ANNALES
POLONICI MATHEMATICI
101.1 (2011)

# Sharp norm estimate of Schwarzian derivative for a class of convex functions

by Stanisława Kanas (Rzeszów and Lublin) and
Toshiyuki Sugawa (Sendai)

**Abstract.** We establish a sharp norm estimate of the Schwarzian derivative for a function in the classes of convex functions introduced by Ma and Minda [Proceedings of the Conference on Complex Analysis, Int. Press, 1992, 157–169]. As applications, we give sharp norm estimates for strongly convex functions of order $\alpha$, $0 < \alpha < 1$, and for uniformly convex functions.

**1. Background and main result.** Let $\mathscr{A}$ be the class of analytic functions $f$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$, and let $\mathscr{S}$ be the class of univalent functions in $\mathscr{A}$. The Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

and its norm (the hyperbolic sup-norm)

$$\|S_f\| = \sup_{z\in\mathbb{D}} (1 - |z|^2)^2 |S_f(z)|$$

play an important role in the theory of Teichmüller spaces. Key results concerning the Schwarzian derivative are summarized in the following theorem.

Theorem 1.1 (Nehari [N1], Kühnau [K], Ahlfors–Weill [AW]). *Let $f \in \mathscr{A}$. If $f$ is univalent, then $\|S_f\| \le 6$. Conversely, if $\|S_f\| \le 2$, then $f$ is univalent. Moreover, let $0 \le k < 1$. If $f$ extends to a $k$-quasiconformal mapping of the Riemann sphere $\widehat{\mathbb{C}}$ then $\|S_f\| \le 6k$. Conversely, if $\|S_f\| \le 2k$, then $f$ extends to a $k$-quasiconformal mapping of $\widehat{\mathbb{C}}$.*

Here, a mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called *$k$-quasiconformal* if $f$ is a sense-preserving homeomorphism of $\widehat{\mathbb{C}}$ and

2010 *Mathematics Subject Classification*: Primary 30C45; Secondary 30F60.
*Key words and phrases*: strongly convex function, uniformly convex function, quasiconformal extension.

DOI: 10.4064/ap101-1-8  

has locally integrable partial derivatives on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$ with $|f_{\bar{z}}| \le k|f_z|$ a.e. The best reference to the above theorem is Lehto's book [L2].

The universal Teichmüller space $\mathscr{T}$ can be identified with the set of Schwar- zian derivatives of univalent analytic functions on $\mathbb{D}$ with quasiconformal extensions to $\widehat{\mathbb{C}}$. It is known that $\mathscr{T}$ is a bounded domain in the Banach space of analytic functions on $\mathbb{D}$ with finite hyperbolic sup norm (see [L2]).

In connection with Teichmüller spaces, it is an interesting problem to estimate the norm of the Schwarzian derivatives for typical subclasses of univalent functions. A function $f \in \mathscr{A}$ is called *starlike* (resp. *convex*) if $f$ is univalent and the image $f(\mathbb{D})$ is starlike with respect to the origin (resp. convex). The classes of starlike and convex functions are denoted by $\mathscr{S}^*$ and $\mathscr{K}$, respectively. It is well known that $f \in \mathscr{A}$ is starlike (resp. convex) if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$ (resp. $\mathrm{Re}[1 + zf''(z)/f'(z)] > 0$). These notions have been refined and generalized in many ways (see [G1]).

In the present note, we are mainly concerned with strongly starlike and convex functions. A function $f \in \mathscr{A}$ is called *strongly starlike* (resp. *strongly convex*) of order $\alpha$ $(0 < \alpha < 1)$ if $|\arg[zf'(z)/f(z)]| < \pi\alpha/2$ (resp. $|\arg[1 + zf''(z)/f'(z)]| < \pi\alpha/2$) in $|z| < 1$. The classes of strongly starlike and convex functions of order $\alpha$ will be denoted by $\mathscr{S}^*_\alpha$ and $\mathscr{K}_\alpha$, respectively. See [Su2] for geometric characterizations of functions in $\mathscr{S}^*_\alpha$.

