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On local-in-time existence for the Dirichlet problem for equations of compressible viscous fluids

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Abstract. The local existence of solutions for the compressible Navier–Stokes equations with the Dirichlet boundary conditions in the L_p-framework is proved. Next an almost-global-in-time existence of small solutions is shown. The considerations are made in Lagrangian coordinates. The result is sharp in the L_p-approach, because the velocity belongs to $W_r^{2,1}$ with $r > 3$.

1. Introduction. In this paper we consider the motion of viscous compressible fluids in a bounded domain $\Omega \subset \mathbb{R}^3$ described by the Navier–Stokes equations

$$\begin{aligned} &\varrho(v_t + v\cdot\nabla v) + \mu\Delta v - \nu\nabla\operatorname{div} v + \nabla p = \varrho f && \text{in } \Omega^T,\\ &\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega^T,\\ &v = \alpha && \text{on } S^T,\\ &v|_{t=0} = v_0, \quad \varrho|_{t=0} = \varrho_0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega^T = \Omega\times[0,T]$, $S = \partial\Omega$, $S^T = S\times[0,T]$, $v(x,t)$ is the velocity of the fluid, $\varrho(x,t)$ the density, $p = p(\varrho)$ the pressure, $f(x,t)$ the external force, μ and ν the constant viscosity coefficients which satisfy the thermodynamic restrictions

$$\mu > 0, \quad \nu > \tfrac{1}{3}\mu, \tag{1.2}$$

and the dot denotes the scalar product in $\mathbb{R}^3$.

To prove the existence of solutions to (1.1) we introduce the Lagrangian coordinates as the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \Omega. \tag{1.3}$$

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Solving (1.3) we obtain

$$x = \xi + \int_0^t u(\xi, \tau)\, d\tau \equiv x(\xi, t) \tag{1.4}$$

which is the relation between the Eulerian x and Lagrangian ξ coordinates and $u(\xi,t) = v(x(\xi,t),t)$. Here we need to add an extra condition $v \cdot \overline{n}|_{S^T} = 0$ ($\overline{n}$ is the normal vector to S) which ensures that this transformation preserves the domain ($\{x \mapsto \xi\} : \Omega \to \Omega$).

In Lagrangian coordinates, (1.1) reads

$$\begin{aligned} &\eta u_t - \mu \Delta_u u - \nu \nabla_u \operatorname{div}_u u + \nabla_u q = \eta g && \text{in } \Omega^T, \\ &\eta_t + \eta \operatorname{div}_u u = 0 && \text{in } \Omega^T, \\ &u = \alpha && \text{on } S^T, \\ &u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0 && \text{in } \Omega, \end{aligned} \tag{1.5}$$

where $\eta(\xi,t) = \varrho(x(\xi,t),t)$, $q = p(\eta)$, $g(\xi,t) = f(x(\xi,t),t)$, $\nabla_u = \frac{\partial \xi_i}{\partial x} \partial_{\xi_i}$, $\Delta_u = \nabla_u^2$, $\operatorname{div}_u = \nabla_u \cdot$ and the summation convention over repeated indices is used.

The first result of the paper is the following.

THEOREM 1.1. *Let* $r > 3, S \in W_r^{2-1/r}$, $f \in L_r(0,T; W_\infty^1(\Omega))$, $\alpha \in C^3(S^T)$, $v_0 \in W_r^{2-2/r}(\Omega)$, $\varrho_0 \in W_r^1(\Omega)$, $\alpha \cdot \overline{n} = 0$ *and moreover*

$$0 < 2a \le \varrho_0(x) \le b < \infty.$$

Then there exists $T_0 > 0$ *such that for* $T \le T_0$ *there exists a unique solution of* (1.5) *such that* $u \in W_r^{2,1}(\Omega^T)$, $\eta \in W_r^{1,0}(\Omega^T)$, $\eta_t \in W_r^{1,0}(\Omega^T)$ *and*

$$\begin{aligned} &\|u\|_{W_r^{2,1}(\Omega^T)} + \|\eta\|_{W_r^{1,0}(\Omega^T)} + \|\eta_t\|_{W_r^{1,0}(\Omega^T)} \\ &\le c(\|f\|_{L_r(\Omega^T)} + \|\alpha\|_{W_r^{2-1/r,1-1(2r)}(S^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\varrho_0\|_{W_r^1(\Omega)} \\ &\qquad + \|f\|_{L_r(0,T;W_\infty^1(\Omega))} + \|\alpha\|_{C^3(S^T)}) \end{aligned} \tag{1.6}$$

and

$$0 < a \le \eta(x,t) \le 2b < \infty \tag{1.7}$$

for $(x,t) \in \Omega^T$.

The next result concerns the time continuity of solutions which are close to equilibrium states

$$\nabla p(\overline{\varrho}(x)) = \overline{\varrho}(x) f(x), \tag{1.8}$$

where

$$0 < \overline{\varrho}_* \le \overline{\varrho}(x) \le \overline{\varrho}^* < \infty$$

for each $x \in \Omega$, $f = \nabla \varphi$ and $\overline{\varrho}_*, \overline{\varrho}^*$ are given constants.

