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The hyper-order of solutions of certain linear complex differential equations

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Abstract. We prove some theorems on the hyper-order of solutions of the equation $f^{(k)} - e^{Q} f = a(1 - e^{Q})$, where Q is an entire function, which is a polynomial or not, and a is an entire function whose order can be larger than 1. We improve some results by J. Wang and X. M. Li.

1. Introduction and main results. We assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [6, 9, 15, 17]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ $(r \to \infty,\ r \notin E)$.

Let f and g be two nonconstant meromorphic functions and let a be a complex number. We say that f and g *share* a *CM* provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, let $b \not\equiv \infty$ be a nonconstant meromorphic function such that $T(r,b) = S(r,f)$ and $T(r,b) = S(r,g)$. If $f - b$ and $g - b$ share 0 CM, we say that f and g *share* b *CM*. In this paper, we also need the following definitions.

Definition 1. For a nonconstant entire function f, the *order* $\sigma(f)$, *lower order* $\mu(f)$, *hyper-order* $\sigma_2(f)$ and *lower hyper-order* $\mu_2(f)$ are defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

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$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r},$$
$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r},$$

respectively. Here and in what follows, $M(r,f) = \max_{|z|=r} |f(z)|$.

In 1977, L. A. Rubel and C. C. Yang [12] proved that if an entire function f shares two distinct complex numbers CM with its derivative f', then $f = f'$. What is the relation between f and f' if the entire function f shares one complex number a CM with its derivative f'? In 1996, R. Brück [2] made a conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer, and if f and f' share one complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. In [2], R. Brück proved this conjecture for $a = 0$, and also for $a \neq 0$ and $N(r, 1/f') = S(r,f)$. In 1998, G. G. Gundersen and L. Z. Yang [5] proved that the conjecture is true for $a \neq 0$, provided that $\sigma(f) < \infty$. In 1999, L. Z. Yang [16] proved that if a nonconstant entire function f and one of its derivatives $f^{(k)}$ share one complex number a ($\neq 0$) CM, where $\sigma(f) < \infty$ and k is a positive integer, then $f - a = c(f^{(k)} - a)$ for some complex number $c \neq 0$. In 2004, J. P. Wang proved the following theorem.

Theorem A (see [14]). *Let f be a nonconstant entire function of finite order, let P be a polynomial of degree $p \geq 1$, and let k be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some complex number $c \neq 0$.*

Regarding Theorem A, it is natural to ask what can be said if the order of f is infinite. In [10], X. M. Li and C. C. Gao got the following result.

Theorem B (see [10]). *Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a nonconstant solution of the equation*

$$f^{(k)} - Q_1 = e^P (f - Q_2),$$

then $\sigma_2(f) = \deg P$.

Regarding Theorem B, what can be said if a nonconstant entire function f and one of its derivative $f^{(k)}$ share an entire function a which is a small function of f? In [13], J. Wang and X. M. Li proved the following theorem.

Theorem C (see [13]). *If f is a nonconstant solution of the differential equation $f^{(k)} - a_1 = (f - a_2)e^Q$, where a_1 and a_2 are two entire functions such that $\sigma(a_j) < 1$ ($j = 1, 2$), k is a positive integer, and Q is a polynomial, then $\mu_2(f) = \sigma_2(f) = \deg Q$.*

From Theorem C, we know that the order of a_j ($j = 1, 2$) must be less than 1. What can be said if the order of a_j is not less than 1 under the hypothesis of Theorem C? In this paper, we prove the following theorem.

THEOREM 1. *If f is a nonconstant solution of the differential equation*

$$f^{(k)} - a = (f - a)e^{Q}, \tag{1.1}$$

where a is an entire function, Q is a polynomial with $\deg Q < \sigma(a) < \infty$ *and k is a positive integer, then $\mu_2(f) = \sigma_2(f) = \deg Q$.*

REMARK 1. From the proof of Theorem 1, we will see that if Q is a constant, then $\deg Q = 0$, thus, $\infty > \sigma(a) > 0$; if Q is a nonconstant polynomial, then $\deg Q \geq 1$, thus, $\infty > \sigma(a) > 1$.

