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A stochastic model of symbiosis

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Abstract. We consider a system of stochastic differential equations which models the dynamics of two populations living in symbiosis. We prove the existence, uniqueness and positivity of solutions. We analyse the long-time behaviour of both trajectories and distributions of solutions. We give a biological interpretation of the model.

1. Introduction. Relations between two populations living in symbiosis can be described by the following system of differential equations [3] (Gause and Witt, 1935)

$$x' = (a_1 + b_1 y - c_1 x)x, \qquad y' = (a_2 + b_2 x - c_2 y)y. \tag{1}$$

This model does not take into account the random influence of the environment and the following stochastic model would be more realistic:

$$dX(t) = ((a_1 + b_1 Y(t) - c_1 X(t))\,dt + \rho_{11}\,dW_1(t) + \rho_{12}\,dW_2(t))X(t), \tag{2}$$

$$dY(t) = ((a_2 + b_2 X(t) - c_2 Y(t))\,dt + \rho_{21}\,dW_1(t) + \rho_{22}\,dW_2(t))Y(t), \tag{3}$$

where a_i, b_i, c_i $(i = 1, 2)$ are positive constants, and $W_1(t)$, $W_2(t)$ are two independent standard Wiener processes. The stochastic processes $X(t)$ and $Y(t)$ represent, respectively, the first and the second population. The constants a_i $(i = 1, 2)$ are ideal growth rates, b_i $(i = 1, 2)$ are symbiosis coefficients, c_i $(i = 1, 2)$ are death rates and ρ_{ij} $(i, j = 1, 2)$ are the coefficients of environmental stochastic perturbations of populations.

The classical model of symbiosis described by (1) is well known (see [1], [9]). If the coefficients of system (1) are positive and satisfy the inequality $b_1 b_2 < c_1 c_2$ then the sizes of both populations converge to a unique equilibrium. Otherwise, if $b_1 b_2 \geq c_1 c_2$ then the sizes go to infinity. In the literature we can find other models of symbiosis. V. A. Kostitzin [6] considered a more complicated model of symbiosis in which, besides free-living individuals in two associated species, we have the third population of symbiotic couples.

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In [14] A. Rescigno and W. Richardson considered a more general model of symbiosis than (1). They replaced the growth rates $a_1 + b_1Y(t) - c_1X(t)$, $a_2+b_2X(t)-c_2Y(t)$ by functions f_1, f_2 which satisfy specific conditions. We can also study stochastic versions of these models using similar methods.

The aim of this paper is to study the long-time behaviour of both trajectories and distributions of the solutions of system (2), (3). First we show the existence, uniqueness, positivity and non-extinction property of the solutions. Next we prove that the probability distributions of the process $(X(t), Y(t))$ are absolutely continuous with respect to the Lebesgue measure. Let $U(x, y, t)$ be the density of the distribution of $(X(t), Y(t))$. We give a sufficient and a necessary condition for the asymptotic stability of system (2), (3), i.e. the convergence of $U(x, y, t)$ to an invariant density $U_*(x, y)$. If this system is not asymptotically stable then we prove that $\lim_{t\to\infty} Y(t) = 0$ a.e. We also show that in this case $\lim_{t\to\infty} X(t) = 0$ a.e. or the probability distributions of the process $X(t)$ converge weakly to some probability measure.

In order to prove asymptotic stability of system (2), (3) we use the theory of integral Markov semigroups developed in [11], [12], [15] and [18]. If this system is not asymptotically stable then we use the comparison theorem for one-dimensional stochastic differential equations and the ergodic theorem to show that $\lim_{t\to\infty} Y(t) = 0$ a.e. A similar technique was applied to study the asymptotic behaviour of a stochastic prey-predator model [16], [17], a stochastic competition model [20] and a stochastic SIR model [19].

In the model described by equations (2), (3) the random noise is proportional to the number of individuals in the population. This occurs when the noise is caused, for example, by an epidemic disease or weather conditions. We can consider another model of symbiosis in which the random noise affects each individual separately. In this case we assume that the mean value of the stochastic perturbation is zero. From the central limit theorem it follows that the stochastic perturbation is gaussian and proportional to $\sqrt{X(t)}$ and $\sqrt{Y(t)}$ respectively. Then we obtain the system

$$\begin{aligned}
(4)\qquad dX(t) &= (a_1 + b_1Y(t) - c_1X(t))X(t)dt \\
&\quad + (\rho_{11}\, dW_1(t) + \rho_{12}\, dW_2(t))\sqrt{X(t)}, \\
(5)\qquad dY(t) &= ((a_2 + b_2X(t) - c_2Y(t))Y(t)dt + (\rho_{21}\, dW_1(t) \\
&\quad + \rho_{22}\, dW_2(t))\sqrt{Y(t)}.
\end{aligned}$$

The existence, uniqueness, positivity and non-extinction property of the solutions of system (4), (5) follow from general theorems which can be found in [21]. Some technical details, for instance, the construction of a Khasminskiĭ function, will be more complicated. Some models of symbiosis with other stochastic perturbations, for example $X^2(t)$ (see [8]), are also considered.

2. Mathematical results and their interpretation. In this section we formulate the main result of our paper. We will study the stochastic model described by (2), (3) assuming that $b_1 b_2 < c_1 c_2$. This is a natural assumption, because in the deterministic model (1) we observe that if $b_1 b_2 \geq c_1 c_2$ then the sizes of both populations go to infinity. In other words, in the deterministic case too much symbiosis causes an unlimited growth of populations. In the stochastic model we can observe similar effects. Moreover, we will assume that the random noise for both populations is proportional to the number of individuals and is weakly correlated, i.e. $\rho_{11}\rho_{22} - \rho_{12}\rho_{21} \neq 0$. The asymptotic behaviour of system (2), (3) depends on the constants b_1, b_2, c_1, c_2, $\rho_1 = \sqrt{\rho_{11}^2 + \rho_{12}^2}$, $\rho_2 = \sqrt{\rho_{21}^2 + \rho_{22}^2}$, $\tilde{a}_1 = a_1 - \rho_1^2/2$, $\tilde{a}_2 = a_2 - \rho_2^2/2$.