From (1.1) with $\alpha = 0$, $f = \nabla\varphi$ and (1.8) we obtain the following system for perturbations in Eulerian coordinates:

$$
\begin{aligned}
&\varrho(v_t + v\cdot\nabla v) + \mu\varDelta v - \nu\nabla\operatorname{div} v + \gamma\nabla\sigma = \sigma f - \sigma\nabla\gamma && \text{in } \varOmega^T,\\
&\sigma_t + \operatorname{div}(\varrho v) = 0 && \text{in } \varOmega^T,\\
&v = 0 && \text{on } S^T,\\
&v|_{t=0} = v_0, \quad \sigma|_{t=0} = \varrho_0 - \overline{\varrho} && \text{in } \varOmega,
\end{aligned}
\tag{1.9}
$$

where

$$\sigma(x,t) = \varrho(x,t) - \overline{\varrho}(x)$$

and γ is defined by the relation

$$p(\varrho) - p(\overline{\varrho}) = \sigma\int_0^1 p'(\varrho + s(\overline{\varrho} - \varrho))\,ds = \sigma\gamma.$$

System (1.9) in Lagrangian coordinates reads

$$
\begin{aligned}
&\eta u_t - \mu\varDelta_u u - \nu\nabla_u \operatorname{div}_u u + \gamma\nabla_u\chi = \chi g - \chi\nabla_u\gamma && \text{in } \varOmega^T,\\
&\chi_t + \eta\operatorname{div}_u u = -\overline{\eta}_t && \text{in } \varOmega^T,\\
&u = 0 && \text{on } S^T,\\
&u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0 - \overline{\varrho} && \text{in } \varOmega,
\end{aligned}
\tag{1.10}
$$

where

$$\chi(\xi,t) = \eta(\xi,t) - \overline{\eta}(\xi,t)$$

and $\overline{\eta}(\xi,t) = \overline{\varrho}(x(\xi,t))$.

We prove the almost-global-in-time existence of solutions to (1.10).

THEOREM 1.2. *Let $r > 3$, $f = \nabla\varphi \in W^1_\infty(\varOmega)$, $\varrho_0 - \overline{\varrho} \in W^1_r(\varOmega)$, $u_0 \in W^{2-2/r}_r(\varOmega)$, $p(\varrho) = a\varrho^\kappa$ with $a > 0$, $\kappa > 1$ and $\varrho_0 \ge \overline{\varrho}_*/2$. Let $T > 0$ be given. Then there exist $\overline{M}_1(T)$ and $M_2(T, M_1)$, where $M_2(T, M_1) \to 0$ as $T \to \infty$ and $M_1 \to 0$, such that for $M_1 \le \overline{M}_1(T)$, if*

$$\|v_0\|_{W^{2-2/r}_r(\varOmega)} + \|\varrho_0 - \overline{\varrho}\|_{W^1_r(\varOmega)} \le M_2(T, M_1),$$

then there exist solutions of (1.10) such that $u \in W^{2,1}_r(\varOmega\times[0,T])$, $\chi \in V_r(\varOmega\times[0,T])$ and

$$\|u\|_{W^{2,1}_r(\varOmega\times[0,T])} + \|\eta\|_{V_r(\varOmega\times[0,T])} \le M_1.$$

Moreover,

$$\eta(\xi,t) \ge \overline{\varrho}_*/4$$

for each $(\xi,t) \in \varOmega\times[0,T]$.

To prove these results we need solvability of the following linear problem in the L_r-framework:

$$
\begin{aligned}
&Cu_t - \mu\Delta u - \nu\nabla \operatorname{div} u + A\nabla\eta = F,\\
&\eta_t + B \operatorname{div} u = H,\\
&u|_{S_T} = G,\\
&u|_{t=0} = u_0, \qquad \eta|_{t=0} = \eta_0,
\end{aligned}
\tag{1.11}
$$

where $A(x,t), B(x,t), C(x,t)$ are positive functions which belong to $C^\alpha(\Omega\times[0,T])$.

For system (1.11) we have proved the following result (see [3]).

THEOREM 1.3. *Let* $r > 3$, $S \in W_r^{2-1/r}$, $F \in L_r(\Omega_T)$, $H \in W_r^{1,0}(\Omega_T)$, $G \in W_r^{2-1/r,1-1/(2r)}(S_T)$, $u_0 \in W_r^{2-2/r}(\Omega)$, $\eta_0 \in W_r^1(\Omega)$ *and*

$$0 < C_* \le A(x,t), B(x,t), C(x,t) \le C^* < \infty.$$

Moreover, A, B, C *belong to* $C^\alpha(\Omega^T)$, $\alpha > 0$. *Then there exists a unique solution of problem* (1.11) *such that* $u \in W_r^{2,1}(\Omega^T)$, $\eta \in W_r^{1,0}(\Omega^T)$, $\eta_t \in W_r^{1,0}(\Omega^T)$ *and*

$$
\begin{aligned}
&\|u\|_{W_r^{2,1}(\Omega^T)} + \|\eta\|_{W_r^{1,0}(\Omega^T)} + \|\eta_t\|_{W_r^{1,0}(\Omega^T)}\\
&\quad\le c(T,C^*,C_*)[\|f\|_{L_r(\Omega^T)} + \|H\|_{W_r^{1,0}(\Omega^T)} + \|G\|_{W_r^{2-1/r,1-1(2r)}(S^T)}\\
&\qquad + \|u_0\|_{W_r^{2-2/r}(\Omega)} + \|\eta_0\|_{W_r^{1,0}(\Omega)}],
\end{aligned}
\tag{1.12}
$$

where $c(T,\cdot,\cdot)$ *is an increasing function of* T.