Define $\gamma(\beta)$ for $0 < \beta < 1$ by

$$\gamma(\beta) = \frac{2}{\pi}\arctan\left[\tan\frac{\pi\beta}{2} + \frac{\beta}{(1+\beta)^{(1+\beta)/2}(1-\beta)^{(1-\beta)/2}\cos(\pi\beta/2)}\right].$$

Note that $\gamma(\beta)$ increases from 0 to 1 when $\beta$ varies from 0 to 1. Mocanu [Mo] found the following relation.

Theorem 1.2 (Mocanu). *$\mathscr{K}_{\gamma(\beta)} \subset \mathscr{S}^*_\beta$ for $0 < \beta < 1$.*

In other words, $\mathscr{K}_\alpha \subset \mathscr{S}^*_{\gamma^{-1}(\alpha)}$ for $0 < \alpha < 1$, where $\gamma^{-1}$ denotes the inverse function of $\gamma : [0, 1] \to [0, 1]$. For sharp or improved relations of this kind, see the paper [KS2] of the present authors.

We summarize important properties of strongly starlike functions as follows.

Theorem 1.3. *A strongly starlike function $f$ of order $\alpha \in (0, 1)$ extends to a $\sin(\pi\alpha/2)$-quasiconformal mapping of $\widehat{\mathbb{C}}$ and therefore $\|S_f\| \le 6\sin(\pi\alpha/2)$.*

The first part is due to Fait, Krzyż and Zygmunt [FKZ] and the second one is obtained from the first in combination with Theorem 1.1 (as was pointed out by Chiang [Ch]).

By Theorems 1.2 and 1.3, we see that a function $f \in \mathscr{K}_\alpha$ extends to a $\sin(\pi\gamma^{-1}(\alpha)/2)$-quasiconformal mapping of $\widehat{\mathbb{C}}$ and satisfies $\|S_f\| \leq 6\sin(\pi\gamma^{-1}(\alpha)/2)$. On the other hand, we have the following norm estimate for convex functions.

THEOREM 1.4. *A convex function $f$ satisfies $\|S_f\| \leq 2$. The bound is sharp.*

This result was repeatedly proved in the literature (see [Rob], [N2], [L1]), and was refined by Suita [Sui] in the following form: the *sharp* inequality $\|S_f\| \leq 8\alpha(1-\alpha)$ holds for a function $f \in \mathscr{A}$ with $\mathrm{Re}[1 + zf''(z)/f'(z)] > \alpha$ and $1/2 \leq \alpha < 1$.

Obviously, the estimate $\|S_f\| \leq 6\sin(\pi\gamma^{-1}(\alpha)/2)$ for $f \in \mathscr{K}_\alpha$ is not better than Theorem 1.4 when $\alpha$ is close to 1. We will give a sharp norm estimate for $f \in \mathscr{K}_\alpha$.

MAIN THEOREM 1.5. *Let $f$ be a strongly convex function of order $\alpha$ for $0 < \alpha < 1$. Then the sharp inequality $\|S_f\| \leq 2\alpha$ holds.*

Define a function $f_\alpha \in \mathscr{K}_\alpha$ by the relation

$$1 + \frac{zf_\alpha''(z)}{f_\alpha'(z)} = \left(\frac{1+z^2}{1-z^2}\right)^\alpha.$$

Then a simple computation gives

$$f_\alpha(z) = z + \alpha z^3/3 + \alpha^2 z^5/5 + \alpha(1+8\alpha^2)z^7/63 + \cdots$$

and thus $S_{f_\alpha}(0) = 2\alpha$. Therefore, we see that $\|S_{f_\alpha}\| = 2\alpha$.

Combining Theorem 1.5 with the Ahlfors–Weill theorem (Theorem 1.1), we obtain the following result.

COROLLARY 1.6. *A function $f \in \mathscr{K}_\alpha$ extends to an $\alpha$-quasiconformal mapping of $\widehat{\mathbb{C}}$ for $0 < \alpha < 1$.*

By using Mathematica Ver. 7, we found that $\sin(\pi\gamma^{-1}(\alpha)/2) < \alpha$ when $0 < \alpha < 0.3354$ (see Figure 1). Therefore, the corollary gives a better bound only when $\alpha > 0.3355$, though it has the obvious merit of simplicity.

