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On a nonlocal problem for fractional integrodifferential inclusions in Banach spaces

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Abstract. This paper investigates a class of fractional functional integrodifferential inclusions with nonlocal conditions in Banach spaces. The existence of mild solutions of these inclusions is determined under mixed continuity and Carathéodory conditions by using strongly continuous operator semigroups and Bohnenblust–Karlin's fixed point theorem.

1. Introduction. Fractional differential equations have attracted much attention due to their wide applications in engineering, economics and other fields. For details, see the monographs [HI], [MR], [P], the papers [DF], [E], [IM], [JA], [L], [LV], [M], [YG] and the references therein. The nonlocal Cauchy problem was considered by Byszewski [BL], [B], [BA], and the importance of nonlocal initial conditions in different fields has been discussed in [AM], [DE], [LL], [NT], [Y], [YP] and the references therein. Recently, some classes of abstract fractional differential and integrodifferential equations with nonlocal conditions have been investigated by Mophou and N'Guérékata [MN], [MG]; Balachandran and Trujillo [BT], Chang et al. [CK] and Yan [YA] have studied the existence of mild solutions of fractional functional integrodifferential equations with nonlocal conditions in Banach spaces by using fractional calculus and Schaefer's fixed point theorem. On the other hand, realistic problems arising from economics, optimal control and so on can be modeled as differential inclusions, so differential inclusions are widely investigated by many authors (see [BH], [D], [EI], [F], [H], [HP], [S] and references therein). Since many partial fractional differential and integrodifferential equations provide a unifying structure for the study of fractional differential equations, we feel that there is a real need to discuss the nonlocal Cauchy problem for fractional differential inclusions in abstract spaces.

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In this paper, we shall study a class of fractional functional integrodifferential inclusions with a nonlocal condition in Banach spaces, of the form

$$(1.1)\qquad \frac{d^\beta x(t)}{dt^\beta} \in Ax(t) + F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\,ds\Big), \qquad t \in J,$$

$$(1.2)\qquad x(0) + g(x) = x_0,$$

where $J = [0,b]$, $0 < \beta < 1$, the state $x(\cdot)$ takes values in a Banach space X with the norm $|\cdot|$ and A generates a strongly continuous semigroup $T(t)$ in X; $F : J \times X^{n+1} \to 2^X \setminus \{\emptyset\}$ is a multivalued map, $h : \Delta \times X \to X$, $\Delta = \{(t,s) : 0 \le s \le t \le b\}$, $g : C(J,X) \to X$, $\sigma_i : J \to J$, $i = 1, \ldots, n+1$, are given functions to be specified later.

Motivated by the above mentioned works, our goal in this paper is to establish the existence of mild solutions of (1.1)–(1.2) by using Bohnenblust–Karlin's fixed point theorem. We only assume continuity and Carathéodory conditions. The rest of this paper is organized as follows. In Section 2, we introduce some notation and necessary preliminaries. In Section 3, we give our main results. In Section 4, an example is given to illustrate our results. Finally in Section 5, we apply the preceding technique to a control problem.

2. Preliminaries. In this section, we introduce some basic definitions, notation and lemmas which are used throughout this paper.

Let $C(J,X)$ denote the Banach space of continuous functions from J into X with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\}.$$

A measurable function $x : J \to X$ is *Bochner integrable* if $|x|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [YO]). $L^1(J,X)$ denotes the linear space of equivalence classes of all measurable functions $x : J \to X$, normed by

$$\|x\|_{L^1} = \int_0^b |x(t)|\,dt \qquad \text{for all } x \in L^1(J,X).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \to 2^X \setminus \{\emptyset\}$ is *convex* (*closed*) *valued* if $G(X)$ is convex (closed) for all $x \in X$; and G is *bounded on bounded sets* if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for any bounded subset B of X, that is, $\sup_{x\in B} \sup\{|y| : y \in G(x)\} < \infty$.

G is called *upper semicontinuous* (u.s.c.) on X if, for each $x_0 \in X$, $G(x_0)$ is a nonempty, closed subset of X and, for each open subset N of

X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq N$.

The multivalued operator G is called *compact* if $\overline{G(X)}$ is a compact subset of X, and *completely continuous* if $G(D)$ is relatively compact for every bounded subset D of X. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps see the books of Deimling [D] and Hu and Papageorgiou [HP].

Now, we recall the properties of fractional calculus which are used in this paper (see [HI], [MR], [P]).