From Theorem 1 we get the following corollary which improves Theorem 1 of [5].

COROLLARY 1. *If f is a nonconstant solution of the differential equation* (1.1), *where a is an entire function, Q is a nonconstant polynomial with* $\deg Q < \sigma(a) < \infty$ *and k is a positive integer, then $\mu_2(f) = \sigma_2(f) =$* $\deg Q \geq 1$, *and f is an entire function of infinite order.*

From Theorem 1 we also get the following two corollaries which improve Theorem A.

COROLLARY 2. *Let f be a nonconstant solution of the differential equation* (1.1), *where a is an entire function, Q is a polynomial with $\infty > \sigma(a) >$* $\deg Q$ *and k is a positive integer. If $\mu_2(f) < \infty$ and $\mu_2(f)$ is not a positive integer, then $f^{(k)} - a = c(f - a)$ for some complex number $c \neq 0$.*

COROLLARY 3. *Let f be a nonconstant solution of the differential equation* (1.1), *where a is an entire function, Q is a polynomial with $\infty > \sigma(a) >$* $\deg Q$ *and k is a positive integer. If $\mu(f) < \infty$, then $f^{(k)} - a = c(f - a)$ for some complex number $c \neq 0$.*

In Theorem 1, $Q(z)$ is assumed to be a polynomial. What can be said if $Q(z)$ is a transcendental entire function? In [11], the authors proved the following theorem, assuming that f satisfies a certain additional condition and $a = z$.

THEOREM D (see [11]). *Let Q be a transcendental entire function and k be a positive integer. If f is a solution of the equation*

$$\frac{f^{(k)} - z}{f - z} = e^{Q} \tag{1.2}$$

and there exists a positive integer l $(2 \leq l \leq k)$ such that $m(r, 1/f^{(l)}) =$ $O\{\log rT(r,f)\}$ *$(r \to \infty,\ r \notin E)$, where E is a set of finite linear measure, then $\sigma_2(f) = \infty$.*

We continue this study using the method of [2] and get the following theorem, assuming that $\sigma(Q) < 1/2$.

THEOREM 2. *Let Q be a transcendental entire function with $\sigma(Q) < 1/2$, a be an entire function of finite order and k be a positive integer. If f is a solution of the equation*

$$\frac{f^{(k)} - a}{f - a} = e^{Q}, \tag{1.3}$$

then $\sigma_2(f) = \infty$.

From Theorem 2 we get the following corollary.

COROLLARY 4. *Let Q be a transcendental entire function with $\sigma(Q) < 1/2$ and k be a positive integer. If f is a solution of the equation*

$$\frac{f^{(k)} - z}{f - z} = e^{Q}, \tag{1.4}$$

then $\sigma_2(f) = \infty$.

Comparing Theorem D with Corollary 4 suggests asking about the relationship between the condition $m(r, 1/f^{(l)}) = O(\log rT(r, f))$ $(r \to \infty,$ $r \notin E)$ (in Theorem D) and the condition $\sigma(Q) < 1/2$ (in Corollary 4). It is an interesting question for further study.

2. Lemmas. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We define $\mu(r) = \max\{|a_n|r^n : n = 0, 1, 2, \ldots\}$ and set $\nu(r, f) = \max\{m : \mu(r) = |a_m|r^m\}$, the *central index* of f (see [5]).