For some reason, the second author [Su1] was even led to expect that each function in $\mathscr{S}_\alpha^*$ might extend to an $\alpha$-quasiconformal mapping of $\widehat{\mathbb{C}}$. This was recently disproved by Yuliang Shen [S] for every $0 < \alpha < 1$.

Goodman [G2] introduced the class $\mathcal{UCV}$ of uniformly convex functions. Here, a function $f \in \mathscr{A}$ is called *uniformly convex* if every (positively oriented) circular arc of the form $\{z \in \mathbb{D} : |z - \zeta| = r\}$, $\zeta \in \mathbb{D}$, $0 < r < |\zeta| + 1$, is mapped by $f$ univalently onto a convex arc. In particular, $\mathcal{UCV} \subset \mathscr{K}$. See also [KW], [KS1] and [K] for $k$-uniform convexity $(0 \leq k < \infty)$, a more refined notion of convexity, and related results. We have the following sharp norm estimate for $\mathcal{UCV}$.

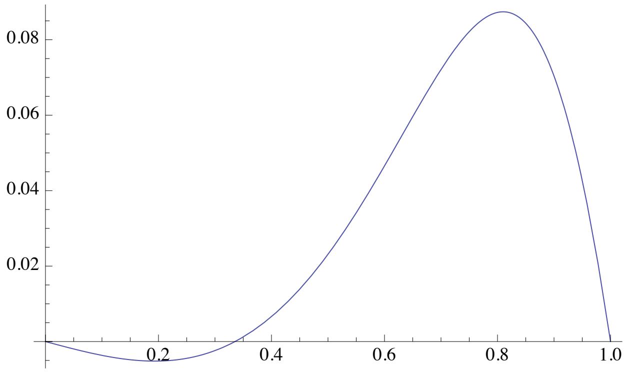


Fig. 1. Graph of $\sin(\pi\gamma^{-1}(\alpha)/2) - \alpha$

Main Theorem 1.7. *Let $f$ be a uniformly convex function. Then the sharp inequality $\|S_f\| \le 8/\pi^2$ holds. In particular, $f$ extends to a $4/\pi^2$-quasiconformal mapping of $\widehat{\mathbb{C}}$.*

In [K] the first author observed that a uniformly convex function extends to a $k_1$-quasiconformal mapping of $\widehat{\mathbb{C}}$, where $k_1 = \sin(\pi\gamma^{-1}(1/2)/2) \approx 0.52311$. Therefore, the above bound $4/\pi^2 \approx 0.40528$ is slightly better. (Note also that the numerical computation of $K(1) = (1+k_1)/(1-k_1) \approx 3.19387$ in [K] was incorrect.)

In Section 2, we provide a principle leading to a sharp norm estimate of the Schwarzian derivative for a subclass of $\mathscr{K}$ given in a specific way. By making use of it, we prove Theorem 1.5 in Section 3 and Theorem 1.7 in Section 4.

**2. General norm estimate for convex functions.** Ma and Minda [MM1] introduced a unifying way of treatment of various subclasses of $\mathscr{K}$. Let $\varphi$ be an analytic function on $\mathbb{D}$ with $\varphi(0) = 1$. The class $\mathscr{K}(\varphi)$ is defined to be the set of functions $f \in \mathscr{A}$ with $1 + zf''(z)/f'(z) \prec \varphi(z)$. Here, an analytic function $g$ on $\mathbb{D}$ is said to be *subordinate* to another $h$ and denoted by $g \prec h$ or $g(z) \prec h(z)$ if $g = h \circ \omega$ for an analytic function $\omega$ on $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega| < 1$. When $h$ is univalent, it is useful to note that $g \prec h$ if and only if $g(0) = h(0)$ and $g(\mathbb{D}) \subset h(\mathbb{D})$.