DEFINITION 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number p $(> \mu)$ such that $f(t) = t^p k(t)$, where $k \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

DEFINITION 2.2. The *Riemann–Liouville fractional integral operator* of order $\mu > 0$, of a function $f \in C_\zeta$, $\zeta \geq -1$, is defined as

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s)\, ds, \qquad t > 0. \tag{2.1}$$

DEFINITION 2.3. If $f \in C_{-1}^m$ and m is a positive integer, then we can define the *fractional derivative* of $f(t)$ *in the Caputo sense* as

$$\frac{d^\mu f(t)}{dt^\mu} = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} f^m(s)\, ds \tag{2.2}$$

for $m-1 < \mu \leq m$, $m \in \mathbb{N}$.

REMARK 2.4. If $0 < \mu \leq 1$, then

$$\frac{d^\mu f(t)}{dt^\mu} = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-s)^{-\mu} f'(s)\, ds,$$

where $f'(s) = df(s)/ds$ and f is an abstract function with values in X.

DEFINITION 2.5. A continuous function $x(t)$ satisfying the integral inclusion

$$x(t) \in T(t)[x_0 - g(x)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) \times F\Big(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))\,d\tau\Big)\,ds \tag{2.3}$$

is called a *mild solution* of problem (1.1)–(1.2) on J.

LEMMA 2.6 (Bohnenblust and Karlin [BK]). *Let X be a Banach space, and D a nonempty subset of X, which is bounded, closed, and convex. Suppose $G : D \to 2^X \setminus \{\emptyset\}$ is u.s.c. with closed, convex values, and such that $G(D) \subset D$ and $\overline{G(D)}$ compact. Then G has a fixed point.*

3. Main results. In this section, we state and prove an existence theorem for problem (1.1)–(1.2). Let us list the following hypotheses:

(H1) $T(t),\ t > 0$, is a compact semigroup and there exists a constant $M \ge 1$ such that $|T(t)| \le M$.

(H2) For each $(t,s) \in \Delta$, the function $h(t,s,\cdot) : X \to X$ is continuous and for each $x \in X$ the function $h(\cdot,\cdot,x) : \Delta \to X$ is strongly measurable.

(H3) $F : J \times X^{n+1} \to BCC(X)$ is measurable in $t \in J$ for each $(x_1,\ldots,x_{n+1}) \in X^{n+1}$, u.s.c. with respect to $(x_1,\ldots,x_{n+1}) \in X^{n+1}$ for each $t \in J$, and for $x \in C(J,X)$ the set

$$S_{F,x} = \Big\{ f \in L^1(J,X) : f(t) \in F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\,ds\Big)\Big\}$$

is nonempty.

(H4) For each $r > 0$ and $x \in C(J,X)$ with $\|x\|_\infty \le r$, there exists a positive function $l_r : J \to \mathbb{R}^+$ such that

$$\sup\Big\{ |f| : f(t) \in F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\,ds\Big)\Big\} \le l_r(t)$$

for a.e. $t \in J$.

(H5) The function $s \mapsto (t-s)^{\beta-1} l_r(s)$ belongs to $L^1([0,t],\mathbb{R}^+)$ and there is a $\gamma > 0$ such that

$$\liminf_{r\to\infty} \frac{1}{r} \int_0^t (t-s)^{\beta-1} l_r(s)\,ds = \gamma < \infty.$$

(H6) $\sigma_i : J \to J,\ i = 1,\ldots,n+1$, are continuous functions.

(H7) The function $g(\cdot) : C(J,X) \to X$ is continuous and there exists a $\delta \in (0,b)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in C := C(J,X)$ with $\phi = \psi$ on $[\delta, b]$.

(H8) There is a constant $c > 0$ such that

$$0 \le \limsup_{\|\phi\|_\infty \to \infty} \frac{|g(\phi)|}{\|\phi\|_\infty} \le c, \qquad \phi \in C. \tag{3.1}$$

THEOREM 3.1. *If hypotheses* (H1)–(H8) *are satisfied, then the nonlocal Cauchy problem* (1.1)–(1.2) *has at least one mild solution on* J, *provided that*

$$M\left(c + \frac{\gamma}{\Gamma(\beta)}\right) < 1. \tag{3.2}$$

Proof. We transform the problem (1.1)–(1.2) into a fixed point problem. The multivalued map $P : C(J,X) \to 2^{C(J,X)}$ defined by

$$P(x) := \Big\{\rho \in C(J,X) : \rho(t) = T(t)[x_0 - g(x)] + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}T(t-s)f(s)\,ds : f \in S_{F,x}\Big\}$$

has a fixed point. This fixed point is then a mild solution of (1.1)–(1.2).