LEMMA 1 (see [9]). *Let $g : (0, \infty) \to \mathbb{R}$ and $h : (0, \infty) \to \mathbb{R}$ be increasing functions such that $g(r) \le h(r)$ outside an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

LEMMA 2 (see [8]). *If f is an entire function, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r}. \tag{2.1}$$

LEMMA 3 (see [3]). *If f is a transcendental entire function, then*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r}. \tag{2.2}$$

LEMMA 4 (see [13]). *If f is an entire function of infinite order, then*

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r}. \tag{2.3}$$

LEMMA 5 (see [9]) *Suppose that all the coefficients* $a_0(\not\equiv 0),\ a_1, \dots, a_{n-1}$ *and* $g\ (\not\equiv 0)$ *of the nonhomogeneous linear differential equation*

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f' + a_0 f = g \tag{2.4}$$

are entire functions. Then all the solutions of (2.4) *are entire functions.*

LEMMA 6 (see [1]). *Let* $h(z)$ *be an entire function of order* $\sigma(h) = \alpha < 1/2$, $A(r) = \inf_{|z|=r} \log|h(z)|$ *and* $B(r) = \sup_{|z|=r} \log|h(z)|$. *If* $\beta < \alpha < 1$, *then*

$$\underline{\log \operatorname{dens}}\{r : A(r) > \cos(\pi\alpha)B(r)\} \geq 1 - \beta/\alpha. \tag{2.5}$$

REMARK 2. In Lemma 6, the *lower logarithmic density* of a set E is defined by

$$\underline{\log \operatorname{dens}}\, E = \liminf_{r\to\infty} \frac{\lambda(E \cap [1, r])}{\log r}, \tag{2.6}$$

where $\lambda(E \cap [1, r])$ is the logarithmic measure of $E \cap [1, r]$.

REMARK 3. By the definition of the logarithmic measure and logarithmic density of a set E, we know that if $\underline{\log \operatorname{dens}}\, E > 0$, then the logarithmic measure of E is infinite.

LEMMA 7. *Let* f, a *be two entire functions with* $\sigma(a) = \sigma(f)$ *and* $\{z_r\}$ *be a sequence of points such that* $|z_r| = r$ *and* $|f(z_r)| = M(r, f)$. *Then*

$$0 \leq \lim_{r\to\infty} \left|\frac{a(z_r)}{f(z_r)}\right| \leq A, \tag{2.7}$$

where A *is a finite positive number.*

Proof. Suppose that $\lim_{r\to\infty} |a(z_r)/f(z_r)| = \infty$. Then, for any positive number B, there exists r_0 such that

$$\frac{|a(z_r)|}{M(r, f)} = \left|\frac{a(z_r)}{f(z_r)}\right| > B \tag{2.8}$$

for $|z_r| = r > r_0$. From (2.8) we have

$$BM(r, f) < |a(z_r)| \leq M(r, a) \tag{2.9}$$

for $|z_r| = r > r_0$. By Definition 1 and (2.9), we have $\sigma(f) < \sigma(a)$, a contradiction. This completes the proof.

REMARK 4. The following example shows that $\lim_{r\to\infty} |a(z_r)/f(z_r)|$ can be zero in Lemma 7.

EXAMPLE. Let $f(z) = e^z$ and $a(z) = e^{-z}$. Obviously, $f(z)$ gets the maximum modulus and $a(z)$ gets the minimum modulus on the circle $|z| = r$ when $z \in \mathbb{R}^+$. Thus, we have $\lim_{r_n\to\infty} |a(z_{r_n})/f(z_{r_n})| = 0$ for the sequence $\{z_{r_n}\} \subset \mathbb{R}^+$.

3. Proofs of theorems

Proof of Theorem 1. By Lemma 5, f is an entire function. Suppose that $f(z)$ is a nonconstant polynomial. Then from (1.1) we have

$$a = \frac{f^{(k)} - e^Q f}{1 - e^Q}. \tag{3.1}$$

Hence $\sigma(a) \leq \deg Q$, which contradicts the hypothesis. Next we suppose that f is a transcendental entire function. We discuss the following two cases.

CASE 1. Suppose that e^Q is a constant, say $c \neq 0$. Then (1.1) can be rewritten as

$$f^{(k)} - a = c(f - a). \tag{3.2}$$

If $\sigma(f) < \infty$, then $\mu_2(f) = \sigma_2(f) = \deg Q = 0$, which yields the conclusion of Theorem 1.