Let

$$P_\alpha(z) = \left(\frac{1+z}{1-z}\right)^\alpha$$

for a constant $\alpha > 0$. Then $P_\alpha$ maps $\mathbb{D}$ univalently onto the sector $|\arg w| < \pi\alpha/2$ for $0 < \alpha \le 1$. Thus, $\mathscr{K}(P_\alpha) = \mathscr{K}_\alpha$ for $0 < \alpha < 1$ and $\mathscr{K}(P_1) = \mathscr{K}$.

Ma and Minda [MM2] and Rønning [Ron] found the following characterization of the class $\mathcal{UCV}$. A function $f \in \mathscr{A}$ is uniformly convex if and only if $\operatorname{Re}[1 + zf''(z)/f'(z)] > |zf''(z)/f'(z)|,\ z \in \mathbb{D}$. Noting that the function

$$P(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \tag{2.1}$$

maps $\mathbb{D}$ univalently onto the domain $\{w : \operatorname{Re} w > |w-1|\}$, we have $\mathcal{UCV} = \mathscr{K}(P)$ (see [Ron, p. 191]).

We first give a sharp norm estimate for the class $\mathscr{K}(\varphi)$. To this end, we consider the quantity

$$F(s,t) = \frac{(1-t^2)^2}{2t^2}A(s) + (1-t^2)\left(1 - \frac{s^2}{t^2}\right)B(s),$$

where

$$A(s) = \sup_{|z|=s} |2z\varphi'(z) + 1 - \varphi(z)^2| \quad \text{and} \quad B(s) = \sup_{|z|=s} |\varphi'(z)|.$$

Define

$$N(\varphi) = \sup_{0<s<t<1} F(s,t).$$

Then we have the following.

Main Theorem 2.1. *Let $\varphi$ be an analytic function on the unit disk with $\varphi(0) = 1$. Then the sharp inequality $\|S_f\| \le N(\varphi)$ holds for $f \in \mathscr{K}(\varphi)$.*

Note that $\varphi$ is not required to satisfy $\operatorname{Re}\varphi > 0$ here, though there is no guarantee that $N(\varphi)$ is finite in this general case.

*Proof.* Denote by $\mathscr{W}$ the set of analytic functions $\omega$ on $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega| < 1$.

Let $f \in \mathscr{K}(\varphi)$. Since $1 + zf''(z)/f'(z) \prec \varphi(z)$ by definition, we have $f''(z)/f'(z) = (\varphi(\omega(z)) - 1)/z$ for an $\omega \in \mathscr{W}$. Set $w = \omega(z)$ for a fixed $z \in \mathbb{D}$. Then the Schwarzian derivative $S_f$ can be expressed by

$$S_f(z) = \frac{\varphi'(w)\omega'(z)}{z} - \frac{\varphi(w)^2 - 1}{2z^2}.$$

We now recall Dieudonné's lemma (cf. [D]): for a fixed pair of points $z, w \in \mathbb{D}$ with $|w| \le |z|$, one has

$$\{\omega'(z) : \omega \in \mathscr{W},\ \omega(z) = w\} = \left\{v \in \mathbb{C} : \left|v - \frac{w}{z}\right| \le \frac{|z|^2 - |w|^2}{|z|(1-|z|^2)}\right\}.$$

This means that $\zeta = \omega'(z) - w/z$ varies over the closed disk

$$|\zeta| \le (t^2 - |w|^2)/t(1-t^2)$$

for fixed $|z| = t < 1$. Then we can write

$$S_f(z) = \frac{\varphi'(w)}{z}\left(\zeta + \frac{w}{z}\right) - \frac{\varphi(w)^2 - 1}{2z^2}$$
$$= \frac{2w\varphi'(w) + 1 - \varphi(w)^2}{2z^2} + \zeta \cdot \frac{\varphi'(w)}{z}.$$

