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Hausdorff dimension of invariant measures related to Poisson driven stochastic differential equations

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Abstract. It is shown that the Hausdorff dimension of an invariant measure generated by a Poisson driven stochastic differential equation is greater than or equal to 1.

1. Introduction. We consider a stochastic differential equation of the form

$$d\xi(t) = a(\xi(t))dt + \int_{\Theta} \sigma(\xi(t),\theta)\,\mathcal{N}_p(dt,d\theta), \qquad t \geq 0, \tag{1.1}$$

with the initial condition

$$\xi(0) = \xi_0 \tag{1.2}$$

where $(\xi(t))_{t\geq 0}$ is a stochastic process with values in a separable Banach space X. We make the following five assumptions:

i. The coefficient $a\colon X \to X$ is Lipschitzian,

$$\|a(x) - a(y)\| \leq l_a \|x - y\| \quad \text{for } x, y \in X.$$

ii. $(\Theta, \mathcal{G}, \kappa)$ is a probability space.

iii. The perturbation coefficient $\sigma\colon X \times \Theta \to X$ is $\mathcal{B}_X \times \mathcal{G}/\mathcal{B}_X$-measurable and

$$\|\sigma(x,\cdot) - \sigma(y,\cdot)\|_{L^2(\kappa)} \leq l_\sigma \|x - y\| \quad \text{for } x, y \in X.$$

iv. There are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a sequence $(t_n)_{n\geq 0}$ of nonnegative random variables and a sequence $(\theta_n)_{n\geq 1}$ of random elements with values in the space Θ. The variables $\Delta t_n = t_n - t_{n-1}$ ($t_0 = 0$) are nonnegative, independent and equally distributed with density function $\lambda e^{-\lambda t}$ for $t \geq 0$. The elements θ_n are independent, equally distributed with distribution κ. The sequences $(t_n)_{n\geq 0}$ and

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$(\theta_n)_{n\geq 1}$ are also independent. It is well known that the mapping

$$\Omega \ni \omega \mapsto p(\omega) = (t_n(\omega), \theta_n(\omega))_{n\geq 1}$$

defines a stationary Poisson point process. Moreover, for every measurable set $Z \subset (0,\infty) \times \Theta$ the variable

$$\mathcal{N}_p(Z) = \operatorname{card}\{n\colon (t_n, \theta_n) \in Z\}$$

is Poisson distributed and

$$\mathbb{E}(\mathcal{N}_p((0,t] \times G)) = \lambda t \kappa(G) \quad \text{for } t \in (0,\infty),\ G \in \mathcal{G},$$

where $\mathbb{E}$ denotes expectation with respect to the probability $\mathbb{P}$.

v. For every $\mu \in \mathcal{M}_1$ there is an X-valued random vector ξ_μ defined on Ω, independent of p and having distribution μ.

Recently equation (1.1) was considered for example in [LT, MS, S, T]. It is well known [GS] that equations (1.1) and (1.2) define a semigroup of Markov operators $(P^t)_{t\geq 0}$ acting on the space of all Borel measures on X. J. Myjak and T. Szarek [MS] gave sufficient conditions for the existence of a unique invariant measure with respect to $(P^t)_{t\geq 0}$. They also proved that the lower capacity of this measure is greater than or equal to 1. T. Szarek [S] showed that the Hausdorff dimension of this measure is greater than or equal to $\log 2/\log 3$. In this paper we will show that the Hausdorff dimension of the invariant distribution with respect to $(P^t)_{t\geq 0}$ is greater than or equal to 1. A similar result, but with much stronger assumptions, can be obtained from Theorem 5.1.1 of [H].

2. Preliminaries. Let $(X, \|\cdot\|)$ be a separable Banach space. We denote by $B(x,r)$ the open ball with center at $x \in X$ and radius $r > 0$, and by $\mathcal{B}_X$ the family of all Borel subsets of X.