Let $\{\delta_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0,b)$ with $\lim_{n\to\infty} \delta_n = 0$. To prove the above, we show that the inclusion

$$\frac{d^\beta x(t)}{dt^\beta} \in Ax(t) + F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))ds\Big), \qquad t \in J, \tag{3.3}$$

$$x(0) + T(\delta_n)g(x) = x_0, \tag{3.4}$$

has at least one mild solution $x_n \in C(J,X)$.

For fixed $n \in \mathbb{N}$, let $P_n : C(J,X) \to 2^{C(J,X)}$ be defined by

$$P_n(x) := \Big\{\rho_n \in C(J,X) : \rho_n(t) = T(t)[x_0 - T(\delta_n)g(x)] + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}T(t-s)f(s)\,ds : f \in S_{F,x}\Big\}$$

for $t \in [0,b]$. It is easy to see that a fixed point of P_n is a mild solution of the nonlocal Cauchy problem (3.3)–(3.4). We now show that P_n satisfies all the conditions of Lemma 2.6. The proof will be given in several steps.

STEP 1. *$P_n(x)$ is convex for each $x \in C(J,X)$.*

In fact, if $\rho_n^1, \rho_n^2 \in P_n(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in J$ we have

$$\rho_n^i(t) = T(t)[x_0 - T(\delta_n)g(x)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f_i(s)\, ds, \quad i = 1, 2. \tag{3.5}$$

Let $0 \le \lambda \le 1$. Then for each $t \in J$ we have

$$(\lambda\rho_n^1 + (1-\lambda)\rho_n^2)(t) = T(t)[x_0 - T(\delta_n)g(x)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s)(\lambda f_1(s) + (1-\lambda) f_2(s))\, ds.$$

Since $S_{F,x}$ is convex (because F has convex values) we have

$$\lambda\rho_n^1 + (1-\lambda)\rho_n^2 \in P_n(x).$$

Step 2. *$P_n(B_q) \subset B_q$ for some positive number q, where for each constant $q > 0$,*

$$B_q := \{x \in C([0,b], X) : \|x\|_\infty \le q\}.$$

Clearly B_q is a bounded closed convex set in $C(J,X)$. Assume, to derive a contradiction, that for each positive number q, there exists a function $x^q \in B_q$ with $\|P_n(x^q)\| := \{\|\rho_n^q\|_\infty : \rho_n^q \in P_n(x^q)\} > q$ and

$$\rho_n^q(t) = T(t)[x_0 - T(\delta_n)g(x^q)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f_q(s)\, ds \tag{3.6}$$

for some $f_q \in S_{F,x^q}$. From condition (H8), we conclude that there exist positive constants ϵ and γ_1 such that, for all $\|\phi\|_\infty > \gamma_1$,

$$|g(\phi)| \le (c+\epsilon)\|\phi\|_\infty, \quad M\left(c + \epsilon + \frac{\gamma}{\Gamma(\beta)}\right) < 1. \tag{3.7}$$

Let

$$E_1 = \{\phi : \|\phi\|_\infty \le \gamma_1\}, \quad E_2 = \{\phi : \|\phi\|_\infty > \gamma_1\},$$
$$C_1 = \max\{|g(\phi)| : \phi \in E_1\}.$$

Then

$$|g(\phi)| \le C_1 + (c+\epsilon)\|\phi\|_\infty. \tag{3.8}$$

It follows from (H1), (H3), (H4), (3.6) and (3.8) that for each $t \in [0, b]$ we have

$$\begin{aligned} q &< \|P_n(x^q)\| \\ &\le |T(t)[x_0 - T(\delta_n)g(x^q)]| + \frac{1}{\Gamma(\beta)}\Big|\int_0^t (t-s)^{\beta-1}T(t-s)f_q(s)\,ds\Big| \\ &\le |T(t)x_0| + |T(t+\delta_n)g(x^q)| + \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f_q(s)|\,ds \\ &\le M[\|x_0| + C_1 + (c+\epsilon)q] + \frac{M}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}l_q(s)\,ds. \end{aligned}$$

Dividing by q and taking the lower limit as $q \to +\infty$, we have

$$M\left(c + \epsilon + \frac{\gamma}{\Gamma(\beta)}\right) \ge 1,$$

which contradicts (3.7). Hence there exists a positive number q such that $P_n(B_q) \subset B_q$.