Therefore, for $|z| = t < 1$, we have the sharp inequality

$$|S_f(z)| \le \frac{|2w\varphi'(w) + 1 - \varphi(w)^2|}{2t^2} + \frac{t^2 - |w|^2}{t^2(1-t^2)}|\varphi'(w)|$$

for $f \in \mathscr{K}(\varphi)$. This can be expressed by writing

$$\sup_{f\in\mathscr{K}(\varphi)} |S_f(z)| = \sup_{s\le t}\sup_{|w|=s}\left[\frac{|2w\varphi'(w) + 1 - \varphi(w)^2|}{2t^2} + \frac{t^2 - |w|^2}{t^2(1-t^2)}|\varphi'(w)|\right]$$
$$= \sup_{s\le t}\left[\frac{A(s)}{2t^2} + \frac{t^2 - s^2}{t^2(1-t^2)}B(s)\right].$$

Hence, we have

$$\sup_{f\in\mathscr{K}(\varphi)} (1-t^2)|S_f(z)| = \sup_{s<t} F(s,t)$$

for any fixed point $z \in \mathbb{D}$ with $|z| = t$, as required. ∎

As we will see below, we often have the relations

$$A(s) = 2s\varphi'(s) + 1 - \varphi(s)^2 \quad \text{and} \quad B(s) = \varphi'(s)$$

for $0 \le s < 1$. Then, by a simple computation, we obtain the expression

$$F(s,t) = \frac{(1-t^2)^2}{2t^2}(1 - \varphi(s)^2) + \frac{(1-t^2)(1-s)(s+t^2)}{t^2}\varphi'(s). \tag{2.2}$$

Observe that $F(s,t)$ is described in terms of $s$ and $t^2$ in this case.

**3. Proof of Theorem 1.5.** We begin with the following properties of the functions $P_\alpha$.

LEMMA 3.1. *The functions $P_\alpha(z)$ and $Q_\alpha(z) = 2zP'_\alpha(z) + 1 - P_\alpha(z)^2$ have non-negative Taylor coefficients about $z = 0$ for $0 < \alpha \le 1$.*

*Proof.* Since

$$\log\frac{1+z}{1-z} = 2\sum_{n=1}^{\infty}\frac{z^{2n-1}}{2n-1},$$

the function $P_\alpha(z) = \exp(\alpha \log\frac{1+z}{1-z})$ has positive Taylor coefficients. Note also that, by this expression, $P_\alpha$ satisfies the differential equation

$$\frac{P'_\alpha(z)}{P_\alpha(z)} = \frac{2\alpha}{1-z^2}, \tag{3.1}$$

and therefore

$$P''_\alpha(z) = \frac{2(\alpha+z)}{1-z^2} P'_\alpha(z). \tag{3.2}$$

We next use the expansion

$$P_\alpha(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

Since $a_1 = 2\alpha$ and $P_\alpha$ maps $\mathbb{D}$ univalently onto a convex domain for $0 < \alpha \le 1$, by a theorem of Löwner (cf. [D]) we have

$$0 \le a_n \le 2\alpha \quad (n = 1, 2, \dots). \tag{3.3}$$

By using (3.1) and (3.2), we now have the expression

$$Q'_\alpha(z) = 2zP''_\alpha(z) + 2P'_\alpha(z)(1 - P_\alpha(z)) = 2P'_\alpha(z)\left(1 - P_\alpha(z) + \frac{2z(\alpha+z)}{1-z^2}\right).$$

Since $P'_\alpha(z)$ has positive Taylor coefficients and

$$1 - P_\alpha(z) + \frac{2z(\alpha+z)}{1-z^2} = \sum_{n=1}^{\infty}(2\alpha - a_{2n-1})z^{2n-1} + \sum_{n=1}^{\infty}(2 - a_{2n})z^{2n},$$

the required assertion for $Q_\alpha$ is deduced from (3.3). ■

We are now ready to prove our main theorem.

*Proof of Theorem 1.5.* In view of Theorem 2.1, we need to show that $N(P_\alpha) = 2\alpha$.

By Lemma 3.1, we can apply (2.2) for $\varphi = P_\alpha$:

$$\begin{aligned} F(s,t) &= \frac{(1-t^2)^2}{2t^2}(1 - P_\alpha(s)^2) + \frac{(1-t^2)(1-s)(s+t^2)}{t^2} P'_\alpha(s) \\ &= \frac{(1-t^2)^2}{2t^2}(1 - P_\alpha(s)^2) + 2\alpha \frac{(1-t^2)(s+t^2)}{t^2(1+s)} P_\alpha(s). \end{aligned}$$

Here, we have used (3.1).