Let $\mathcal{M}$ be the family of all finite Borel measures on X. Then $\mathcal{M}_{\text{sig}}$ denotes the family of finite signed measures, and $\mathcal{M}_1$ the set of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$. The elements of $\mathcal{M}_1$ will be called *distributions*. Given $\mu \in \mathcal{M}$ we define the *support* of μ by the formula

$$\operatorname{supp}\mu = \{x \in X\colon \mu(B(x,r) > 0 \text{ for } r > 0\}.$$

Let $C(X)$ be the space of bounded continuous functions $f\colon X \to \mathbb{R}$ with the supremum norm. We will use the abbreviation

$$\langle f, \mu\rangle = \int_X f(x)\,\mu(dx).$$

For $A \subset X$ and $s, \delta > 0$ define

$$\mathcal{H}^s_\delta(A) = \inf\Big\{\sum_{i=1}^\infty (\operatorname{diam} U_i)^s\colon A \subset \bigcup_{i=1}^\infty U_i,\ \operatorname{diam} U_i \leq \delta\Big\}$$

and

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A).$$

The value

$$\dim_{\mathrm{H}} A = \inf\{s > 0\colon \mathcal{H}^s(A) = 0\}$$

is called the *Hausdorff dimension* of the set A. The *Hausdorff dimension* of a measure $\mu \in \mathcal{M}_1$ is defined by the formula

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A\colon A \in \mathcal{B}_X,\ \mu(A) = 1\}.$$

For a given $\mu \in \mathcal{M}$ we define the *lower pointwise dimension* of μ at $x \in X$ by

$$\underline{\mathrm{d}}_x\mu = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

(here $\log 0 = -\infty$).

LEMMA 2.1. *Let μ be a distribution. If $A \subseteq X$ is such that $\mu(A) = 1$ then*

$$\forall_{x\in A}\ \underline{\mathrm{d}}_x\mu \ge \delta \ \Rightarrow\ \dim_{\mathrm{H}} A \ge \delta.$$

The proof can be found in [Y] (it was formulated in the case when $X = \mathbb{R}^n$ but it remains valid for any separable Banach space).

By a solution of (1.1), (1.2) we mean a process $(\xi(t))_{t\ge 0}$ with values in X such that with probability one the following two conditions are satisfied:

- Each sample path is a right continuous function such that for every $t > 0$ the limit $\xi(t-) = \lim_{s\nearrow t} \xi(s)$ exists,
- $\xi(t) = \xi_0 + \int_0^t a(\xi(s))\,ds + \int_0^t \int_\Theta \sigma(\xi(s-),\theta)\,\mathcal{N}_p(ds,d\theta)$ for $t \ge 0$.

It is easy to give an explicit formula for the solution of (1.1), (1.2). Consider the ordinary differential equation

$$y'(t) = a(y(t)) \quad \text{for } t \ge 0 \tag{2.1}$$

and denote by $y(t) = S^t(x)$, $t \in \mathbb{R}$, the solution of (2.1) satisfying the initial condition $y(0) = x$. Then for every fixed $p = (t_i,\theta_i)_{i\in\mathbb{N}}$ the solution is given by the formula

$$\begin{aligned} \xi_x(t_n) &= \xi_x(t_n-) + \sigma(\xi_x(t_n-),\theta_n) && \text{for } n \in \mathbb{N}\ (\xi_x(0) = x), \\ \xi_x(t) &= S^{t-t_n}(\xi_x(t_n)), && \text{for } n \in \mathbb{N}_0,\ t_n \le t < t_{n+1}. \end{aligned} \tag{2.2}$$

Define

$$U^t f(x) = \int_\Omega f(\xi_x(t)(\omega))\,\mathbb{P}(d\omega) \quad \text{for } t \ge 0,\ f \in C(X). \tag{2.3}$$

The classical theory of equation (1.1) ensures that $(U^t)_{t\ge 0}$ is a continuous semigroup of bounded linear operators on $C(X)$. Analogously, for given $\mu \in \mathcal{M}_1$ we may find a solution $\xi_\mu(t)$, $t \ge 0$ of (1.1), (1.2) such that $\xi_\mu(0)$

has distribution μ. For every $t \geq 0$ we define $P^t\mu$ to be the distribution of $\xi_\mu(t)$.

The operators P^t and U^t satisfy the duality condition

$$\langle f, P^t\mu\rangle = \langle U^t f, \mu\rangle \quad \text{for } f \in C(X),\ \mu \in \mathcal{M}_1. \tag{2.4}$$

The operator U^t, $t \geq 0$, defined by (2.3) may be extended to all nonnegative Borel functions. Then condition (2.4) is also satisfied. The operators P^t are defined independently of the choice of $\xi_\mu(0)$ and form a semigroup acting on $\mathcal{M}_1$. Moreover using (2.4) the semigroup $(P^t)_{t\geq 0}$ can be extended to $\mathcal{M}_{\mathrm{sig}}$.