THEOREM 1. *Let $b_1 b_2 < c_1 c_2$. If $(X(t), Y(t))$ is a solution of system (2), (3), then for every $t > 0$ the distribution of $(X(t), Y(t))$ has a density $U(t, x, y)$.*

(I) *If $\tilde{a}_1 > 0$ and $\tilde{a}_2 > 0$ then there exists a unique invariant density $U_*(x, y)$ such that*

$$\lim_{t\to\infty} \iint_{\mathbb{R}_+^2} |U(x, y, t) - U_*(x, y)| \, dx \, dy = 0. \tag{6}$$

(II) *If $\tilde{a}_1 < 0$ and $\tilde{a}_2 < 0$ then*

$$\lim_{t\to\infty} X(t) = 0 \text{ a.e.} \quad \text{and} \quad \lim_{t\to\infty} Y(t) = 0 \text{ a.e.}$$

(III) *If $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ and $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 > 0$ then there exists a unique invariant density $U_*(x, y)$ such that*

$$\lim_{t\to\infty} \iint_{\mathbb{R}_+^2} |U(x, y, t) - U_*(x, y)| \, dx \, dy = 0. \tag{7}$$

(IV) *If $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ and $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 < 0$ then $\lim_{t\to\infty} Y(t) = 0$ a.e. and the distribution of the process $X(t)$ converges weakly to the measure which has the density $f_*(x) = C x^{2\tilde{a}_1/\rho_1^2 - 1} e^{-2c_1 x/\rho_1^2}$, where $C = (2c_1/\rho_1^2)^{2\tilde{a}_1/\rho_1^2} / \Gamma(2\tilde{a}_1/\rho_1^2)$.*

REMARK 1. In every case the support of the invariant density U_* is $\mathbb{R}_+^2$. By the *support* of a measurable function f we simply mean the set

$$\operatorname{supp} f = \{(x, y) \in X : f(x, y) \neq 0\}.$$

REMARK 2. In the statement of Theorem 1 we omit some cases. For example, we do not take into account the case in which $\tilde{a}_1 < 0$, $\tilde{a}_2 > 0$, $\tilde{a}_2 b_1 + \tilde{a}_1 c_2 > 0$, because it is symmetrical to (III), nor the case when $\tilde{a}_1 < 0$, $\tilde{a}_2 > 0$, $\tilde{a}_2 b_1 + \tilde{a}_1 c_2 < 0$ which is symmetrical to (IV).

REMARK 3. Theorem 1 has an interesting biological interpretation. The inequality $b_1 b_2 < c_1 c_2$ means that the influence of symbiosis is smaller than the suppression of growth connected with the population size. This condition guarantees that the population cannot go to infinity in finite time and cannot die out in finite time. From (I) it follows that if stochastic perturbations are small for both populations then the sizes of the populations stabilize. However, if the stochastic perturbation is large for one population then this population can die out. It turns out that if the stochastic perturbation is large for one population and small for the second population but symbiosis coefficients are large then both populations survive. The main difference between the deterministic and stochastic models is that a large stochastic perturbation can cause the extinction of a population.

3. Existence and uniqueness of solution. In this section we prove that system (2), (3) has a unique and positive global (i.e. with no explosion in finite time) solution for any given initial value. In the proof we use the idea developed in [10], [8], [19]. First we introduce some notation. Let

$$\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_i > 0 \text{ for all } i = 1, 2\}.$$

We will study system (2), (3) with the initial condition $(X(0), Y(0)) \in \mathbb{R}^2_+$. We have the following result.

THEOREM 2. *If $b_1 b_2 < c_1 c_2$ then for any initial condition $(X(0), Y(0)) \in \mathbb{R}^2_+$ there is a unique solution $(X(t), Y(t))$ of system (2), (3) for $t \geq 0$ and the solution remains in $\mathbb{R}^2_+$ with probability 1, that is, $(X(t), Y(t)) \in \mathbb{R}^2_+$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of equations (2), (3) are locally Lipschitz continuous, for any given initial condition $(X(0), Y(0)) \in \mathbb{R}^2_+$ there is a unique local solution $(X(t), Y(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. In order to show that this solution is global, we will show that $\tau_e = \infty$ a.e. Let $k_0 > 0$ be sufficiently large such that

$$(X(0), Y(0)) \in [1/k_0, k_0]^2.$$

For each integer $k > k_0$ we define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : (X(t), Y(t)) \notin [1/k, k]^2\} \tag{8}$$

and we set $\inf \emptyset = \infty$. Clearly (τ_k) is an increasing sequence. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s. then $\tau_e = \infty$ a.s. and consequently $(X(t), Y(t)) \in \mathbb{R}^2_+$ for all $t \geq 0$ a.s. In other words, to complete the proof we need to show that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\omega \in \Omega : \tau_\infty(\omega) \leq T\} > \varepsilon.$$

Consequently, there exists an integer $k_1 \geq k_0$ such that

$$P\{\omega \in \Omega : \tau_k(\omega) \leq T\} \geq \varepsilon \quad \text{for all } k \geq k_1.$$

Define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$V(x,y) = b_2 x_0 F(x/x_0) + b_1 y_0 F(y/y_0), \tag{9}$$

where $F(x) = x - \ln x - 1$ and

$$x_0 = \frac{a_2 b_1 + a_1 c_2}{c_1 c_2 - b_1 b_2}, \qquad y_0 = \frac{a_1 b_2 + a_2 c_1}{c_1 c_2 - b_1 b_2}.$$

If $(X(t), Y(t)) \in \mathbb{R}_+^2$, the Itô formula shows that

$$\begin{aligned} dV(X(t),Y(t)) = \big[&b_2(a_1 + b_1 Y(t) - c_1 X(t))(X(t) - x_0) \\ &+ b_1(a_2 + b_2 X(t) - c_2 Y(t))(Y(t) - y_0) \\ &+ \tfrac{1}{2}\rho_1^2 b_2 x_0 + \tfrac{1}{2}\rho_2^2 b_1 y_0\big]\,dt \\ &+ b_2(X(t) - x_0)\rho_{11}\,dW_1(t) + b_2(X(t) - x_0)\rho_{12}\,dW_2(t) \\ &+ b_1(Y(t) - y_0)\rho_{21}\,dW_1(t) + b_1(Y(t) - y_0)\rho_{22}\,dW_2(t). \end{aligned}$$

