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Bundle functors with the point property which admit prolongation of connections

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Abstract. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor with the point property $F(pt) = pt$, where pt is a one-point manifold. We prove that F is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-canonical construction D of general connections $D(\Gamma)$ on $Fp : FY \to FM$ from general connections Γ on fibred manifolds $p : Y \to M$.

1. Introduction. *A general connection* on a fibred manifold $p : Y \to M$ is a smooth section $\Gamma : Y \to J^1Y$ of the first jet prolongation of Y, which can also be interpreted as the lifting map (denoted by the same symbol) $\Gamma : Y \times_M TM \to TY$, or as the connection projection affinor $\Gamma : TY \to VY \subset TY$ on Y, or as the horizontal distribution $\Gamma \subset TY$ with $\Gamma \oplus VY = TY$, [3].

Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor and let $\Gamma : Y \times_M TM \to TY$ be a general connection on $p : Y \to M$. If F preserves products, then Γ induces a connection $\mathcal{F}\Gamma$ on $Fp : FY \to FM$. More precisely, there is the canonical flow identification $TFM = FTM$ and the product preserving identification $F(Y \times_M TM) = FY \times_{FM} FTM$, and the lifting map of $\mathcal{F}\Gamma$ is $F\Gamma : FY \times_{FM} TFM \to TFY$ (modulo the identifications). We recall that the connection $\mathcal{F}\Gamma$ has been constructed by I. Kolář [2] in the case of higher order velocities functor and then by J. Slovák [5] in the general case.

A simple example of a bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ with the *point property* $F(pt) = pt$ (where pt is a one-point manifold) which is not product preserving is the vector second order tangent functor $T^{(2)} = (J^2(-.\mathbb{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$. In [4], we gave a negative answer to the question (formulated by I. Kolář) about the existence of natural operators D transforming general connections Γ on fibred manifolds $p : Y \to M$ into general connections $D(\Gamma)$ on $T^{(2)}p : T^{(2)}Y \to T^{(2)}M$. In fact, in [4], we proved the following general result.

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Theorem A. *A vector bundle functor $F : \mathcal{M}f \to \mathcal{VB}$ with the point property is product preserving if and only if for any m and n there is an $\mathcal{FM}_{m,n}$-natural operator D transforming general connections Γ on (m,n)-dimensional fibred manifolds $p : Y \to M$ into general connections $D(\Gamma)$ on $Fp : FY \to FM$.*

In [4], to prove Theorem A we used essentially an $\mathcal{M}f_m$-canonical identification $F(M \times \mathbb{R}^n) = \ker(F\,\mathrm{pr}_1 : F(M \times \mathbb{R}^n) \to FM) \oplus_M FM$, defined by means of operations in the vector bundle $F(M \times \mathbb{R}^n)$. For other (not vector) bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ with the point property we do not know similar identifications. So, it seems that the proof from [4] works in the vector bundle functor situation only.

The purpose of the present note is to extend the result of Theorem A to all (not necessarily vector) bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ with the point property.

In the present note we use the terminology and notations from the book [3]. In particular, $\mathcal{M}f$ is the category of all manifolds and all maps, $\mathcal{FM}$ is the category of all fibred manifolds and fibred maps and $\mathcal{FM}_{m,n}$ is the category of all fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred local diffeomorphisms. All manifolds and maps are assumed to be of class C^∞.

2. The main result. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor with the point property, which means that $F(pt) = pt$, where pt is a one-point manifold.

Given a manifold M we have a canonical section $e_M : M \to FM$ of $FM \to M$ given by $e_M(x) \in \mathrm{im}(Fx)$, $x \in M$, where $x : pt \to M$ is the constant map. (Since F satisfies the point property, $\mathrm{im}(Fx)$ is a one-point set. Consequently, $e_M(x)$ is well-defined.) Hence $M \subset FM$ modulo the embedding $e_M : M \to FM$. Then given a fibred manifold $p : Y \to M$ we have obvious fibred manifolds $Fp : FY \to FM$ and $\tilde{F}Y := (Fp)^{-1}(M) \to Y$, the restriction of the bundle functor projection $FY \to Y$. If $f : Y \to Y_1$ is a fibred map between fibred manifolds $p : Y \to M$ and $p_1 : Y_1 \to M_1$ with the base map $\underline{f} : M \to M_1$ we have the fibred map $Ff : FY \to FY_1$ between the fibred manifolds $Fp : FY \to FM$ and $Fp_1 : FY_1 \to FM_1$ covering $F\underline{f} : FM \to FM_1$. Since $Ff(\tilde{F}Y) \subset \tilde{F}Y_1$, we have (by restriction of Ff) the fibred map $\tilde{F}f : \tilde{F}Y \to \tilde{F}Y_1$ covering $f : Y \to Y_1$. Consequently, we have the bundle functor $\tilde{F} : \mathcal{FM} \to \mathcal{FM}$.