Lemma 2.2. *For $\mu \in \mathcal{M}_1$, $A \in B(X)$ and $t \geq 0$,*

$$P^t\mu(A) \geq e^{-\lambda t} \int_X \mathbb{1}_A(S^t(x))\,\mu(dx).$$

For the proof see [MS, Lemma 5.1].

A measure $\mu \in \mathcal{M}$ is called *invariant* with respect to $(P^t)_{t\geq 0}$ if $P^t\mu = \mu$ for $t \geq 0$.

3. Main theorem. Suppose that there exists a measure $\mu_* \in \mathcal{M}_1$ invariant with respect to $(P^t)_{t\geq 0}$. We define the sequence $(D_n)_{n\in\mathbb{N}}$ of sets by the formula

$$D_n := \{S^{1/n}(x) : x \in \operatorname{supp}\mu_*\} \quad \text{for } n \in \mathbb{N}.$$

Lemma 3.1.

$$\dim_{\mathrm{H}}\mu_* = \inf\Big\{\dim_{\mathrm{H}} A : A \in \mathcal{B}_X,\ A \subseteq \bigcup_{n\in\mathbb{N}} D_n,\ \mu_*(A) = 1\Big\}.$$

Proof. By Lemma 2.2 we have

$$\mu_*(D_n) = P^{1/n}\mu_*(D_n) \geq e^{-\lambda/n} \int_X \mathbb{1}_{D_n}(S^{1/n}(x))\,\mu_*(dx) = e^{-\lambda/n}.$$

Consequently, $\mu_*(\bigcup_{n\in\mathbb{N}} D_n) = 1$ and

$$\begin{aligned}\dim_{\mathrm{H}}\mu_* &= \inf\{\dim_{\mathrm{H}} A : A \in \mathcal{B}_X,\ A \subseteq X,\ \mu_*(A) = 1\}\\ &= \inf\Big\{\dim_{\mathrm{H}} A : A \in \mathcal{B}_X,\ A \subseteq \bigcup_{n\in\mathbb{N}} D_n,\ \mu_*(A) = 1\Big\}. \blacksquare\end{aligned}$$

Theorem 3.2. *If $a(x) \neq 0$ for every $x \in X$ and there exists $\beta > 0$ such that*

$$e^{-\beta t}\|x-y\| \leq \|S^t(x) - S^t(y)\| \leq e^{\beta t}\|x-y\| \quad \text{for } x, y \in X,\ t \geq 0, \tag{3.1}$$

then $\dim_{\mathrm{H}}\mu_* \geq 1$.

Proof. Let $x \in \bigcup_{n\in\mathbb{N}} D_n$. We will prove that

$$\forall_{\gamma\in(0,1)}\exists_{K>0} \quad \mu_*(B(x,r)) \leq Kr^{1-\gamma} \quad \text{for } r \in (0,\infty).$$

Pick $\gamma \in (0,1)$. Let $\eta \in (0,1/2)$ be such that

$$3^{1-\gamma} \le (1+\eta)^{-1}(3-2\eta)(1-\eta)^2. \tag{3.2}$$

There exists $n \in \mathbb{N}$ such that $x \in D_n$. From the definition of S^t it follows that

$$\lim_{t\to 0} \frac{\|x - S^t(x)\|}{t} = a(x). \tag{3.3}$$

For abbreviation set $a = a(x)$. Let $r_0 \in (0, \min\{4, a/(4n)\})$ be such that

$$e^{4r_0\beta/a} \le 1+\eta, \qquad e^{-4r_0\lambda/a} \ge 1-\eta, \qquad e^{-4r_0\beta/a} \ge 1-\eta \tag{3.4}$$

and

$$\forall_{w \le 8r_0/a} \quad \frac{3}{4}a \le \frac{\|x - S^w(x)\|}{w} \le \frac{5}{4}a. \tag{3.5}$$