Next we suppose that $\sigma(f) = \infty$. Then

$$M(r, f) \to \infty \quad \text{as } r \to \infty. \tag{3.3}$$

Let $M(r, f) = |f(z_r)|$, where $z_r = re^{i\theta(r)}$, $\theta(r) \in [0, 2\pi)$. From (3.3) and Wiman–Valiron theory (see [9]), there exists a subset $F \subset (1, \infty)$ with finite logarithmic measure such that for some z_r satisfying $|z_r| = r \notin F$ and $M(r, f) = |f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)) \tag{3.4}$$

as r $(\notin F) \to \infty$. From the condition $\sigma(a) < \infty$ and Definition 1, we see that there exists an infinite sequence z_{r_n} such that

$$\lim_{r_n \to \infty} \frac{\log\log M(r_n, f)}{\log r_n} = \limsup_{r_n \to \infty} \frac{\log\log M(r_n, f)}{\log r_n} = \infty \tag{3.5}$$

and

$$\lim_{r_n \to \infty} \left|\frac{a(z_{r_n})}{f(z_{r_n})}\right| = \lim_{r_n \to \infty} \frac{|a(z_{r_n})|}{M(r_n, f)} = 0. \tag{3.6}$$

Since (3.2) can be rewritten as

$$c = \frac{f^{(k)}/f - a/f}{1 - a/f}, \tag{3.7}$$

from (3.4)–(3.7) we have

$$c = \left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^k (1 + o(1)) \tag{3.8}$$

as r_n $(\notin F) \to \infty$. Proceeding as in the proof of Lemma 2.5 in [13] and applying (3.5), we get

$$\lim_{r_n\to\infty}\frac{\log\log M(r_n,f)}{\log r_n}=\lim_{r_n\to\infty}\frac{\log\nu(r_n,f)}{\log r_n}=\infty, \tag{3.9}$$

which contradicts (3.8).

CASE 2. Suppose that e^Q is a nonconstant entire function. Then $\sigma(e^Q)=\deg Q\geq 1$. We discuss the following two subcases:

SUBCASE 2.1. Suppose that $\sigma(f)=\infty$. Then we have

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log\nu(r,f)}{\log r}=\infty. \tag{3.10}$$

Let

$$Q:=q_nz^n+q_{n-1}z^{n-1}+\cdots+q_1z+q_0, \tag{3.11}$$

where q_n $(\neq 0)$, $q_{n-1},\dots,q_1,q_0$ are complex numbers.

From (3.11) we get $\lim_{|z|\to\infty}|Q/(q_nz^n)|=1$. Hence there exists $r_0>0$ such that $|Q/(q_nz^n)|>1/e$ for $|z|>r_0$. Combining this with (1.1) we get

$$\begin{aligned}n\log r+\log|q_n|-1&<\log|\log e^Q|\leq|\log\log e^Q|\\&=\left|\log\log\frac{f^{(k)}-a}{f-a}\right|=\left|\log\log\frac{f^{(k)}/f-a/f}{1-a/f}\right|\end{aligned} \tag{3.12}$$

when $|z|>r_0$. Since $\sigma(a)<\infty$ and $\sigma(f)=\infty$, from (3.4) and (3.5) we get

$$\lim_{r_n\to\infty}\left|\frac{a(z_{r_n})}{f(z_{r_n})}\right|=\lim_{r_n\to\infty}\frac{|a(z_{r_n})|}{M(r_n,f)}=0. \tag{3.13}$$