Since $b_1 b_2 < c_1 c_2$, we have

$$b_2(a_1 + b_1 Y(t) - c_1 X(t))(X(t) - x_0) + b_1(a_2 + b_2 X(t) - c_2 Y(t))(Y(t) - y_0) \le 0.$$

We therefore obtain

$$\begin{aligned} EV(X(\tau_k \wedge T), Y(\tau_k \wedge T)) &\le V(X(0),Y(0)) + \alpha E(\tau_k \wedge T) \\ &\le V(X(0),Y(0)) + \alpha T, \end{aligned} \tag{10}$$

where $\alpha = \frac{1}{2}\rho_1^2 b_2 x_0 + \frac{1}{2}\rho_2^2 b_1 y_0$. Set $\Omega_k = \{\omega \in \Omega : \tau_k(\omega) \le T\}$ for $k \ge k_1$. Then $P(\Omega_k) \ge \varepsilon$. Note that for every $\omega \in \Omega_k$ there is some component of $(X(\tau_k(\omega)), Y(\tau_k(\omega))$ which equals either k or $1/k$, and hence by (9),

$$V(X(\tau_k(\omega)), Y(\tau_k(\omega))) \ge b_2 x_0 \left(\frac{k}{x_0} - \ln\frac{k}{x_0} - 1\right)$$

or

$$V(X(\tau_k(\omega)), Y(\tau_k(\omega))) \ge b_1 y_0 \left(\frac{k}{y_0} - \ln\frac{k}{y_0} - 1\right)$$

or

$$V(X(\tau_k(\omega)), Y(\tau_k(\omega))) \ge b_2 x_0 \left(\frac{1}{kx_0} - \ln\frac{1}{kx_0} - 1\right)$$

or

$$V(X(\tau_k(\omega)), Y(\tau_k(\omega))) \ge b_1 y_0 \left(\frac{1}{ky_0} - \ln\frac{1}{ky_0} - 1\right).$$

Let $C(k)$ be the minimum of the four right hand sides above. Then we have $\lim_{k\to\infty} C(k) = \infty$. From (10),

$$V(X(0),Y(0)) + \alpha T \ge E(1_{\Omega_k} V(X(\tau_k(\omega)), Y(\tau_k(\omega))) \ge \varepsilon C(k),$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$ leads to the contradiction

$$\infty > V(X(0),Y(0)) + \alpha T = \infty,$$

so we must have $\tau_\infty = \infty$ a.s. ∎

4. Markov semigroups. Now we recall some definitions and theorems concerning Markov semigroups.

Let (X,Σ,m) be a σ-finite measure space. Denote by D the subset of the space $L^1=L^1(X,\Sigma,m)$ consisting of all densities, i.e.

$$D=\{f\in L^1 : f\ge 0,\ \|f\|=1\}.$$

A linear mapping $P:L^1\to L^1$ is called a *Markov operator* if $P(D)\subset D$.

The Markov operator P is called an *integral* operator if there exists a measurable function $k:X\times X\to[0,\infty)$ such that

$$Pf(x)=\int\limits_X k(x,y)f(y)\,m(dy) \tag{11}$$

for every density f. The function k is called a *kernel* of the operator P. One can check that from the condition $P(D)\subset D$ it follows that

$$\int\limits_X k(x,y)\,m(dx)=1 \tag{12}$$

for almost all $y\in X$.

A family $\{P(t)\}_{t\ge0}$ of Markov operators which satisfies these conditions:

(a) $P(0)=\mathrm{Id}$,
(b) $P(t+s)=P(t)P(s)$ for $s,\,t\ge0$,
(c) for each $f\in L^1$ the function $t\mapsto P(t)f$ is continuous with respect to the L^1 norm,

is called a *Markov semigroup*. A Markov semigroup $\{P(t)\}_{t\ge0}$ is called *integral* if for each $t>0$, the operator $P(t)$ is an integral Markov operator.

We also need two definitions concerning the asymptotic behaviour of a Markov semigroup. A density f_* is called *invariant* if $P(t)f_*=f_*$ for each $t>0$. The Markov semigroup $\{P(t)\}_{t\ge0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty}\|P(t)f-f_*\|=0 \quad \text{for } f\in D.$$

A Markov semigroup $\{P(t)\}_{t\ge0}$ is called *sweeping* with respect to a set $A\in\Sigma$ if for every $f\in D$,

$$\lim_{t\to\infty}\int\limits_A P(t)f(x)\,m(dx)=0. \tag{13}$$

THEOREM 3 ([13]). *Let $\{P(t)\}_{t\ge0}$ be an integral Markov semigroup. Assume that the semigroup $\{P(t)\}_{t\ge0}$ has only one invariant density f_*. If $f_*>0$ a.e. then the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable. In particular, if the integral Markov semigroup $\{P(t)\}_{t\ge0}$ has a positive kernel and an invariant density then $\{P(t)\}_{t\ge0}$ is asymptotically stable.*

From Theorem 3 we obtain the following result.

COROLLARY 1 ([16]). *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $k(t,x,y)$ for $t>0$, which satisfies* (12) *for all $y \in X$. Assume that for every $f \in D$,*

$$\int_0^\infty P(t)f\,dt > 0 \quad a.e. \tag{14}$$

Then the semigroup is asymptotically stable or sweeping with respect to compact sets.

A Markov semigroup $\{P(t)\}_{t\geq 0}$ that is asymptotically stable or sweeping from a sufficiently large family of sets (e.g. from all compact sets) is said to satisfy the *Foguel alternative*.

If we know that the semigroup satisfies the Foguel alternative we can exclude sweeping by showing that there exists a Khasminskiĭ function, defined as follows. Let $\mathcal{A}$ be the infinitesimal generator of a Markov semigroup $\{P(t)\}_{t\geq 0}$. Let $\mathcal{R} = (I - \mathcal{A})^{-1}$ be the resolvent operator at point 1. Let $V : X \to [0,\infty)$ be a measurable function. Set

$$D_V = \Big\{ f \in D : \int_X f(x)V(x)\,m(dx) < \infty \Big\}.$$

Then V is called a *Khasminskiĭ function* for the Markov semigroup $\{P(t)\}_{t\geq 0}$ and a set $Z \in \Sigma$ if there exist $M > 0$ and $\varepsilon > 0$ such that

$$\int_X V(x)\mathcal{R}f(x)\,dm(x) \leq \int_X (V(x) - \varepsilon) f(x)\,dm(x) + \int_Z M\mathcal{R}f(x)\,dm(x)$$

for every $f \in D_V$. It is difficult to find a Khasminskiĭ function using the definition. In our case we can apply the following theorem.