Set

$$K = \max\left\{\frac{4\lambda}{\eta(1-\eta)^2 a}, \frac{4}{r_0}\right\}.$$

For every $r \ge r_0/4$ we have

$$\mu_*(B(x,r)) \le 1 \le Kr.$$

We define

$$r_* := \inf\{r' > 0\colon \mu_*(B(x,r)) \le K r^{1-\gamma} \text{ for } r \ge r'\}.$$

Of course $r_* \le r_0/4$. We will show that $r_* = 0$. Suppose, contrary to our claim, that $r_* > 0$. Let $\hat{r} \in (r_*/3, r_*)$ be such that

$$\mu_*(B(x,\hat{r})) > K\hat{r}^{1-\gamma}. \tag{3.6}$$

Set $r := \hat{r}(1-\eta)^{-2}$. We have

$$r \le \frac{r_*}{(1-\eta)^2} \le \frac{r_0}{4} \cdot 4 = r_0.$$

From (3.5) and continuity of the semigroup S^t it follows that there exists $b \in [3a/4, 5a/4]$ such that

$$b = \frac{\|x - S^{2r/b}(x)\|}{2r/b}.$$

Set $t := 2r/b$. We have $\|x - S^t(x)\| = 2r$ and

$$t \le \frac{2r_0}{b} \le \frac{2a}{4n} \cdot \frac{4}{3a} < \frac{1}{n}.$$

From (3.4) it follows that

$$\hat{r} = r(1-\eta)^2 \le re^{-8r_0\beta/a} \le re^{-6r\beta/b} = re^{-2\beta t}. \tag{3.7}$$

Choose $x_0 \in S^{-t}(x)$ and define $x_1 = S^t(x)$. From (3.1) it follows that

$$S^t(y) \in B(x, re^{-2\beta t}) \;\Rightarrow\; y \in B(x_0, re^{-\beta t}) \quad \text{for } y \in X.$$

Using Lemma 2.2 and inequality (3.7) we obtain

$$1-\mu_*(B(x,\hat{r}))\geq e^{-\lambda t}-e^{-\lambda t}\int\limits_X \mathbb{1}_{B(x,re^{-2\beta t})}S^t(y)\,\mu_*(dy)$$
$$\geq e^{-\lambda t}-e^{-\lambda t}\mu_*(B(x_0,re^{-\beta t})).$$

Since $1-e^{-\lambda t}\leq \lambda t$ we have

$$\mu_*(B(x,\hat{r}))\leq \mu_*(B(x_0,re^{-\beta t}))+\lambda t. \tag{3.8}$$

We will show that

$$\mu_*(B(x_0,re^{-\beta t}))\geq (1-\eta)\mu_*(B(x,\hat{r})). \tag{3.9}$$

Indeed, suppose towards a contradiction that

$$\mu_*(B(x_0,re^{-\beta t}))<(1-\eta)\mu_*(B(x,\hat{r})).$$

Then by (3.8) we have

$$\mu_*(B(x,\hat{r}))\leq (1-\eta)\mu_*(B(x,\hat{r}))+\lambda t$$

and consequently

$$\mu_*(B(x,\hat{r}))\leq \frac{\lambda t}{\eta}\leq K\cdot\frac{(1-\eta)^2at}{4}\leq K\cdot(1-\eta)^2r=K\hat{r}<K\hat{r}^{1-\gamma},$$

which contradicts (3.6).

By (3.1) we have

$$e^{-\beta t}\|x_0-x\|\leq \|x-x_1\|\leq e^{\beta t}\|x_0-x\|.$$

Moreover, since $2t\leq 8r_0/a$, from (3.5) it follows that

$$\|x_0-x_1\|\geq e^{-\beta t}\|x-S^{2t}(x)\|\geq \frac{3}{2}ate^{-\beta t}=3\frac{a}{b}re^{-\beta t}\geq 2re^{-\beta t}.$$

Therefore the sets $B(x_0,re^{-\beta t})$, $B(x,re^{-\beta t})$ and $B(x_1,re^{-\beta t})$ are mutually disjoint and all contained in $B(x,3re^{\beta t})$. Thus

$$\mu_*(B(x,3re^{\beta t}))\geq \mu_*(B(x_0,re^{-\beta t}))+\mu_*(B(x,re^{-\beta t}))$$
$$+\mu_*(B(x_1,re^{-\beta t})). \tag{3.10}$$