STEP 3. *P_n sends bounded sets to equicontinuous sets in $C(J, X)$.*

Let $0 < t_1 < t_2 \le b$, and let B_q be a bounded set as in Step 2. For each $x \in B_q$ and $\rho_n \in P_n(x)$, there exists $f \in S_{F,x}$ such that for each $t \in J$,

$$\rho_n(t) = T(t)[x_0 - T(\delta_n)g(x)] + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}T(t-s)f(s)\,ds. \tag{3.9}$$

In view of (3.9) and (H1)–(H4), we have

$$\begin{aligned} &|(\rho_n(t_2) - \rho_n(t_1)| \\ &\quad\le |[T(t_2) - T(t_1)][x_0 - T(\delta_n)g(x)]| \\ &\qquad + \frac{1}{\Gamma(\beta)}\int_0^{t_1-\varepsilon} |[(t_2-s)^{\beta-1}T(t_2-s) - (t_1-s)^{\beta-1}T(t_1-s)]f(s)|\,ds \\ &\qquad + \frac{1}{\Gamma(\beta)}\int_{t_1-\varepsilon}^{t_1} |[(t_2-s)^{\beta-1}T(t_2-s) - (t_1-s)^{\beta-1}T(t_1-s)]f(s)|\,ds \\ &\qquad + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} |(t_2-s)^{\beta-1}T(t_2-s)f(s)|\,ds \\ &\quad\le |T(t_2) - T(t_1)[x_0 - T(\delta_n)g(x)]| \\ &\qquad + \frac{1}{\Gamma(\beta)}\int_0^{t_1-\varepsilon} [(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}]|T(t_2-s)|l_q(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}\int_0^{t_1-\varepsilon}(t_1-s)^{\beta-1}|T(t_1-s)|\,|T(t_2-t_1)-I|l_q(s)\,ds\\
&+\frac{1}{\Gamma(\beta)}\int_{t_1-\varepsilon}^{t_1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]|T(t_2-s)|l_q(s)\,ds\\
&+\frac{1}{\Gamma(\beta)}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\beta-1}|T(t_1-s)|\,|T(t_2-t_1)-I|l_q(s)\,ds\\
&+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}|T(t_2-s)|l_q(s)\,ds\\
\le{}& |[T(t_2)-T(t_1)][x_0-T(\delta_n)g(x)]|\\
&+\frac{M}{\Gamma(\beta)}\int_0^{t_1-\varepsilon}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]l_q(s)\,ds\\
&+\frac{M}{\Gamma(\beta)}|T(t_2-t_1)-I|\int_0^{t_1-\varepsilon}(t_1-s)^{\beta-1}l_q(s)\,ds\\
&+\frac{M}{\Gamma(\beta)}\int_{t_1-\varepsilon}^{t_1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]l_q(s)\,ds\\
&+\frac{M}{\Gamma(\beta)}|T(t_2-t_1)-I|\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\beta-1}l_q(s)\,ds\\
&+\frac{M}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}l_q(s)\,ds.
\end{aligned}
$$

The right-hand side of the above inequality tends to zero independently of $x \in B_q$ as $t_2 - t_1 \to 0$ and ε is sufficiently small, since the compactness of $T(t)$ $(t > 0)$ implies the continuity in the uniform operator topology. Thus P_n sends B_q to an equicontinuous family of functions.

STEP 4. *The set $W(t) = \{\rho_n(t) : \rho_n \in P_n(B_q)\}$ is relatively compact in X.*

We note that $W(0)$ is relatively compact in X. Let $0 < t \le s \le b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $x \in B_q$, we define

$$
\begin{aligned}
(\rho_n^\varepsilon x)(t) &= T(t)[x_0 - T(\delta_n)g(x)] + \frac{1}{\Gamma(\beta)}\int_0^{t-\varepsilon}(t-s)^{\beta-1}T(t-s)f(s)\,ds\\
&= T(t)[x_0 - T(\delta_n)g(x)] + \frac{T(\varepsilon)}{\Gamma(\beta)}\int_0^{t-\varepsilon}(t-s)^{\beta-1}T(t-s-\varepsilon)f(s)\,ds
\end{aligned}
$$

