcal{C}(\Delta, N^+)$.*
(4) *$\mathcal{O}_J(\Delta, M)$ is relatively compact in $\mathcal{C}(\Delta, N^+)$.*

Here $\mathcal{O}_J[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$ denotes the set of $\tilde{f} \in \mathcal{O}_J(\Delta, N)$ such that $\tilde{f}$ is the unique extension of $f \in \mathcal{O}_J(\Delta^*, M)$.

Proof. (1)⇒(2). Suppose that the family $\mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$ is not evenly continuous at (z_0, p) in $\Delta \times N^+$. We only have to consider the case $z_0 = 0$. There exist sequences (f_n) in $\mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$ and (z_n) in Δ^* such that $z_n \to 0$, $f_n(0) \to p$ and $f_n(z_n) \to q \in N^+$ with $p \neq q$. By continuity, there is a sequence (w_n) in Δ^* converging to 0 such that $f_n(w_n) \to p$. Since $p \neq q$, either p or q is in $\overline{M} = R_M^J(N)$. Using Theorem 2.4(2), we get a contradiction since either p or q is a J-hyperbolic point.

(2)⇒(3). We have $\mathcal{O}_J[\Delta, N; \mathcal{O}_J(\Delta^*, M)] \subset \mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$.

(3)⇒(4). We have $\mathcal{O}_J(\Delta, M) \subset \mathcal{O}_J[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$.

(4)⇒(1). Let $p \in \overline{M}$. Using Theorem 2.4, we will prove that there exists a neighborhood U of p such that

$$\sup\{|f'(0)|_G : f \in \mathcal{O}_J(\Delta, M),\ f(0) \in U\} < \infty.$$

Otherwise, there is a sequence (f_n) in $\mathcal{O}_J(\Delta, M)$ such that $f_n(0) \to p$ and $|f_n'(0)|_G \to \infty$. Let U be a neighborhood of p; there is an $r \in]0, 1[$ such that $f_n(\Delta_r) \subset U$. If we choose U sufficiently small, there exists a positive constant α such that $|f_n'(z)|_G \leq \alpha$ in Δ_r, and we get a contradiction.

Remark 2.9. An almost complex manifold (M, J) is hyperbolic if and only if $\mathcal{O}_J(\Delta, M)$ is relatively compact in $\mathcal{C}(\Delta, M^+)$. This generalizes a result of Abate [1] for complex manifolds.

We close this section by showing extension and convergence Noguchi-type theorem for pseudoholomorphic curves with values in a submanifold (not necessarily relatively compact).

Theorem 2.10. *Let (M, J) be a hyperbolic embedded almost complex submanifold of an almost complex manifold (N, J). Then:*

(1) *Each $f \in \overline{\mathcal{O}_J(\Delta^*, M)}$ extends to $\tilde{f} \in \mathcal{C}(\Delta, N^+)$.*
(2) *If (f_n) is a sequence in $\mathcal{O}_J(\Delta^*, M)$ and $f_n \to f$, then $\tilde{f}_n \to \tilde{f}$.*

Proof. (1) Let $f \in \overline{\mathcal{O}_J(\Delta^*, M)}$. There exists a sequence (f_n) in $\mathcal{O}_J(\Delta^*, M)$ converging to f. Each f_n extends to $\tilde{f}_n$ in $\mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$. By Theorem 2.8, there exists a subsequence (f_{n_k}) converging to $g \in \mathcal{C}(\Delta, N^+)$. Hence $g = f$ on Δ^* so $g = \tilde{f}$.

(2) Let $(\tilde{f}_{\varphi(n)})$ be an arbitrary subsequence of $(\tilde{f}_n)$. Since $\tilde{f}_{\varphi(n)}$ is in the family $\mathcal{C}[\Delta, N; \mathcal{O}_J(\Delta^*, M)]$, using Theorem 2.8, we can extract a further subsequence converging to a map $g \in \mathcal{C}(\Delta, N^+)$ coinciding with f on Δ^*. Hence, $\tilde{f} = g$ and we conclude finally that the sequence $(\tilde{f}_n)$ converges to $\tilde{f}$ uniformly on each compact subset.