By substituting (3.4) and (3.13) into (3.12) we have

$$n\log|z_{r_n}|+\log|q_n|-1\leq\left|\log\log\left(\frac{\nu(r_n,f)}{z_{r_n}}\right)^k(1+o(1))\right|, \tag{3.14}$$

as $|z_{r_n}|=r_n$ $(>r_0)\to\infty$, $r_n\notin F$. Since

$$\begin{aligned}&\log\left(\frac{\nu(r_n,f)}{z_{r_n}}\right)^k(1+o(1))\\&\quad=k\left(1-\frac{\log r_n}{\log\nu(r_n,f)}-\frac{i\theta(r_n)}{\log\nu(r_n,f)}\right)\log\nu(r_n,f)+o(1)\end{aligned} \tag{3.15}$$

as $|z_{r_n}|=r_n\to\infty$, $r_n\notin F$, from (3.4), (3.5), Lemma 3 and the condition $\theta(r_n)\in[0,2\pi)$ we get

$$\begin{aligned}n&\leq\limsup_{r\to\infty}\frac{|\log\log(\nu(r,f)/z_r)^k(1+o(1))|}{\log r}\\&\leq\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_2(f).\end{aligned} \tag{3.16}$$

From (3.11) we have $\sigma(e^Q)=\deg Q=n$. Thus, $\sigma(e^Q)=n\leq\sigma_2(f)$.

On the other hand, from (1.1), we have

$$|Q(z)| = |\log e^Q| = \left|\log \frac{f^{(k)}/f - a/f}{1 - a/f}\right|. \tag{3.17}$$

Substituting (3.4), (3.5) and (3.13) into (3.17) we get

$$e^Q = \left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^k (1 + o(1)), \tag{3.18}$$

as $|z_{r_n}| = r_n \to \infty,\ r_n \notin F$. From (3.18) we get

$$|Q(z_{r_n})| = k|\log \nu(r_n, f) - \log r_n - i\theta(r_n)|(1 + o(1)) \tag{3.19}$$

as $|z_{r_n}| = r_n \to \infty,\ r_n \notin F$. By (3.18), we have

$$\limsup_{r_n\to\infty} \frac{\log\log\left(\frac{\nu(r_n,f)}{|z_{r_n}|}\right)^k (1 + o(1))}{\log r_n} \le \limsup_{r_n\to\infty} \frac{\log\log M(r_n, e^Q)}{\log r_n}. \tag{3.20}$$

Since

$$\limsup_{r_n\to\infty} \frac{\log\log \nu(r_n, f)}{\log r_n} = \limsup_{r_n\to\infty} \frac{\log\log(\nu(r_n, f)^k/|z_{r_n}|^k)}{\log r_n} \tag{3.21}$$

and

$$\limsup_{r_n\to\infty} \frac{\log\log(\nu(r_n, f)^k/2r_n^k)}{\log r_n} \le \limsup_{r_n\to\infty} \frac{\log\log(\nu(r_n, f)/|z_{r_n}|)^k(1 + o(1))}{\log r_n}. \tag{3.22}$$

From (3.20)–(3.22) and Lemma 3, we get

$$\sigma_2(f) \le \sigma(e^Q) = n. \tag{3.23}$$

Combining (3.23) with (3.16), we have $\sigma_2(f) = \deg Q = n$.

Additionally, from (3.12), (3.18) and the conditions $z_r = re^{i\theta(r)},\ \theta(r) \in [0, 2\pi),\ |z_r| = r$, we get

$$\begin{aligned} n\log|z_r| + \log|q_n| - 1 &\le \log|Q(z_r)| \\ &\le |\log\log e^{Q(z_r)}| \quad (r > r_0) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \log e^Q &= k(\log \nu(r, f) - \log r - i\theta(r) + o(1)) \\ &= k(\log \nu(r, f) - \log r)(1 + o(1)) \end{aligned} \tag{3.25}$$

as $r \to \infty,\ r \notin F$. From (3.24), (3.25) and Lemma 4, we get

$$n \le \liminf_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r} = \mu_2(f). \tag{3.26}$$

Since $\mu_2(f) \le \sigma_2(f)$, we have $\mu_2(f) = \sigma_2(f) = \deg Q = n$. Thus, f satisfies our conclusion.

Subcase 2.2. Suppose that $\sigma(f) < \infty$.