for some $f \in S_{F,x}$. Using the compactness of $T(t)$ for $t > 0$, we deduce that the set $W_\varepsilon(t) = \{\rho_n^\varepsilon(t) : \rho_n^\varepsilon \in P_n(B_q)\}$ is pre-compact in X for every ε with $0 < \varepsilon < t$. Moreover for every $x \in B_q$ we have

$$\begin{aligned}|\rho(t) - \rho_n^\varepsilon(t)| &\le \frac{1}{\Gamma(\beta)} \int_{t-\varepsilon}^{t} (t-s)^{\beta-1} |T(t-s) f(s)| \, ds \\ &\le \frac{M}{\Gamma(\beta)} \int_{t-\varepsilon}^{t} (t-s)^{\beta-1} l_q(s) \, ds.\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $W(t) = \{\rho_n(t) : \rho_n \in P_n(B_q)\}$, and $W(t)$ is a relatively compact in X.

STEP 5. *P_n has a closed graph.*

Let $x^{(m)} \to x^*$ $(m \to \infty)$, $\rho_n^{(m)} \in P_n(x^{(m)})$, $x^{(m)} \in B_q$ and $\rho_n^{(m)} \to \rho_n^*$. We shall show that $\rho_n^* \in P_n(x^*)$. Now $\rho_n^{(m)} \in P_n(x^{(m)})$ means that there exists $f^{(m)} \in S_{F,x^{(m)}}$ such that, for each $t \in J$,

$$\rho_n^{(m)}(t) = T(t)[x_0 - T(\delta_n) g(x^{(m)})] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f^{(m)}(s) \, ds.$$

We must prove that there exists $f^* \in S_{F,x^*}$ such that, for each $t \in J$,

$$\rho_n^*(t) = T(t)[x_0 - T(\delta_n) g(x^*)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f^*(s) \, ds.$$

Clearly,

$$\begin{aligned}&\|(\rho_n^{(m)}(t) - T(t)[x_0 - T(\delta_n) g(x^{(m)})]) \\ &\qquad - (\rho_n^*(t) - T(t)[x_0 - T(\delta_n) g(x^*)])\|_\infty \to 0 \qquad \text{as } m \to \infty.\end{aligned}$$

Consider the continuous linear operator $\Phi : L^1(J, X) \to C(J, X)$ defined by

$$f \mapsto (\Phi f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s) \, ds.$$

We can see that Φ is linear and continuous. Indeed,

$$\|\Phi f\|_\infty \le \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l_q(s) \, ds.$$

From (H4) and [LO] , it follows that $\Phi \circ S_F$ has closed graph. Also, from the definition of Φ,

$$\rho_n^{(m)} - T(t)[x_0 - T(\delta_n) g(x^{(m)})] \in \Phi(S_{F,x^{(m)}}).$$

Since $x^{(m)} \to x^*$, it follows that

$$\rho_n^*(t) - T(t)[x_0 - T(\delta_n)g(x^*)] = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f^*(s)\, ds$$

for some $f^* \in S_{F,x^*}$.

As a consequence of Steps 1 to 5, together with the Arzelà–Ascoli theorem, we conclude that P_n is a compact multivalued map, u.s.c. with convex closed values. In view of Lemma 2.6, P_n has a fixed point x_n in B_q, which is in turn a mild solution of (3.3)–(3.4). Thus,

$$x_n(t) = T(t)[x_0 - T(\delta_n)g(x_n)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f_n(s)\, ds \tag{3.10}$$

for all $t \in [0,b]$, and some $f_n \in S_{F,x_n}$.

Next we will show that the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact in $C(J,X)$.

Step 6. $\{x_n : n \in \mathbb{N}\}$ *is equicontinuous on* J.