THEOREM 4 ([12]). *Let K be a compact set in $\mathbb{R}^2$. Assume that there exists a C^2-function $V : \mathbb{R}^2 \to [0,\infty)$ such that*

$$\sup_{x \notin K} \mathcal{A}^* V(x) < 0.$$

Then V is a Khasminskiĭ function for the Markov semigroup $\{P(t)\}_{t\geq 0}$ and the set K.

5. Properties of trajectories. In order to investigate the properties of the solution of system (2), (3) we substitute $X(t) = e^{\xi(t)}$, $Y(t) = e^{\eta(t)}$. Then by the Itô formula we obtain

$$d\xi(t) = (\tilde{a}_1 + b_1 e^{\eta(t)} - c_1 e^{\xi(t)})dt + \rho_{11}\,dW_1(t) + \rho_{12}\,dW_2(t), \tag{15}$$

$$d\eta(t) = (\tilde{a}_2 + b_2 e^{\xi(t)} - c_2 e^{\eta(t)})dt + \rho_{21}\,dW_1(t) + \rho_{22}\,dW_2(t). \tag{16}$$

In this section we prove parts (II) and (IV) of Theorem 1. We will use the following property of solutions of a one-dimensional stochastic equation.

Consider the stochastic equation

$$dX(t) = \sigma(X(t))\,dW(t) + b(X(t))\,dt.$$

Let

$$s(x) = \int_0^x \exp\left\{-\int_0^y \frac{2b(r)}{\sigma^2(r)}\,dr\right\} dy.$$

If $s(-\infty) > -\infty$ and $s(\infty) = \infty$ then $\lim_{t\to\infty} X(t) = -\infty$ a.e.

The proof of parts (II) and (IV) of Theorem 1 is divided into the following lemmas.

LEMMA 1. *Let* $(\xi(t), \eta(t))$ *be a solution of system* (15), (16). *Assume that* $\rho_1 \neq 0$ *and* $\rho_2 \neq 0$. *If* $b_1 b_2 < c_1 c_2$, $\tilde{a}_1 < 0$ *and* $\tilde{a}_2 < 0$ *then*

$$\lim_{t\to\infty} \xi(t) = -\infty \quad \textit{and} \quad \lim_{t\to\infty} \eta(t) = -\infty \ \textit{a.e.}$$

Proof. Let

$$\widetilde{W}(t) = \frac{\rho_{11}}{\rho_1} W_1(t) + \frac{\rho_{12}}{\rho_1} W_2(t), \qquad \overline{W}(t) = \frac{\rho_{21}}{\rho_2} W_1(t) + \frac{\rho_{22}}{\rho_2} W_2(t).$$

Then $\widetilde{W}(t)$, $\overline{W}(t)$ are two standard Wiener processes. We define

$$W(t) = \frac{c_2\rho_{11} + b_1\rho_{21}}{\rho} W_1(t) + \frac{c_2\rho_{12} + b_1\rho_{22}}{\rho} W_2(t),$$

where

$$\rho = \sqrt{(c_2\rho_{11} + b_1\rho_{21})^2 + (c_2\rho_{12} + b_1\rho_{22})^2}.$$

Then $W(t)$ is also a standard Wiener process. Multiplying (15) by c_2 and (16) by b_1 and adding the two equations we have

$$d(c_2\xi(t) + b_1\eta(t)) = (\tilde{a}_1 c_2 + \tilde{a}_2 b_1 + (b_1 b_2 - c_1 c_2)e^{\xi(t)})dt + \rho dW(t).$$

Since $b_1 b_2 < c_1 c_2$, from the comparison theorem ([4, Lemma 4, p. 120]) we get

$$d(c_2\xi(t) + b_1\eta(t)) \le (\tilde{a}_1 c_2 + \tilde{a}_2 b_1)dt + \rho dW(t).$$

Consequently,

$$\lim_{t\to\infty} (c_2\xi(t) + b_1\eta(t)) = -\infty \qquad \text{a.e.}$$

Thus for arbitrarily small $\varepsilon > 0$ there exist t_0 and a set Ω_ε such that $\mathrm{Prob}(\Omega_\varepsilon) > 1 - \varepsilon$ and $\xi(t) < -\frac{b_1}{c_2}\eta(t)$ for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. It follows that

$$d\eta(t) \le (\tilde{a}_2 + b_2 e^{-\frac{b_1}{c_2}\eta(t)})dt + \rho_2\, d\overline{W}(t). \tag{17}$$

Consider the equation

$$d\bar{\eta}(t) = (\tilde{a}_2/2 + b_2 e^{-\frac{b_1}{c_2}\bar{\eta}(t)})dt + \rho_2\, d\overline{W}(t). \tag{18}$$

The Fokker–Planck equation corresponding to (18) has a stationary density

$$f_*(x) = C \exp \frac{2}{{\rho_2}^2}\bigg(\frac{\tilde{a}_2}{2}\,x - \frac{b_2c_2}{b_1}\,e^{-\frac{b_1}{c_2}x}\bigg),$$

where C is some constant. From the ergodic theorem ([4, Theorem 2, p. 141]) it follows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-\frac{b_1}{c_2}\bar{\eta}(s)}\,ds = \int_{-\infty}^{\infty} f_*(x)e^{-\frac{b_1}{c_2}x}\,dx. \tag{19}$$

Since $f_*'(x) = 2/\rho_2^2(\tilde{a}_2/2 + b_2e^{-\frac{b_1}{c_2}x})f_*(x)$, we have

$$\int_{-\infty}^{\infty} f_*(x)e^{-\frac{b_1}{c_2}x}\,dx = -\frac{\tilde{a}_2}{2b_2} > 0. \tag{20}$$