If $\sigma(f) > \sigma(a)$, from (3.6) we get

$$\lim_{r_n\to\infty} \left|\frac{a(z_{r_n})}{f(z_{r_n})}\right| = \lim_{r_n\to\infty} \frac{|a(z_{r_n})|}{M(r_n, f)} = 0. \tag{3.27}$$

By a similar argument to Subcase 2.1, we get $n \le \sigma_2(f) = 0$ (see (3.16)). Since Q is a nonconstant polynomial, we have $n \ge 1$. We get a contradiction.

If $\sigma(f) < \sigma(a)$, from (3.1) we get $\sigma(a) \le \max\{\sigma(f), \sigma(e^Q)\}$. This contradicts our hypothesis.

If $\sigma(f) = \sigma(a)$, by Lemma 7 we have

$$0 \le \lim_{r\to\infty} \left|\frac{a(z_r)}{f(z_r)}\right| \le A \tag{3.28}$$

for any sequence $\{z_r\}$, where A is a positive number.

Suppose that $\lim_{r\to\infty} |a(z_r)/f(z_r)| \ne 1$ for some sequence $\{z_r\}$. By (3.12)–(3.16) and (3.28) we have $n \le \sigma_2(f) = 0$. Since Q is a nonconstant polynomial, we have $n \ge 1$, a contradiction.

Suppose now that $\lim_{r\to\infty} |a(z_r)/f(z_r)| = 1$ for some sequence $\{z_r\}$. Equation (1.1) can be rewritten as

$$f^{(k)} - e^Q f = a(1 - e^Q). \tag{3.29}$$

Thus,

$$\frac{f^{(k)}}{f} - e^Q = \frac{a}{f}\,(1 - e^Q). \tag{3.30}$$

So, we have

$$|e^Q| < \left|\frac{a}{f}\right| \left|1 - e^Q\right| + \left|\frac{f^{(k)}}{f}\right| + O(1). \tag{3.31}$$

Since the order of f is finite, by (3.4), Lemma 3 and $\lim_{r\to\infty} |a(z_r)/f(z_r)| = 1$, we get

$$\frac{\log\log|e^Q|}{\log r} < \frac{\log\log|e^Q|}{\log r} \tag{3.32}$$

for the sequence $\{z_r\}$ with $|z_r| = r\ (\notin F) \to \infty$, a contradiction.

Thus, the proof of Theorem 1 is complete.

Proof of Theorem 2. From Lemma 5, we know that f is an entire function. Suppose that $\sigma_2(f) < \infty$. If $\sigma(f) < \infty$, from (1.3) we have $\sigma(e^Q) \le \max\{\sigma(f), \sigma(a)\} < \infty$. Since $Q(z)$ is a transcendental entire function, we have $\sigma(e^Q) = \infty$, a contradiction. Hence $\sigma(f) = \infty$. As f is a nonconstant entire function we have

$$M(r,f)\to\infty \quad \text{as } r\to\infty. \tag{3.33}$$

From (3.33) and Wiman–Valiron theory (see [9]), there exists a subset $F\subset(1,\infty)$ with finite logarithmic measure such that for some points z_r satisfying $|z_r|=r\notin F$ and $M(r,f)=|f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)}=\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1)) \tag{3.34}$$

as $r\ (\notin F)\to\infty$. By the condition $\sigma(a)<\infty$ and Definition 1, there exists an infinite sequence z_{r_n} such that

$$\lim_{r_n\to\infty}\frac{\log\log M(r_n,f)}{\log r_n}=\limsup_{r_n\to\infty}\frac{\log\log M(r_n,f)}{\log r_n}=\infty \tag{3.35}$$

and

$$\lim_{r_n\to\infty}\left|\frac{a(z_{r_n})}{f(z_{r_n})}\right|=\lim_{r_n\to\infty}\frac{|a(z_{r_n})|}{M(r_n,f)}=0. \tag{3.36}$$