For $\varepsilon > 0$ and $x_n \in B_q$, there exists a constant $\eta > 0$ such that for all $t \in (0,b]$ and $\xi \in (0,\eta)$ with $t+\xi \le b$, we have

$$\begin{aligned}
&|x_n(t+\xi) - x_n(t)| \\
&\quad \le |[T(t+\xi) - T(t)][x_0 - T(\delta_n)g(x_n)]| \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_t^{t+\xi} |(t+\xi-s)^{\beta-1} T(t+\xi-s) f_n(s)|\, ds \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_0^t |(t+\xi-s)^{\beta-1} T(t+\xi-s) - (t-s)^{\beta-1} T(t-s) f_n(s)|\, ds \\
&\quad \le |[T(t+\xi) - T(t)][x_0 - T(\delta_n)g(x_n)]| + \frac{M}{\Gamma(\beta)} \int_t^{t+\xi} (t+\xi-s)^{\beta-1} l_q(s)\, ds \\
&\qquad + \frac{1}{\Gamma(\beta)} \int_0^t |(t+\xi-s)^{\beta-1} T(t+\xi-s) - (t-s)^{\beta-1} T(t-s)| l_q(s)\, ds.
\end{aligned}$$

Using the compact semigroup property, we get

$$|[T(t+\xi) - T(t)][x_0 - T(\delta_n)g(x_n)]| < \varepsilon/3, \tag{3.11}$$

$$\int_t^{t+\xi} (t+\xi-s)^{\beta-1} l_q(s)\, ds < \frac{\Gamma(\beta)}{3M}\varepsilon, \tag{3.12}$$

$$\int_0^t |(t+\xi-s)^{\beta-1}T(t+\xi-s)-(t-s)^{\beta-1}T(t-s)|l_q(s)\,ds<\frac{\Gamma(\beta)}{3}\varepsilon. \tag{3.13}$$

Thus by (3.11)–(3.13) one has

$$|x_n(t+\xi)-x_n(t)|<\varepsilon.$$

Therefore, $\{x_n(t):n\in\mathbb{N}\}$ is equicontinuous for $t\in(0,b]$. Clearly $\{x_n(0):n\in\mathbb{N}\}$ is equicontinuous.

STEP 7. *$\{x_n(t):n\in\mathbb{N}\}$ is relatively compact in X.*

Let $t\in(0,b]$ and $\varepsilon>0$. Since x_n is a mild solution for (3.3)–(3.4) and condition (H1) holds, there exists $\xi\in(0,t)$ such that

$$\begin{aligned}|x_n(t)-T(\xi)x_n(t-\xi)| &\le \frac{1}{\Gamma(\beta)}\int_0^{t-\xi}|[(t-s)^{\beta-1}-(t-\xi-s)^{\beta-1}]T(t-s)f_n(s)|\,ds\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{t-\xi}^{t}|(t-s)^{\beta-1}T(t-s)f_n(s)|\,ds\\ &\le \frac{M}{\Gamma(\beta)}\int_0^{t-\xi}[(t-\xi-s)^{\beta-1}-(t-s)^{\beta-1}]l_q(s)\,ds\\ &\quad+\frac{M}{\Gamma(\beta)}\int_{t-\xi}^{t}(t-s)^{\beta-1}l_q(s)\,ds<\varepsilon\end{aligned}$$

for all $n\in\mathbb{N}$. Combining the above inequality with the compactness of the operator $T(\xi)$, one finds that $\{x_n(t):n\in\mathbb{N}\}$ is relatively compact in X.

Set

$$\tilde{x}_n(t):=\begin{cases}x_n(t) & \text{if } t\in[\delta_n,b],\\ x_n(\delta_n) & \text{if } t\in[0,\delta_n].\end{cases}$$

Using condition (H7), we obtain

$$g(x_n)=g(\tilde{x}_n),$$

where $\tilde{x}_n(t)=x_n(t)$ for $t\in[\delta_n,b]$. On the other hand, in Steps 6 and 7, applying the Arzelà–Ascoli theorem again one obtains the relative compactness of $\{\tilde{x}_n:n\in\mathbb{N}\}$ in $C((0,b],X)$. Therefore there exists a subsequence of $\{\tilde{x}_n:n\in\mathbb{N}\}$, denoted again by $\{\tilde{x}_n:n\in\mathbb{N}\}$, and a function $x\in C((0,b],X)$ such that

$$\tilde{x}_n\to x \quad \text{as } n\to\infty.$$

Therefore, by the continuity of $T(t)$ and g, we get

$$x_n(0)=x_0-T(\delta_n)g(x_n)=x_0-T(\delta_n)g(\tilde{x}_n)\to x_0-g(x)=x(0) \quad \text{as } n\to\infty.$$

Thus the sequence $\{x_n(0):n\in\mathbb{N}\}$ is relatively compact.