From (19), (20) we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-\frac{b_1}{c_2}\bar{\eta}(s)}\,ds = -\frac{\tilde{a}_2}{2b_2}. \tag{21}$$

In view of (17) we have

$$\eta(t) \le \bar{\eta}(t) + \frac{\tilde{a}_2}{2}t + C_1,$$

where $\bar{\eta}(t)$ is a solution of equation (18) and C_1 is some constant. Therefore

$$\eta(t) \le \tilde{a}_2 t + b_2\int_0^t e^{-\frac{b_1}{c_2}\bar{\eta}(s)}\,ds + \rho_2\overline{W}(t) + C_1.$$

From this and (21), $\lim_{t\to\infty}\lim\overline{W}(t)/t = 0$, we obtain

$$\limsup_{t\to\infty} \frac{\eta(t)}{t} \le \frac{\tilde{a}_2}{2} < 0.$$

Consequently, $\lim_{t\to\infty}\eta(t) = -\infty$ a.e. Using the symmetry of system (15), (16) we analogously show that $\lim_{t\to\infty}\xi(t) = -\infty$ a.e. ∎

LEMMA 2. *Let* $(\xi(t),\eta(t))$ *be a solution of system* (15), (16)*. Assume that* $\rho_1 \ne 0$ *and* $\rho_2 \ne 0$*. If* $b_1b_2 < c_1c_2$, $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ *and* $\tilde{a}_1b_2 + \tilde{a}_2c_1 < 0$ *then* $\lim_{t\to\infty}\eta(t) = -\infty$ *a.e and the distribution of the process* $\xi(t)$ *converges weakly to a measure which has the density*

$$f_*(x) = C\exp\bigg(\frac{2}{\rho_1^2}\,(\tilde{a}_1x - c_1e^x)\bigg).$$

Proof. As in the proof of Lemma 1, we show that $\lim_{t\to\infty}\eta(t) = -\infty$ a.e. Thus for arbitrary $\varepsilon > 0$ there exist t_0 and a set Ω_ε such that $\mathrm{Prob}(\Omega_\varepsilon) > 1-\varepsilon$ and $b_1e^{\eta(t,\omega)} \le \varepsilon$ for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. Therefore

$$(\tilde{a}_1 - c_1e^{\xi(t)})dt + \rho_1 d\widetilde{W}(t) \le d\xi(t) \le (\tilde{a}_1 + \varepsilon - c_1e^{\xi(t)})dt + \rho_1 d\widetilde{W}(t),$$

where $\widetilde{W}(t) = \frac{\rho_{11}}{\rho_1} W_1(t) + \frac{\rho_{12}}{\rho_1} W_2(t)$. Let $\xi_\varepsilon(t)$ be a solution of the equation

$$d\xi_\varepsilon(t) = (\tilde{a}_1 + \varepsilon - c_1 e^{\xi_\varepsilon(t)})dt + \rho_1 d\widetilde{W}(t)$$

with the initial condition $\xi_\varepsilon(0) = \xi(0)$. Assume that $\xi_0(t)$ is a solution of the equation

$$d\xi_0(t) = (\tilde{a}_1 - c_1 e^{\xi_0(t)})dt + \rho_1 d\widetilde{W}(t)$$

with the initial condition $\xi_0(0) = \xi(0)$. Thus from the comparison theorem ([4, Lemma 4, p. 120]) it follows that $\xi_0(t) \le \xi(t) \le \xi_\varepsilon(t)$ a.e. Denote by $F_{\xi_0(t)}$, $F_{\xi(t)}$, $F_{\xi_\varepsilon(t)}$, respectively, the distributions of the processes $\xi_0(t)$, $\xi(t)$, $\xi_\varepsilon(t)$. Then $F_{\xi_0(t)}(x) \ge F_{\xi(t)}(x) \ge F_{\xi_\varepsilon(t)}(x)$ for $x \in \mathbb{R}$. Every Markov semigroup connected with a diffusion process of a non-degenerate type is an integral semigroup which has a positive kernel. From Theorem 3 it follows that the densities of the process $\xi_0(t)$ converge in L^1 to an invariant density f_* and the densities of the process $\xi_\varepsilon(t)$ converge in L^1 to an invariant density

$$f_\varepsilon^*(x) = C \exp\left(\frac{2}{\rho_1^2} \left((\tilde{a}_1 + \varepsilon)x - c_1 e^x \right) \right),$$

where C is some constant. In particular, $f_* = f_0^*$. Let

$$F_\varepsilon^*(x) = \int_{-\infty}^{x} f_\varepsilon^*(s)\, ds.$$

Thus $F_{\xi_\varepsilon(t)}$ uniformly converges to F_ε^* as $t \to \infty$ and F_ε^* uniformly converges to F_0^* as $\varepsilon \to 0$. Therefore the distribution of the process $\xi(t)$ converges weakly to the measure which has the density f_*. ■

6. Asymptotic stability. Let $(\xi(t), \eta(t))$ be a solution of (15), (16) such that the distribution of $(\xi(0), \eta(0))$ is absolutely continuous with density $v(x, y)$. Then the random variable $(\xi(t), \eta(t))$ has a density $u(x, y, t)$ and u satisfies the Fokker–Planck equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\rho_1^2 \frac{\partial^2 u}{\partial x^2} + (\rho_{11}\rho_{21} + \rho_{12}\rho_{22}) \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2}\rho_2^2 \frac{\partial^2 u}{\partial y^2} - \frac{\partial (f_1 u)}{\partial x} - \frac{\partial (f_2 u)}{\partial y}, \tag{22}$$

where $f_1(x, y) = \tilde{a}_1 + b_1 e^y - c_1 e^x$, $f_2(x, y) = \tilde{a}_2 + b_2 e^x - c_2 e^y$.