Since $F(0,t) = 2\alpha(1-t^2) \to 2\alpha$ as $t \to 0$, it is enough to show that $F(s,t) \le 2\alpha$ when $0 < s < t < 1$. Letting $x = 1 - t^2$, we see that

$$\begin{aligned} &F(s,t) \le 2\alpha \\ &\quad \Leftrightarrow\ (1 - P_\alpha(s)^2)x^2 + \frac{4\alpha x(1+s-x)}{1+s} P_\alpha(s) \le 4\alpha(1-x) \\ &\quad \Leftrightarrow\ \left(P_\alpha(s)^2 + \frac{4\alpha}{1+s}P_\alpha(s) - 1\right)x^2 - 4\alpha(1 + P_\alpha(s))x + 4\alpha \ge 0. \end{aligned}$$

The left-hand side in the last inequality can be regarded as a quadratic polynomial in $x$ of the form

$$Kx^2 - 4Mx + 4L = K\left(x - \frac{2M}{K}\right)^2 + \frac{4}{K}(KL - M^2)$$

with $K > 0$. We now compute $KL - M^2$ as follows:

$$\begin{aligned} h(s) &:= \alpha\left(P_\alpha(s)^2 + \frac{4\alpha}{1+s}P_\alpha(s) - 1\right) - \alpha^2(1 + P_\alpha(s))^2 \\ &= \alpha\left((1-\alpha)P_\alpha(s)^2 + \frac{2\alpha(1-s)}{1+s}P_\alpha(s) - (1+\alpha)\right). \end{aligned}$$

Since

$$h'(s) = \frac{4\alpha^2(1-\alpha)}{(1+s)^2}(P_{\alpha+1}(s) - 1)P_\alpha(s) > 0$$

for $s > 0$, the function $h(s)$ is increasing in $0 < s < 1$. Thus $h(s) > h(0) = 0$ for $0 < s < 1$. Therefore, $KL - M^2 \ge 0$, which implies $F(s,t) \le 2\alpha$ as expected. ■

Obviously, the above proof covers the case when $\alpha = 1$. Thus, we have obtained yet another proof of Theorem 1.4.

**4. Proof of Theorem 1.7.** We will need the following estimate.

LEMMA 4.1. *For every non-negative integer $n$,*

$$\sum_{\substack{k,l,m\ge 0\\ k+l+m=n}} \frac{1}{(2k+1)(2l+1)(2m+1)} \le 1.$$

*Proof.* We denote by $A_n$ the sum in question. Also, let

$$B_n = \sum_{\substack{k,l\ge 0\\ k+l=n}} \frac{1}{(2k+1)(2l+1)}.$$

By partial fraction decomposition, we observe

$$B_n = \frac{1}{2n+2}\sum_{k+l=n}\left(\frac{1}{2k+1} + \frac{1}{2m+1}\right) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2k+1}.$$

Since $1/(2k+1) \le 1/3$ for $k \ge 1$, we have

$$B_n \le \frac{1}{n+1}\left(1 + \frac{n}{3}\right) \le \frac{2}{3} \quad \text{for } n \ge 1. \tag{4.1}$$

Similarly, by partial fraction decomposition, we have

$$\frac{2n+3}{(2k+1)(2l+1)(2m+1)} = \frac{1}{(2k+1)(2l+1)} + \frac{1}{(2l+1)(2m+1)} + \frac{1}{(2m+1)(2k+1)}$$

for $k+l+m=n$. We now apply it and take into account the symmetry in $k, l, m$ and (4.1) to obtain finally

$$A_n = \frac{3}{2n+3}\sum_{j=0}^{n} B_j \le \frac{3}{2n+3}\left(1+\frac{2}{3}n\right) = 1. \ \blacksquare$$

**Note.** In the previous version of the manuscript, we had a lengthy proof for Lemma 4.1. The second author asked for an elegant proof of it in *Sugaku Seminar*, a mathematical monthly magazine published in Japan. Several readers gave nice proofs as above. For details, see an article (in Japanese) of the second author in Sugaku Seminar 50 (2011), no. 3. The authors would like to express their thanks to the readers of the magazine.