These facts imply the relative compactness of $\{x_n : n \in \mathbb{N}\}$ in $C(J,X)$. Therefore, without loss of generality, we may suppose that

$$x_n \to x_* \in C(J,X) \quad \text{as } n \to \infty.$$

Obviously, $x_* \in B_q$, and taking limits in (3.10) one has

$$x_*(t) = T(t)[x_0 - g(x_*)] + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f_*(s)\, ds \tag{3.14}$$

for $t \in J$, and some $f_* \in S_{F,x_*}$, which implies that x_* is a mild solution of problem (1.1)–(1.2), and the proof of Theorem 3.1 is complete. ∎

Remark 3.2. (H2)–(H4) are satisfied if there exist positive functions $a_1, a_2 : J \to \mathbb{R}^+$ such that

$$\Big|F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\, ds\Big)\Big|$$
$$\le a_1(t)|x| + a_2(t) \quad \text{for almost all } t \in J \text{ and all } x \in X,$$

or there exist positive functions $b_1, b_2 : J \to \mathbb{R}^+$ and $\theta_1 \in (0,1)$ such that

$$\Big|F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\, ds\Big)\Big|$$
$$\le b_1(t)|x|^{\theta_1} + b_2(t) \quad \text{for almost all } t \in J \text{ and all } x \in X.$$

(H7)–(H8) are satisfied if there exist constants b_1, b_2 such that

$$|g(\phi)| \le b_1 + b_2\|\phi\|_\infty, \qquad \phi \in C(J,X), \tag{3.15}$$

or there exist constants $c_1, c_2, \mu_1 \in [0,1)$ such that

$$|g(\phi)| \le c_1 + c_2\|\phi\|_\infty^{\mu_1}, \qquad \phi \in C(J,X). \tag{3.16}$$

Remark 3.3. As compared to the previous research on nonlocal Cauchy problems, we no longer require the Lipschitz continuity of F. Indeed, we only require that F satisfies the Carathéodory condition. Moreover, we consider the case in which g is continuous but without imposing severe compactness conditions and convexity. So these results have more practical applications.

4. Example. Consider the nonlocal fractional order partial functional integrodifferential inclusion

$$\frac{d^\beta x(t)}{dt^\beta} \in \frac{\partial^2}{\partial x^2} z(t,x) + \frac{1}{1+t^2}\Big[\sqrt{z(\sin t, x)} + \sin z(t,x) \int_0^t e^{-z(\sin s, x)}\, ds\Big], \tag{4.1}$$

$$z(t,0) = z(t,\pi) = 0, \tag{4.2}$$

$$(4.3)\qquad z(0,x)+\sum_{i=1}^{p} c_i\sqrt[3]{z(t_i,x)}=z_0(x),\qquad 0\le t\le 1,\ 0\le x\le \pi,$$

where $0<t_1<\cdots<t_p\le 1$ and c_i are constants, $z_0\in X=L^2([0,\pi])$ and $z_0(0)=z_0(\pi)=0$.

Let $X=L^2([0,\pi])$ and let the operator $A:X\to X$ be given by $Au=u''$ with

$$D(A):=\{u\in X: u''\in X,\ u(0)=u(\pi)=0\}.$$

Then

$$Au=\sum_{n=1}^{\infty} n^2(u,u_n)u_n,\qquad u\in D(A),$$

where $u_n(x)=\sqrt{2/\pi}\sin(nx)$, $n=1,2,\ldots$, is the orthogonal set of eigenfunctions of A. It is well known that A is the infinitesimal generator of a compact analytic semigroup $T(t),t>0$, in X and is given by

$$T(t)u=\sum_{n=1}^{\infty}\exp(-n^2t)(u,u_n)u_n,\qquad u\in X,$$

where $T(t)$ satisfies hypothesis (H1).

Define $F:[0,1]\times X\times X\to X$, $h:[0,1]\times[0,1]\times X\to X$ and $g: C([0,1],X)\to X$ by

$$F\Big(t,z(\sigma(t)),\int_0^t h(t,s,z(\sigma(s)))\,ds\Big)(x) = \frac{1}{1+t^2}\Big[\sqrt{z(\sin t,x)}+\sin z(t,x)\int_0^t e^{-z(\sin s,x)}\,ds\Big],$$

$$\int_0^t h(t,s,z(\sigma(s)))(x)\,ds=\frac{1}{1+t^2}\sin z(t,x)\int_0^t e^{-z(\sin s,x)}ds,$$

$$g(z)(x)=\sum_{i=1}^{p} c_i\sqrt[3]{z(t_i,x)},\qquad z\in C([0,1],X).$$

It is easy to see that with these choices, the assumptions (H2)–(H8) of Theorem 3.1 are satisfied with the constants $\gamma=1/\beta$, $c=0$. Assume that

$$\frac{M}{\Gamma(\beta+1)}<1.$$

Now the condition (3.2) in Section 3 holds and hence by Theorem 3.1, the nonlocal Cauchy problem (4.1)–(4.3) has a mild solution on $[0,1]$.