Now we introduce a Markov semigroup connected with the Fokker–Planck equation (22). Let $X = \mathbb{R}^2$, Σ be the σ-algebra of Borel subsets of X, and m be the Lebsegue measure on (X, Σ). Let $P(t)v(x, y) = u(x, y, t)$ for $v \in D$. Since the operator $P(t)$ is a contraction on D, it can be extended to a contraction on $L^1(\mathbb{R}^2, \Sigma, m)$. Thus the operators $\{P(t)\}_{t \ge 0}$ form a Markov semigroup. Let $\mathcal{A}$ be the infinitesimal generator of the semigroup $\{P(t)\}_{t \ge 0}$, i.e.

$$\mathcal{A}v = \frac{1}{2}\rho_1^2 \frac{\partial^2 v}{\partial x^2} + (\rho_{11}\rho_{21} + \rho_{12}\rho_{22}) \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2}\rho_2^2 \frac{\partial^2 v}{\partial y^2} - \frac{\partial (f_1 v)}{\partial x} - \frac{\partial (f_2 v)}{\partial y}.$$

The adjoint operator of $\mathcal{A}$ is of the form

$$\mathcal{A}^*v = \frac{1}{2}\,\rho_1^2\,\frac{\partial^2 v}{\partial x^2} + (\rho_{11}\rho_{21}+\rho_{12}\rho_{22})\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{2}\,\rho_2^2\,\frac{\partial^2 v}{\partial y^2} + f_1\,\frac{\partial v}{\partial x} + f_2\,\frac{\partial v}{\partial y}.$$

By $\mathcal{P}(t,x,y,A)$ we denote the transition probability function for the diffusion process $(\xi(t),\eta(t))$, i.e. $\mathcal{P}(t,x,y,A) = \mathrm{Prob}((\xi(t),\eta(t))\in A)$ and $(\xi(t),\eta(t))$ is a solution of (15), (16) with the initial condition $(\xi(0),\eta(0)) = (x,y)$. Since $\rho_{11}\rho_{22}-\rho_{12}\rho_{21}\neq 0$, for each point $(x_0,y_0)\in\mathbb{R}^2$ and $t>0$ the measure $\mathcal{P}(t,x_0,y_0,\cdot)$ is absolutely continuous with respect to the Lebesgue measure. Denote by $k(t,x,y;x_0,y_0)$ the density of $\mathcal{P}(t,x_0,y_0,\cdot)$. Thus

$$P(t)v(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t,x,y;\xi,\eta)v(\xi,\eta)\,d\xi\,d\eta \tag{23}$$

and consequently $\{P(t)\}_{t\ge 0}$ is an integral Markov semigroup. The asymptotic stability of the semigroup $\{P(t)\}_{t\ge 0}$ implies the convergence in L^1 of the densities of the process $(\xi(t),\eta(t))$ to an invariant density. Therefore, instead of proving part (I) and (III) of Theorem 1, we show the asymptotic stability of the semigroup $\{P(t)\}_{t\ge 0}$. As $\rho_{11}\rho_{22}-\rho_{12}\rho_{21}\neq 0$, the semigroup $\{P(t)\}_{t\ge 0}$ connected with (22) has a continuous and positive kernel k. By Corollary 1 this semigroup satisfies the Foguel alternative. In order to exclude sweeping we construct a Khasminskiĭ function. We have the following result.

THEOREM 5. *Assume that $b_1b_2 < c_1c_2$. If $\tilde a_1 > 0$, $\tilde a_2 > 0$ or $\tilde a_1 > 0$, $\tilde a_2 < 0$, $\tilde a_1 b_2 + \tilde a_2 c_1 > 0$ then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

Proof. According to Theorem 4 we will construct a nonnegative C^2-function V and a compact set $K\subset\mathbb{R}^2$ such that

$$\sup_{x\notin K}\mathcal{A}^*V(x) < 0.$$

Using similar arguments to those in [11] one can check that the existence of a Khasminskiĭ function implies that the semigroup is not sweeping from compact sets.

Let $\gamma_1 = \log(c_1/b_1)$, $\gamma_2 = \log(b_2/c_2)$, $\gamma = (\gamma_1+\gamma_2)/2$, $\delta = (\gamma_1-\gamma_2)/2$. As $b_1b_2 < c_1c_2$ we have $\gamma_2<\gamma_1$ and $\delta>0$. If d_1 is a constant such that $0<d_1<\delta$ then we have the inequalities

$$\gamma + d_1 < \gamma_1, \qquad \gamma - d_1 > \gamma_2. \tag{24}$$

First we consider the case when $\tilde a_1>0$, $\tilde a_2>0$. Let $a = \min(\tilde a_1,\tilde a_2)$ and

$$d_2 > \max\left\{\frac{6}{a}\,({\rho_1}^2+{\rho_2}^2), -(\gamma+\gamma_1), \gamma+\gamma_2\right\}.$$

We define

$$
\begin{aligned}
D_1 &= \{(x,y)\in\mathbb{R}^2 : x > z_0+d_1,\ y > z_0+\gamma+d_1,\\
&\qquad\qquad\qquad\qquad\qquad x+\gamma-d_1 < y < x+\gamma+d_1\},\\
D_2 &= \{(x,y)\in\mathbb{R}^2 : x > z_0,\ y = x+\gamma+d_1\},\\
D_3 &= \{(x,y)\in\mathbb{R}^2 : x > z_0+d_1,\ y = x+\gamma-d_1\},\\
D_4 &= \{(x,y)\in\mathbb{R}^2 : x > z_0+d_1,\ y < -z_0-\gamma-d_1,\\
&\qquad\qquad\qquad\qquad\qquad -x-\gamma-d_1 < y < -x-\gamma+d_1\},\\
D_5 &= \{(x,y)\in\mathbb{R}^2 : x > z_0+d_1,\ y = -x-\gamma+d_1\},\\
D_6 &= \{(x,y)\in\mathbb{R}^2 : x > z_0,\ y = -x-\gamma-d_1\},\\
D_7 &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1,\ y < -z_0-\gamma-d_1,\\
&\qquad\qquad\qquad\qquad\qquad x-\gamma-d_2 < y < x-\gamma+d_2\},\\
D_8 &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1+d_2,\ y = x-\gamma-d_2\},\\
D_9 &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1,\ y = x-\gamma+d_2\},\\
D_{10} &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1,\ y > z_0+\gamma+d_1,\\
&\qquad\qquad\qquad\qquad\qquad -x+\gamma-d_1 < y < -x+\gamma+d_1\},\\
D_{11} &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1,\ y = -x+\gamma-d_1\},\\
D_{12} &= \{(x,y)\in\mathbb{R}^2 : x < -z_0,\ y = -x+\gamma+d_1\},\\
D_{13} &= \{(x,y)\in\mathbb{R}^2 : y > z_0+\gamma+d_1,\ y > x+\gamma+d_1,\ y > -x+\gamma+d_1\},\\
D_{14} &= \{(x,y)\in\mathbb{R}^2 : x > z_0+d_1,\ -x-\gamma+d_1 < y < x+\gamma-d_1\},\\
D_{15} &= \{(x,y)\in\mathbb{R}^2 : y < -z_0-\gamma-d_1,\ y < -x-\gamma-d_1,\ y < x-\gamma-d_2\},\\
D_{16} &= \{(x,y)\in\mathbb{R}^2 : x < -z_0-d_1,\ x-\gamma+d_2 < y < -x+\gamma-d_1\},
\end{aligned}
$$