From (1.3) and (3.33)–(3.36), we have

$$e^{Q(z_{r_n})}=\left(\frac{\nu(r_n)}{z_{r_n}}\right)^k(1+o(1))+o(1), \tag{3.37}$$

where $\nu(r_n)$ is the central index of f. Since $\sigma(f)=\infty$, Lemma 2 shows that $\nu(r_n)$ satisfies $\nu(r_n)\ge|z_{r_n}|^N$ for any sufficiently large positive number N, as $|z_{r_n}|=r_n\to\infty, r_n\notin F$. So we have

$$\begin{aligned}|Q(z_{r_n})|&\le\left|\log\left|\left(\frac{\nu(r_n)}{z_{r_n}}\right)^k(1+o(1))+o(1)\right|\right|+2\pi\\&\le k\log\nu(r_n)+o(1)\end{aligned} \tag{3.38}$$

as $|z_{r_n}|=r_n\to\infty,\ r_n\notin F$. By Lemma 3, we have

$$\frac{\log\log\nu(r_n)}{\log r_n}\le\sigma_2(f)+1 \tag{3.39}$$

for sufficiently large r_n. From (3.38) and (3.39), we have

$$|Q(z_{r_n})|\le r^{\sigma_2(f)+1}+O(1) \tag{3.40}$$

as $|z_{r_n}|=r_n\to\infty,\ r_n\notin F$. By Lemma 6, there exists a set $H\subset(1,\infty)$ with infinite logarithmic measure such that

$$|Q(z_{r_n})|\ge M(r_n,Q)^c \tag{3.41}$$

for $|z_{r_n}|=r_n\in H$, where $0<c<1$. From (3.40) and (3.41) we have

$$\frac{M(r,Q)^c}{r^{\sigma_2(f)+1}} \le 1 \tag{3.42}$$

for $r_n \in H \setminus F$ and $|f(z_{r_n})| = M(r_n, f)$. Since Q is transcendental, we have

$$\frac{M(r_n,Q)^c}{r_n^{\sigma_2(f)+1}} \to \infty \tag{3.43}$$

as $r_n \to \infty$, a contradiction. Thus, the proof of Theorem 2 is complete.

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References

[1] P. D. Barry, *On a theorem of Besicovitch*, Quart. J. Math. Oxford 14 (1963), 293–302.

[2] R. Brück, *On entire functions which share one value CM with their first derivative*, Results Math. 30 (1996), 21–24.

[3] Z. X. Chen and C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J. 22 (1999), 273–285.

[4] Z. X. Chen and Z. L. Zhang, *Entire functions sharing fixed points with their higher-order derivatives*, Acta Math. Sinica (Chinese Ser.) 50 (2007), 1213–1222.

[5] G. G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl. 223 (1998), 88–95.

[6] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[7] —, *Slowly growing integral and subharmonic functions*, Comment. Math. Helv. 34 (1960), 75–84.

[8] G. Jank und L. Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel, 1985.

[9] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter, Berlin, 1993.

[10] X. M. Li and C. C. Gao, *Entire functions sharing one polynomial with their derivatives*, Proc. Indian Acad. Sci. Math. 118 (2008), 13–26.

[11] H. F. Liu and D. C. Sun, *On the uniqueness problems of entire functions and their derivatives*, J. Math. Anal. Appl. 348 (2008), 614–619.

[12] L. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, in: Complex Analysis (Lexington, KY, 1976), Lecture Notes in Math. 599, Springer, Berlin, 1977, 101–103.

[13] J. Wang and X. M. Li, *The uniqueness of an entire function sharing a small entire function with its derivatives*, J. Math. Anal. Appl. 354 (2008), 478–489.

[14] J. P. Wang, *Entire functions that share a polynomial with one of their derivatives*, Kodai Math. J. 27 (2004), 144–151.

[15] L. Yang, *Value Distribution Theory and New Research*, Science Press, Beijing, 1982 (in Chinese).

[16] L. Z. Yang, *Solution of a differential equation and its applications*, Kodai Math. J. 22 (1999), 458–464.

[17] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press and Kluwer, Beijing, 2003.

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