3. Examples. We give two examples of almost hyperbolic embedded unbounded domains in $\mathbb{C}^n$.

1. In [5], Gaussier and Sukhov proved the hyperbolic embeddedness of smoothly strictly pseudoconvex Stein domains with almost complex structures sufficiently close to the initial Stein complex structure (see [5, Proposition 4]).

2. The following proposition due to Bertrand [2] yields some almost complex unbounded domains that are hyperbolically embedded in $\mathbb{C}^2$.

PROPOSITION 3.1. *Let D be a domain in an almost complex manifold (M, J). Let $p \in \partial D$ and assume that there is a local peak plurisubharmonic function at p. Then p is a J-hyperbolic point for D.*

Consequently, if each point of the boundary of a hyperbolic domain D has a local peak plurisubharmonic function, then (D, J) is hyperbolically embedded in (M, J).

In [2], the author constructed a local peak J-plurisubharmonic function at each point of the boundary of a J-pseudoconvex domain D of finite D'Angelo type in a four-dimensional almost complex manifold (M, J). The existence of local peak J_{st}-plurisubharmonic functions was first proved by E. Fornæss and N. Sibony in [3]. For almost complex manifolds the existence was proved by S. Ivashkovich and J.-P. Rosay in [9] whenever the domain is strictly J-pseudoconvex.

EXAMPLE. Let $D = \{\rho < 0\}$ be a domain of finite D'Angelo type in $\mathbb{C}^2$. We suppose that ρ is a C^2 defining function of D, J-plurisubharmonic on a neighborhood of D. Then (D, J) is hyperbolically embedded in $(\mathbb{C}^2, J)$.

For the definitions of local peak and finite D'Angelo type domains in the almost complex case see [2].

4. Uniformly normal families of pseudoholomorphic discs. The concept of a normal meromorphic function was introduced by Lehto and Virtanen [19] for the purpose of generalizing the classical Picard theorem. Since that time several authors, including Funahashi [4], Zaidenberg [21] and Joseph–Kwack [11, 12], have extended the higher dimensional Picard theorem to normal mappings. The notion of a normal family can be naturally extended for almost complex manifolds as follows:

DEFINITION 4.1. Let (M, J) and (N, J') be almost complex manifolds.

(1) We say that a family $\mathcal{F} \subset \mathcal{O}_{J,J'}(M, N)$ is *uniformly normal* if $\mathcal{F} \circ \mathcal{O}_J(\Delta, M)$ is relatively compact in $\mathcal{C}(\Delta, N^+)$.
(2) We say that $f \in \mathcal{O}_{J,J'}(M, N)$ is *normal* if $\{f\}$ is uniformly normal.

Here $\mathcal{O}_{J,J'}(M,N)$ denotes the set of (J,J')-holomorphic functions from M to N.

A normal curve behaves like a curve with values in a hyperbolic manifold. The following result explains this behavior (see [7]).

Proposition 4.2. *Let S be a hyperbolic Riemann surface and let (M,J) be an almost complex manifold. Then $\mathcal{F} \subset \mathcal{O}_J(S,M)$ is uniformly normal if and only if there is a length function G on M such that $f^*(G) \leq K_S$ for each $f \in \mathcal{F}$.*

Consequently, a family $\mathcal{F} \subset \mathcal{O}_J(\Delta^*, M)$ is uniformly normal if and only if $\mathcal{F} \circ \mathcal{O}(\Delta^*, \Delta^*)$ is uniformly normal.

Theorem 4.3 below is our first main result in this section; it gives several characterizations of a uniformly normal family of pseudoholomorphic punctured discs. This generalizes a result of Joseph and Kwack (see Theorem 2.1 in [10]).

Theorem 4.3. *Let (M,J) be an almost complex manifold. The following are equivalent for $\mathcal{F} \subset \mathcal{O}_J(\Delta^*, M)$:*

(1) *$\mathcal{F}$ is uniformly normal.*
(2) *There is a length function G on M such that $f^*(G) \leq K_{\Delta^*}$ for each $f \in \mathcal{F}$.*
(3) *If $p \in M$ and (f_n), (z_n) are sequences in $\mathcal{F} \circ \mathcal{O}(\Delta^*, \Delta^*)$, Δ^* respectively such that $z_n \to 0$ and $f_n(z_n) \to p$, then for each U open about p, there is an $r \in]0,1[$ satisfying $f_n(\Delta_r^*) \subset U$ eventually.*