where z_0 is a positive and sufficiently large constant. Next we define the function $z = V_1(x,y)$ for $z \ge z_0$ in the following way:

$$
\begin{gathered}
(x-z)^4 + (y-(z+\gamma))^4 = d_1^4 \quad \text{for } (x,y)\in D_1\cup D_2\cup D_3,\\
(x-z)^4 + (y+(z+\gamma))^4 = d_1^4 \quad \text{for } (x,y)\in D_4\cup D_5\cup D_6,\\
(x+z+d_1-d_2)^4 + (y+z+d_1-d_2+\gamma)^4 = d_2^4 \quad \text{for } (x,y)\in D_7\cup D_8\cup D_9,\\
(x+z)^4 + (y-(z+\gamma))^4 = d_1^4 \quad \text{for } (x,y)\in D_{10}\cup D_{11}\cup D_{12},\\
z = y-\gamma-d_1 \quad \text{for } (x,y)\in D_{13}, \qquad z = x-d_1 \quad \text{for } (x,y)\in D_{14},\\
z = -y-\gamma-d_1 \quad \text{for } (x,y)\in D_{15}, \qquad z = -x-d_1 \quad \text{for } (x,y)\in D_{16}.
\end{gathered}
$$

We show that there exist $\varepsilon > 0$ and a compact set $K_1 \subset \mathbb{R}^2$ such that

$$\mathcal{A}^* V_1(x,y) \le -\varepsilon \quad \text{for } (x,y) \notin K_1.$$

Fix $0 < \varepsilon < a/2$. Let $K_1 = [-z_0-d_1, z_0+d_1] \times [-z_0-\gamma-d_1, z_0+\gamma+d_1]$.

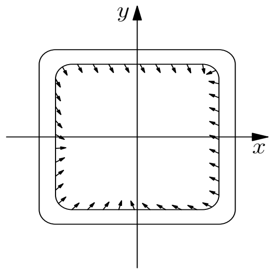

Fig. 1. The level curves of the Khasminskiĭ function if $b_1b_2 < c_1c_2$, $\tilde{a}_1 > 0$ and $\tilde{a}_2 > 0$

We calculate

$$\frac{\partial V_1}{\partial x} = \frac{(x-z)^3}{(x-z)^3 + (y-(z+\gamma))^3}, \qquad \frac{\partial V_1}{\partial y} = \frac{(y-(z+\gamma))^3}{(x-z)^3 + (y-(z+\gamma))^3}$$

and

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial y} = 1, \qquad \frac{\partial V_1}{\partial x} > 0, \qquad \frac{\partial V_1}{\partial y} > 0 \tag{25}$$

for $(x,y) \in D_1$. Moreover,

$$\frac{\partial^2 V_1}{\partial x^2} = \frac{\partial^2 V_1}{\partial y^2} = 3(x-z)^2(y-(z+\gamma))^2 \, \frac{(x-z)^4 + (y-(z+\gamma))^4}{((x-z)^3 + (y-(z+\gamma))^3)^3}$$

and

$$\frac{\partial^2 V_1}{\partial xy} = -\frac{\partial^2 V_1}{\partial x^2} < 0$$

for $(x,y) \in D_1$. Moreover,

$$0 < \frac{\partial^2 V_1}{\partial x^2} = \frac{\partial^2 V_1}{\partial y^2} \le \frac{6}{d_1}. \tag{26}$$

As $b_1 e^{\gamma+d_1} - c_1 < 0$, $b_2 - c_2 e^{\gamma-d_1} < 0$, for sufficiently large z_0 we obtain

$$\begin{aligned} \tilde{a}_1 + b_1 e^y - c_1 e^x &< \tilde{a}_1 + b_1 e^{x+\gamma+d_1} - c_1 e^x = \tilde{a}_1 + e^x(b_1 e^{\gamma+d_1} - c_1) \\ &< \tilde{a}_1 + e^{z_0+d_1}(b_1 e^{\gamma+d_1} - c_1) < -\frac{3}{d_1}\rho_1^2 - \frac{3}{d_1}\rho_2^2 - \varepsilon \end{aligned}$$

and

$$\begin{aligned} \tilde{a}_2 + b_2 e^x - c_2 e^y &< \tilde{a}_2 + b_2 e^x - c_2 e^{x+\gamma-d_1} = \tilde{a}_2 + e^x(b_2 - c_2 e^{\gamma-d_1}) \\ &< \tilde{a}_2 + e^{z_0+d_1}(b_2 - c_2 e^{\gamma-d_1}) < -\frac{3}{d_1}\,\rho_1^2 - \frac{3}{d_1}\,\rho_2^2 - \varepsilon. \end{aligned}$$

From this and from (25), (26) we get

$$\begin{aligned} \mathcal{A}^* V_1(x,y) &\le \frac{3}{d_1}\,\rho_1^2 + \frac{3}{d_1}\,\rho_2^2 - \left(\frac{3}{d_1}\,\rho_1^2 + \frac{3}{d_1}\,\rho_2^2 + \varepsilon\right)\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) \\ &= -\varepsilon \end{aligned} \tag{27}$$

for $(x,y) \in D_1$. Calculations in the other cases are similar or much easier and therefore we omit them.