From (3.1) it follows that

$$y\in B(x,re^{-2\beta t})\;\Rightarrow\; S^t(y)\in B(x_1,re^{-\beta t}).$$

From this and Lemma 2.2 we have

$$\mu_*(B(x_1,re^{-\beta t}))\geq e^{-\lambda t}\int\limits_X \mathbb{1}_{B(x_1,re^{-\beta t})}S^t(y)\,\mu_*(dy)$$
$$\geq e^{-\lambda t}\mu_*(B(x,re^{-2\beta t})).$$

By (3.4) we have

$$1-\eta\leq e^{-4r_0\lambda/a}\leq e^{-\lambda t}. \tag{3.11}$$

Using (3.7) we obtain

$$\mu_*(B(x_1, re^{-\beta t})) \geq (1-\eta)\mu_*(B(x,\hat{r})).$$

By (3.9) and (3.10) we obtain

$$\mu_*(B(x, 3re^{\beta t})) \geq (3-2\eta)\mu_*(B(x,\hat{r})).$$

Consequently, using (3.4) and (3.2),

$$\begin{aligned}\mu_*(B(x,\hat{r})) &\leq \frac{\mu_*(B(x,3re^{\beta t}))}{3-2\eta} \leq \frac{K\cdot 3^{1-\gamma}(1+\eta)^{1-\gamma}r^{1-\gamma}}{3-2\eta}\\ &\leq K\cdot(1-\eta)^2 r^{1-\gamma} \leq K\hat{r}^{1-\gamma},\end{aligned}$$

which contradicts (3.6).

We showed that

$$\forall_{x\in\bigcup_{n\in\mathbb{N}} D_n}\forall_{\gamma\in(0,1)}\exists_{K>0}\forall_{r\in(0,\infty)} \quad \mu_*(B(x,r)) \leq Kr^{1-\gamma}.$$

Thus for every $x \in \bigcup_{n\in\mathbb{N}} D_n$ and $\gamma\in(0,1)$ we have

$$\underline{\mathrm{d}}_x\mu_* = \liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \geq \liminf_{r\to 0}\frac{\log Kr^{1-\gamma}}{\log r} = 1-\gamma.$$

Hence by Lemma 2.1 we obtain

$$\forall_{A\subseteq\bigcup_{n\in\mathbb{N}} D_n,\, \mu(A)=1} \quad \dim_{\mathrm{H}} A \geq 1.$$

Consequently, $\dim_{\mathrm{H}}\mu_* \geq 1$. ∎

Corollary 3.3. *Let $X=\mathbb{R}$. If $a(x)\neq 0$ for every $x\in X$ and there exists $\beta>0$ such that condition* (3.1) *holds, then* $\dim_{\mathrm{H}}\mu_* = 1$.

Proof. From Theorem 3.2 it follows that $\dim_{\mathrm{H}}\mu_* \geq 1$. On the other hand it is well known that in the case when $X=\mathbb{R}$, $\dim_{\mathrm{H}}\mu_* \leq 1$. ∎

Lemma 3.4. *Assume that there exists $\beta>0$ such that*

$$\|S^t(x)-S^t(y)\| \leq e^{\beta t}\|x-y\| \quad \textit{for } x,y\in X,\ t\geq 0$$

and

$$\|q(x,\cdot)-q(y,\cdot)\|_{L^1(\kappa)} \leq l\|x-y\| \quad \textit{for } x,y\in X, \tag{3.12}$$

where $q(x,\theta)=x+\sigma(x,\theta)$ and $l<\exp(-\beta/\lambda)$. Then the semigroup $(P^t)_{t\geq 0}$ given by (2.4) *is asymptotically stable.*

For the proof see [S, Theorem 3.4].

Corollary 3.5. *Assume that $a(x)\neq 0$ for every $x\in X$ and there exists $\beta>0$ such that* (3.1) *is satisfied. If* (3.12) *holds then* $\dim_{\mathrm{H}}\mu_*\geq 1$, *where $\mu_*\in\mathcal{M}_1$ is invariant with respect to $(P^t)_{t\geq 0}$.*

Proof. By Lemma 3.4 the semigroup $(P^t)_{t\geq 0}$ has an invariant distribution μ_*. From Theorem 3.2 it follows that $\dim_{\mathrm{H}}\mu_*\geq 1$. ∎

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