Lemma 4.4. *Let (M,J) be an almost complex manifold and $\mathcal{F} \subset \mathcal{O}_J(\Delta^*, M)$ be a uniformly normal family of pseudoholomorphic curves. Let (z_n) and (f_n) be sequences in Δ^* and $\mathcal{F}$ respectively such that $z_n \to 0$ and $f_n(z_n) \to p$. If $\sigma_n = \{z \in \Delta : |z| = |z_n|\}$, then $f_n(\sigma_n) \to p$.*

Proof. We denote by $\ell(\gamma)$ the length of a piecewise smooth curve γ in M with respect to G. By Proposition 4.2, there exists a length function G on M such that $f^*(G) \leq K_{\Delta^*}$, for each $f \in \mathcal{F}$. It follows that $|f_n'(z)|_G \leq K_{\Delta^*}(z)$ for all $z \in \Delta^*$. Therefore we have $\ell(f_n(\sigma_n)) \leq 2\pi/\big|\log|z_k|\big| \to 0$. This means that $f_n(\sigma_n) \to p$.

Proof of Theorem 4.3. The equivalence between (1) and (2) was proved in Proposition 4.2.

(2)⇒(3). The proof is an easy adaptation of the proof of (1)⇒(2) in Theorem 2.4. We only have to use Lemma 4.4 instead of Lemma 2.5.

(3)⇒(2). Let Q be a compact subset of M and let G be a length function on M. We prove the following claim: there is a positive constant c such that $f^*(G) \leq cK_{\Delta^*}$ on $f^{-1}(Q)$ for every $f \in \mathcal{F}$.

Otherwise, there exist $p_n \in \Delta^*$, $v_n \in T_{p_n}\Delta^*$, $f_n \in \mathcal{F}$ and $q \in Q$ such that $K_{\Delta^*}(p_n, v_n) = 1$, $f_n(p_n) \to q$ and $|f_n'(p_n, v_n)|_G \to \infty$. By the definition of K_{Δ^*}, there is a sequence of holomorphic curves $\varphi_n : \Delta \to \Delta^*$ and a sequence $r_n \in [1/2, 1]$ such that $\varphi_n(0) = p_n$ and $\varphi_n'(0) = r_n v_n$. Let $g_n = f_n \circ \varphi_n \in \mathcal{O}_J(\Delta, M)$. We have $|g_n'(0)|_G = r_n|f_n'(p_n, v_n)|_G \to \infty$. By continuity, there exists a sequence (z_n) in Δ^* such that $z_n \to 0$ and $g_n(z_n) \to q$. Let U and V be relatively compact local coordinate neighborhoods of q such that $\overline{U} \subset V$. Since $g_{n|\Delta^*} \in \mathcal{F} \circ \mathcal{O}(\Delta^*, \Delta^*)$, by hypothesis there is an $r \in]0, 1[$ such that $g_n(\Delta_r^*) \subset U$. Hence, $g_n(\Delta_r) \subset V$. Choosing V sufficiently small, we deduce that the sequence $g_n'(0)$ is bounded and we get a contradiction.

Let Q_n be an exhaustion of open and relatively compact sets in M. By the claim, there exists a sequence c_n such that $f^*(F) \le c_n K_{\Delta^*}$ on $f^{-1}(\overline{Q}_n)$ for every $f \in \mathcal{F}$. Choose a positive continuous function μ on M such that $\mu \le 1/c_n$ on $\overline{Q}_n$. The length function $G = \mu F$ satisfies $f^*(G) \le K_{\Delta^*}$ for each $f \in \mathcal{F}$ (see Lang [18, p. 34]).

By Theorems 2.4(3) and 4.3(3), we obtain the following:

Corollary 4.5. *Let (M, J) be an almost complex manifold and let $\mathcal{F}$ be a uniformly normal family in $\mathcal{O}_J(\Delta^*, M)$. If (f_n) is a sequence in $\mathcal{F}$ and (z_n) is a sequence in Δ^* such that $z_n \to 0$ and $f_n(z_n) \to p \in M$, then*

(1) *Each f_n extends to $\tilde{f}_n \in \mathcal{O}_J(\Delta, M)$.*
(2) *There is an $r \in]0, 1[$ and a subsequence $(\tilde{f}_{n_k})$ of $(\tilde{f}_n)$ such that*

$$\tilde{f}_{n_k} \to f \in \mathcal{O}_J(\Delta_r, M) \quad \text{on } \Delta_r.$$

Corollary 4.6. *Let (M, J) be an almost complex manifold and let $\mathcal{F}$ be a uniformly normal family in $\mathcal{O}_J(\Delta^*, M)$. Then each $f \in \mathcal{F}$ extends to $\tilde{f} \in \mathcal{C}(\Delta, M^+)$.*

Now, we prove a variant of Noguchi-type extension convergence theorems for uniformly normal families of punctured pseudoholomorphic discs.