Now we consider the case when $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ and $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 > 0$. Let $\mu = -2\tilde{a}_2 b_2$, $\nu = \tilde{a}_1 b_2 - \tilde{a}_2 c_1$. It is easy to see that $\mu > 0$ and $\nu > 0$. Let $\alpha = \mu/\nu$, $y_0 = -(3\alpha/2+1)z_0 - \alpha d_1$. As before we define

$$
\begin{aligned}
&E_1 = D_1, \qquad E_2 = D_2, \qquad E_3 = D_3,\\
&E_4 = \{(x,y) \in \mathbb{R}^2 : x > z_0 + d_1,\\
&\qquad\qquad y < y_0,\ -x + z_0 + y_0 < y < -x + z_0 + y_0 + 2d_1\},\\
&E_5 = \{(x,y) \in \mathbb{R}^2 : x > z_0 + d_1,\ y = -x + z_0 + y_0 + 2d_1\},\\
&E_6 = \{(x,y) \in \mathbb{R}^2 : x > z_0,\ y = -x + z_0 + y_0\},\\
&E_7 = \{(x,y) \in \mathbb{R}^2 : x < -z_0 - d_1,\ y \le -z_0\},\\
&E_8 = \{(x,y) \in \mathbb{R}^2 : -z_0 - d_1 \le x < z_0/2,\ y < y_0\},\\
&E_9 = \{(x,y) \in \mathbb{R}^2 : x \ge z_0/2,\ y < y_0,\ y < -x + z_0 + y_0\},\\
&E_{10} = D_{10}, \qquad E_{11} = D_{11}, \qquad E_{12} = D_{12}, \qquad E_{13} = D_{13},\\
&E_{14} = \{(x,y) \in \mathbb{R}^2 : x > z_0 + d_1,\ -x + z_0 + y_0 + 2d_1 < y < x + \gamma - d_1\},\\
&E_{15} = \{(x,y) \in \mathbb{R}^2 : x < -z_0 - d_1,\ -z_0 < y < -x + \gamma - d_1\}.
\end{aligned}
$$

Next we define the function $z = V_2(x,y)$ which is a modification of V_1. Let

$$
\begin{aligned}
&V_2(x,y) = V_1(x,y) \quad \text{for } (x,y) \in E_1 \cup E_2 \cup E_3 \cup E_{10} \cup E_{11} \cup E_{12} \cup E_{13},\\
&(x-z)^4 + (y + \alpha z + (\alpha/2+1)z_0 + (\alpha-1)d_1)^4\\
&\qquad\qquad = d_1^4 \quad \text{for } (x,y) \in E_4 \cup E_5 \cup E_6,\\
&\alpha x + y + \alpha z + z_0 + \alpha d_1 = 0 \quad \text{for } (x,y) \in E_7 \cup E_8,\\
&2z + 2y + 3\alpha z_0 + 2\alpha d_1 = 0 \quad \text{for } (x,y) \in E_9,\\
&z = x - d_1 \quad \text{for } (x,y) \in E_{14}, \qquad z = -x - d_1 \quad \text{for } (x,y) \in E_{15}.
\end{aligned}
$$

We show that there exist $\varepsilon > 0$ and a compact set $K_2 \subset \mathbb{R}^2$ such that

$$\mathcal{A}^* V_2(x,y) \le -\varepsilon \quad \text{for } (x,y) \notin K_2.$$

Fix $0 < \varepsilon < \min(\tilde{a}_1, \mu\tilde{a}_1 + \nu\tilde{a}_2)$. Let $K_2 = [-z_0 - d_1, z_0 + d_1] \times [y_0, z_0 + \gamma + d_1]$. Calculations are similar to the previous case. The main difference is in $E_7 \cup E_8$. If $(x,y) \in E_7 \cup E_8$ then

$$\mathcal{A}^* V_2(x,y) = (1/\mu)(-\mu\tilde{a}_1 - \nu\tilde{a}_2 + (\mu c_1 - \nu b_2)e^x + (\nu c_2 - \mu b_1)e^y). \tag{28}$$

As $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 > 0$, $b_1 b_2 < c_1 c_2$ and $\tilde{a}_2 < 0$ thus $-\mu\tilde{a}_1 - \nu\tilde{a}_2 < 0$, $\mu c_1 - \nu b_2 < 0$ and

$$
\begin{aligned}
\nu c_2 - \mu b_1 &= b_2(\tilde{a}_1 c_2 + \tilde{a}_2 b_1) + \tilde{a}_2(b_1 b_2 - c_1 c_2)\\
&> \frac{b_1 b_2}{c_1}(\tilde{a}_1 b_2 + \tilde{a}_2 c_1) + \tilde{a}_2(b_1 b_2 - c_1 c_2) > 0.
\end{aligned}
$$

Consequently, for $(x,y) \in E_7$ we have

$$\mathcal{A}^* V_2(x,y) \le (1/\mu)(-\mu\tilde{a}_1 - \nu\tilde{a}_2 + (\nu c_2 - \mu b_1)e^{-z_0}) < -\varepsilon, \tag{29}$$

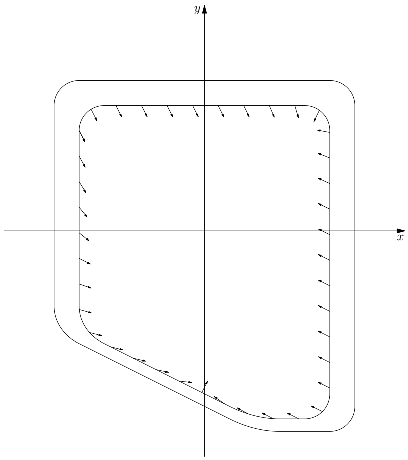

Fig. 2. The level curves of the Khasminskiĭ function if $b_1b_2 < c_1c_2$, $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$, $\tilde{a}_1b_2 + \tilde{a}_2c_1 > 0$

and for $(x, y) \in E_8$ we obtain

$$(30)\quad \mathcal{A}^*V_2(x,y) \le (1/\mu)(-\mu\tilde{a}_1 - \nu\tilde{a}_2 + (\nu c_2 - \mu b_1)e^{-(3\alpha/2+1)z_0-\alpha d_1}) < -\varepsilon,$$

because z_0 is sufficiently large. ∎

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