We next show a result similar to Lemma 3.1.

Lemma 4.2. *The functions $P$ given in* (2.1) *and* $Q(z) = 2zP'(z) + 1 - P(z)^2$ *have non-negative Taylor coefficients about* $z = 0$.

*Proof.* Let

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{2n+1} = \frac{1}{2\sqrt{z}} \log \frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

Then

$$P(z) = 1 + \frac{8}{\pi^2} zG(z)^2. \tag{4.2}$$

Therefore, it is immediate to see that $P(z)$ has positive Taylor coefficients about $z = 0$. Furthermore, we can easily check the formula

$$P'(z) = \frac{8G(z)}{\pi^2(1-z)}. \tag{4.3}$$

We also note that

$$G(z)^3 = \sum_{n=0}^{\infty} A_n z^n,$$

where $A_n$ is the number given in Lemma 4.1. With these facts in mind, we now compute

$$Q(z) = \frac{16}{\pi^2} zG(z)\left(\frac{1}{1-z} - G(z) - \frac{4z}{\pi^2}G(z)^3\right)$$
$$= \frac{64}{\pi^4} z^2 G(z) \sum_{n=0}^{\infty} \left(\frac{\pi^2}{4} \cdot \frac{2n+2}{2n+3} - A_n\right) z^n.$$

Since

$$\frac{\pi^2}{4} \cdot \frac{2n+2}{2n+3} \geq \frac{\pi^2}{6} > 1 \geq A_n$$

for $n = 0, 1, 2, \ldots$ by Lemma 4.1, we see that the Taylor coefficients of $Q$ about $z = 0$ are non-negative. ■

We are now in a position to show the second main result.

*Proof of Theorem 1.7.* We take the same strategy as in the proof of Theorem 1.5. In view of Theorem 2.1, we only need to show that $N(P) = 8/\pi^2$. By Lemma 4.2 and formulae (2.2), (4.2), (4.3), we now have

$$F(s,t) = \frac{(1-t^2)^2}{2t^2}(1 - P(s)^2) + \frac{(1-t^2)(1-s)(s+t^2)}{t^2} P'(s)$$
$$= \frac{8(1-t^2)}{\pi^2 t^2}\left((s+t^2)G(s) - s(1-t^2)G(s)^2 - \frac{4}{\pi^2}s^2(1-t^2)G(s)^3\right)$$
$$= \frac{8x}{\pi^2(1-x)}\left((s+1-x)G(s) - sxG(s)^2 - \frac{4}{\pi^2}s^2 xG(s)^3\right),$$

where we put $x = 1 - t^2$. Since $F(0,t) = 8(1-t^2)/\pi^2 \to 8/\pi^2$ as $t \to 0$, it suffices to show that $F(s,t) \leq 8/\pi^2$ for $0 < s < t < 1$. This is equivalent to the inequality

$$x\left((s+1-x)G(s) - sxG(s)^2 - \frac{4}{\pi^2}s^2 xG(s)^3\right) \leq 1 - x$$
$$\Leftrightarrow \left(G(s) + sG(s)^2 + \frac{4}{\pi^2}G(s)^4\right)x^2 - (1 + (1+s)G(s))x + 1 \geq 0$$

for $0 < x < 1 - s^2$. The left-hand side in the last inequality is of the form $Kx^2 - Mx + L$ with

$$4KL - M^2 = \left(\frac{4sG(s)^2}{\pi}\right)^2 - (1 - (1-s)G(s))^2.$$

It is enough to show $4KL - M^2 \geq 0$. Since $G(s) < 1/(1-s)$, we observe that $4KL - M^2 \geq 0$ if and only if

$$\frac{4sG(s)^2}{\pi} \geq 1 - (1-s)G(s),$$

which is equivalent to

$$G(s) \geq \pi \frac{\sqrt{(1-s)^2 + 16s/\pi} - 1 + s}{8s}. \tag{4.4}$$

Since it is easily checked that $\pi(\sqrt{(1-s)^2 + 16s/\pi} - 1 + s)/8s < 1$ and $G(s) > 1$ for $0 < s < 1$, the inequality (4.4) certainly holds. ∎