Theorem 4.7. *Let (M, J) be an almost complex manifold, $\mathcal{F} \subset \mathcal{O}_J(\Delta^*, M)$ a uniformly normal family, and $\overline{\mathcal{F}}$ its closure in $\mathcal{C}(\Delta^*, M^+)$. Then:*

(1) *$\mathcal{C}[\Delta, M^+; \mathcal{F}]$ is relatively compact in $\mathcal{C}(\Delta, M^+)$.*
(2) *Each $f \in \overline{\mathcal{F}}$ extends to $\tilde{f} \in \mathcal{C}(\Delta, M^+)$.*
(3) *If f_n is a sequence in $\overline{\mathcal{F}}$ and $f_n \to f$, then $\tilde{f}_n \to \tilde{f}$.*

Proof. (1) By Theorem 4.3(2), we only have to prove that the family $\mathcal{C}[\Delta, M^+; \mathcal{F}]$ is evenly continuous at $(0, p)$ in $\Delta \times M^+$. Otherwise, there exist sequences (f_n) in $\mathcal{C}[\Delta, M^+; \mathcal{F}]$ and (z_n) in Δ^* such that $z_n \to 0$, $f_n(0) \to p$ and $f_n(z_n) \to q \in M^+$ with $p \ne q$. By continuity, there is a sequence (w_n) in Δ^* converging to 0 such that $f_n(w_n) \to p$. Using Theorem 4.3(3), we get a contradiction since either p or q is in M.

(2) Let $f \in \overline{\mathcal{F}}$. There exists a sequence (f_n) in $\mathcal{F}$ converging to f. By Corollary 4.6, each f_n extends to $\tilde{f}_n$ in $\mathcal{C}[\Delta, M^+; \mathcal{F}]$, which is relatively compact in $\mathcal{C}(\Delta, M^+)$. Consequently, there exists a subsequence (f_{n_k}) converging to $g \in \mathcal{C}(\Delta, N^+)$. Hence $g = f$ on Δ^*, so $g = \tilde{f}$.

(3) It is similar to the proof of Theorem 2.10(2).

The last result of our paper clarifies the link between the hyperbolic embeddedness and the uniform normality of some family of pseudoholomorphic discs.

THEOREM 4.8. *Let (M, J) be an almost complex submanifold of an almost complex manifold (N, J). The following are equivalent:*

(1) *(M, J) is hyperbolically embedded in (N, J).*
(2) *The canonical injection $M \hookrightarrow N$ is normal.*
(3) *$\mathcal{O}_J(\Delta^*, M)$ is a uniformly normal subfamily of $\mathcal{O}_J(\Delta^*, N)$.*
(4) *There exists a length function G on N such that each $f \in \mathcal{O}_J(\Delta^*, M)$ satisfies $f^*G \le K_{\Delta^*}$.*

Proof. (1)⇔(2). From Theorem 2.8(4).

(2)⇒(3). Obviously, we have $\mathcal{O}_J(\Delta^*, M) \circ \mathcal{O}(\Delta, \Delta^*) \subset \mathcal{O}_J(\Delta, M)$ which is relatively compact in $\mathcal{C}(\Delta, N^+)$. Hence $\mathcal{O}_J(\Delta^*, M)$ is a uniformly normal subfamily of $\mathcal{O}_J(\Delta^*, N)$.

(3)⇔(4). Follows from Theorem 4.3(2).

(4)⇒(1). Taking $\mathcal{F} = \mathcal{O}_J(\Delta^*, M)$, we clearly have $\mathcal{F} \circ \mathcal{O}(\Delta^*, \Delta^*) = \mathcal{F}$. By Theorems 4.3(3) and 2.4, each $p \in \overline{M}$ is a J-hyperbolic point.

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