Example 1. Any general connection $\Theta : TFY \to V(FY \to FM) \subset TFY$ on $Fp : FY \to FM$ induces (by restriction) a general connection $\tilde{\Theta} : T\tilde{F}Y \to V\tilde{F}Y \subset T\tilde{F}Y$ on $\tilde{F}Y \to M$. More precisely, given $v \in \tilde{F}_xY$, $x \in M$, we have $V_v(FY \to FM) = V_v(\tilde{F}Y \to M)$ as $V_v(FY \to FM) =$

$T_v(F_{e_M(x)}Y) = T_v\tilde{F}_xY = V_v(\tilde{F}Y \to M)$. So, given $w \in T_v\tilde{F}Y$ and $v \in \tilde{F}Y \subset FY$, we can put $\tilde{\Theta}(w) := \Theta(w) \in V_v(FY \to FM) = V_v(\tilde{F}Y \to M) \subset T_v\tilde{F}Y$.

The main result of the present note is the following theorem.

THEOREM 1. *Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor with the point property. The following conditions are equivalent.*

(i) *For any non-negative integers m and n there is an $\mathcal{FM}_{m,n}$-canonical construction (an $\mathcal{FM}_{m,n}$-natural operator) D of general connections $D(\Gamma)$ on $Fp : FY \to FM$ from general connections Γ on $\mathcal{FM}_{m,n}$-objects $p : Y \to M$.*

(ii) *For any non-negative integers m and n there is an $\mathcal{FM}_{m,n}$-canonical construction $\tilde{D}$ of general connections $\tilde{D}(\Gamma)$ on $\tilde{F}Y \to M$ from general connections Γ on $\mathcal{FM}_{m,n}$-objects $p : Y \to M$.*

(iii) *For any non-negative integers m and n the bundle functor $\tilde{F} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is of order $(0, s, 0)$ for some finite $s = s(m, n, F)$.*

(iv) *F is product preserving.*

Proof. Condition (i) implies (ii) because of Example 1. More precisely, suppose we have D. Then given a general connection Γ on $p : Y \to M$ we have the general connection $D(\Gamma)$ on $Fp : FY \to FM$. Then (by Example 1 for $\Theta = D(\Gamma)$) we have the general connection $\tilde{D}(\Gamma)$ on $\tilde{F}Y \to M$. Clearly, the construction $\tilde{D}$ is $\mathcal{FM}_{m,n}$-canonical.

Condition (ii) implies (iii) by Corollary 2 in [1] for $\tilde{F} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ inplace of $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$.

Now, we prove that (iii) implies (iv). We recall that a bundle functor $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is said to be of *order* $(0, s, 0)$ if for any $\mathcal{FM}_{m,n}$-map $f : Y \to \overline{Y}$ between $\mathcal{FM}_{m,n}$-objects $p : Y \to M$ and $\overline{Y}$ and any point $w \in GY$ over $y \in Y$, the value $Gf(w)$ depends on the s-jet $j^s_y(f|Y_x)$ at y of the restriction $f|Y_x$ of f to the fibre Y_x of Y over $x = p(y)$. Assume (iii), i.e. that $\tilde{F} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is of order $(0, s, 0)$ for some finite s. Let $\mathbb{R}^{m,n}$ be the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Then $\tilde{F}_0(\mathbb{R}^{m,n}) = F(t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n})(\tilde{F}_0(\mathbb{R}^{m,n}))$ for $t \neq 0$. Letting $t \to 0$ we obtain $\tilde{F}_0(\mathbb{R}^{m,n}) = F(\{0\} \times \mathbb{R}^n) = F\mathbb{R}^n$. Then $\dim(F_{(0,0)}(\mathbb{R}^m \times \mathbb{R}^n)) = \dim(F_0\mathbb{R}^m) + \dim(F_0\mathbb{R}^n)$, and Proposition 38.14 in [3] completes the proof of (iv).

Finally, if $F : \mathcal{M}f \to \mathcal{FM}$ is product preserving, then there is a well-known (mentioned in the Introduction) canonical construction by J. Slovák [4] of general connections $\mathcal{F}\Gamma$ on $Fp : FY \to FM$ from general connections Γ on $p : Y \to M$. Thus (iv) implies (i). ∎

EXAMPLE 2. Let us observe that the assumption "F has the point property" is essential. For example, given a general connection Γ on $Y \to M$ we have the general connection $\mathcal{T}\Gamma$ on $TY \to TM$ (because the tangent bundle functor $T : \mathcal{M}f \to \mathcal{FM}$ is product preserving), and then we have the general connection $\mathcal{T}\Gamma \times \Gamma_o$ on $TY \times \mathbb{R} \to TM \times \mathbb{R}$, where Γ_o is the trivial general connection on $\mathrm{id} : \mathbb{R} \to \mathbb{R}$. On the other hand, the bundle functor $F := T \times \mathbb{R} : \mathcal{M}f \to \mathcal{FM}$ is not product preserving.

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