5. Application. This section is devoted to an application of the argumens of previous sections to the controllability of a fractional functional integrodifferential system with nonlocal conditions in a Banach space X.

More precisely we consider the problem

$$\frac{d^\beta x(t)}{dt^\beta} \in Ax(t) + Bu(t) + F\Big(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))\,ds\Big), \quad t \in J, \tag{5.1}$$

$$x(0) + g(x) = x_0, \tag{5.2}$$

where A, F and g are as in Section 3. Also, the control function u belongs to $L^2(J,U)$ with U a Banach space. Further, B is a bounded linear operator from U to X. Several authors have established controllability results for differential and integrodifferential systems in Banach spaces (see [BG], [G] and the references therein). In the case of a nonlocal condition, a fractional integrodifferential system has recently been studied by Balachandran and Park [BP].

Definition 5.1. A continuous function $x(\cdot) : J \to X$ is said to be a *mild solution* to problem (5.1)–(5.2) if for all $x_0 \in X$, it satisfies the integral equation

$$\begin{aligned} x(t) \in T(t)[x_0 - g(x)] &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) \\ &\times F\Big(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))\,d\tau\Big)\,ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) Bu(s)\,ds. \end{aligned} \tag{5.3}$$

Definition 5.2. The system (5.1)–(5.2) is said to be *controllable* on the interval J if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (5.1)–(5.2) satisfies $x(b) + g(x) = x_1$.

We make the following assumptions:

(B1) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \frac{1}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} T(b-s) Bu(s)\,ds$$

has an induced inverse operator W^{-1} with values in $L^2(J,U) \setminus \operatorname{Ker} W$ and there exist positive constants M_1 such that $|BW^{-1}| \le M_1$.

(B2) The constants M, M_1, b, β satisfy the inequality

$$M\Big(1 + \frac{(1+M)M_1 b^\beta}{\Gamma(\beta+1)}\Big)c + \frac{M\gamma}{\Gamma(\beta)}\Big(1 + \frac{MM_1 b^\beta}{\Gamma(\beta+1)}\Big) < 1. \tag{5.4}$$

Remark 5.3. The construction of the operator W and its inverse is studied by Quinn and Carmichael in [Q].

Theorem 5.4. *Assume that hypotheses* (H1)–(H8), (B1) *and* (B2) *are satisfied. Then system* (5.1)–(5.2) *is controllable on* J.

Proof. Using hypothesis (B1), for an arbitrary function $x(\cdot)$ define the control

$$u_x(t) = W^{-1}\bigg[x_1 - g(x) - T(b)(x_0 - g(x)) \\ - \frac{1}{\Gamma(\beta)}\int_0^b (b-s)^{\beta-1}T(b-s)f(s)\,ds\bigg](t)$$

for some $f \in S_{F,x}$. We shall show that when using the above control the operator $P : C(J,X) \to 2^{C(J,X)}$ defined by

$$\begin{aligned} P(x) := \bigg\{\rho \in C(J,X) : \rho(t) = T(t)[x_0 - g(x)] \\ + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}T(t-s)f(s)\,ds \\ + \frac{1}{\Gamma(\beta)}\int_0^t (t-\theta)^{\beta-1}T(t-\theta)BW^{-1}\bigg[x_1 - g(x) - T(b)(x_0 - g(x)) \\ - \frac{1}{\Gamma(\beta)}\int_0^b (b-s)^{\beta-1}T(b-s)f(s)\,ds\bigg](\theta)d\theta : f \in S_{F,x}\bigg\} \end{aligned}$$

has a fixed point, and then $x(\cdot)$ is a mild solution of system (5.1)–(5.2). Indeed, it is easy to verify that

$$x_1 - g(x) \in (Px)(b),$$

which means that the system is controllable. The remaining part of the proof is similar to that of Theorem 3.1: the operator P has a fixed point which is a mild solution of problem (5.1)–(5.2). Hence, system (5.1)–(5.2) is controllable on the interval J. ∎

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