Define a function $f_0 \in \mathcal{UCV}$ by the relation

$$1 + \frac{zf_0''(z)}{f_0'(z)} = P(z^2) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+z}{1-z}\right)^2.$$

Then we have

$$f_0(z) = z + \frac{4}{3\pi^2}z^3 + \left(\frac{4}{15\pi^2} + \frac{8}{5\pi^4}\right)z^5 + \cdots$$

and thus $S_{f_0}(0) = 8/\pi^2$ so that $\|S_{f_0}\| = 8/\pi^2$.

**Acknowledgements.** The second author was supported in part by JSPS Core-to-Core Program 18005 and JSPS Grant-in-Aid for Exploratory Research, 19654027.

## References

[AW] L. V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. 13 (1962), 975–978.

[Ch] Y. M. Chiang, *Schwarzian derivative and second order differential equations*, Ph.D. thesis, Univ. of London, 1991.

[D] P. L. Duren, *Univalent Functions*, Springer, 1983.

[FKZ] M. Fait, J. G. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. 51 (1976), 279–285.

[G1] A. W. Goodman, *Univalent Functions*, 2 vols., Mariner, 1983.

[G2] —, *On uniformly convex functions*, Ann. Polon. Math. 56 (1991), 87–92.

[K] S. Kanas, *Norm of pre-Schwarzian derivative for the class of k-uniformly convex and k-starlike functions*, Appl. Math. Comput. 215 (2009), 2275–2282.

[KS1] S. Kanas and T. Sugawa, *Conformal representations of the interior of an ellipse*, Ann. Acad. Sci. Fenn. Math. 31 (2006), 329–348.

[KS2] —, —, *Strong starlikeness for a class of convex functions*, J. Math. Anal. Appl. 336 (2007), 1005–1017.

[KW] S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. 105 (1999), 327–336.

[K] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. 48 (1971), 77–105.

[L1] O. Lehto, *Domain constants associated with Schwarzian derivative*, Comment. Math. Helv. 52 (1977), 603–610.

[L2] —, *Univalent Functions and Teichmüller Spaces*, Springer, 1987.

[MM1] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis, Z. Li et al. (eds.), Int. Press, 1992, 157–169.

[MM2] W. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. 57 (1992), 165–175.
[Mo] P. T. Mocanu, *Alpha-convex integral operator and strongly starlike functions*, Studia Univ. Babeş-Bolyai Math. 34 (1989), 18–24.
[N1] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. 55 (1949), 545–551.
[N2] —, *A property of convex conformal maps*, J. Anal. Math. 30 (1976), 390–393.
[Rob] M. S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J. 16 (1969), 97–101.
[Ron] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118 (1993), 189–196.
[S] Y. L. Shen, *Counterexamples concerning quasiconformal extensions of strongly starlike functions*, Acta Math. Sinica Engl. Ser. 23 (2007), 1859–1868.
[Su1] T. Sugawa, *On the norm of the pre-Schwarzian derivatives of strongly starlike functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 52 (1998), 149–157.
[Su2] —, *A self-duality of strong starlikeness*, Kodai Math J. 28 (2005), 382–389.
[Sui] N. Suita, *Schwarzian derivatives of convex functions*, J. Hokkaido Univ. Edu. Sec. IIA 46 (1996), 113–117.

Stanisława Kanas
Department of Mathematics
Rzeszów University of Technology
W. Pola 2
35-959 Rzeszów, Poland
E-mail: skanas@prz.rzeszow.pl
and
Maria Curie-Skłodowska University
20-031 Lublin, Poland

Toshiyuki Sugawa
Graduate School of Information Sciences
Tohoku University
Aoba-ku, Sendai 980-8579, Japan
E-mail: sugawa@math.is.tohoku.ac.jp

*Received 24.5.2010*
*and in final form 28.1.2011* (2223)