The global existence of small regular solutions for problem (1.9) has been shown in [2, 5] for the case when the norms of φ ($f = \nabla\varphi$) are sufficiently small. These results base on energy estimates. In our paper we show existence and almost-global-in-time existence of regular solutions in the L_p-framework. The theorems enable us to prove in [4] the global existence of small solutions to problem (1.9) with "large" φ. Moreover this result is sharp in the L_p-framework.

2. Notation. In our considerations we will need the anisotropic Sobolev–Slobodetskiĭ spaces $W_r^{m,n}(Q^T)$ where $m,n \in \mathbb{R}_+ \cup \{0\}$, $r \ge 1$ and $Q^T = Q\times(0,T)$ with the norm

$$
\begin{aligned}
\|u\|^r_{W_r^{m,n}(Q^T)} &= \int_0^T\int_Q |u(x,t)|^r\,dx\,dt\\
&\quad + \sum_{0\le|m'|\le[|m|]} \int_0^T\int_Q |D_x^{m'}u(x,t)|^r\,dx\,dt
\end{aligned}
\tag{2.1}
$$

$$+ \sum_{|m'|=[|m|]} \int_0^T dt \int_Q \int_Q \frac{|D_x^{m'} u(x,t) - D_{x'}^{m'} u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}} \, dx \, dx$$

$$+ \sum_{0 \le |n'| \le [|n|]} \int_0^T \int_Q |D_t^{n'} u(x,t)|^r \, dx \, dt$$

$$+ \int_Q dx \int_0^T \int_0^T \frac{|D_t^{[n]} u(x,t) - D_{t'}^{[n]}(x,t')|^r}{|t-t'|^{1+r(n-[n])}} \, dt \, dt'.$$

where $s = \dim Q$, $[\alpha]$ is the integral part of α, $D_x^l = \partial_{x_1}^{l_1} \ldots \partial_{x_s}^{l_s}$, where $l = (l_1, \ldots, l_s)$ is a multiindex and $|l| = l_1 + \ldots + l_s$, $l_i \ge 0$, $i = 1, \ldots, s$. For m and n integers, the corresponding terms with differences vanish.

We define the space $V(\Omega^T)$ as the closure

$$V(\Omega^T) = \overline{C^\infty(\Omega^T)}^{\|\cdot\|_{V(\Omega^T)}},$$

where

$$\|f\|_{V(\Omega^T)} = \|f\|_{W_r^{1,0}(\Omega^T)} + \|f_t\|_{W_r^{1,0}(\Omega^T)}. \tag{2.2}$$

In the proof we will use the following results.

PROPOSITION 2.1 (see [1]). *Let $u \in W_r^{m,n}(\Omega^T)$, $m, n \in \mathbb{R}_+$. If*

$$\kappa = \sum_{i=1}^3 \left(\alpha_i + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{m} + \left(\beta + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{n} < 1$$

then

$$\|D_t^\beta D_x^\alpha u\|_{L_q(\Omega^T)} \le \varepsilon \|u\|_{W_r^{m,n}(\Omega^T)} + c(\varepsilon)\|u\|_{L_2(\Omega^T)},$$

where $q \ge r \ge 2$, $\varepsilon \in (0,1)$ and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

PROPOSITION 2.2. *Let $r > 3$. Then $V(\Omega^T) \subset C^\alpha(\Omega^T)$, where $0 < \alpha < 1 - 3/r$, and*

$$\|f\|_{C^\alpha(\Omega^T)} \le c\|f\|_{V(\Omega^T)}.$$

Proof. We have $r > 3$, $0 < \alpha < 1 - 3/r$ and the following imbeddings:

$$f \in L_r(0,T; W_r^1(\Omega)) \subset L_r(0,T; C^\alpha(\Omega)),$$
$$f_t \in L_r(0,T; W_r^1(\Omega)) \subset L_r(0,T; C^\alpha(\Omega)).$$

From these two relations we conclude

$$f \in W_r^1(0,T; C^\alpha(\Omega)) \subset C^{\alpha'}(0,T; C^\alpha(\Omega)),$$

where $0 < \alpha' < 1 - 1/r$. Thus we obtain $f \in C^\alpha(\Omega^T)$.

In our considerations we will treat all imbedding theorems for Sobolev spaces as well known results. All constants are denoted by c.

3. Proof of Theorem 1.1. We prove the existence of solutions of (1.5) by the method of successive approximations. We construct a sequence $\{u_m,\eta_m\}_{m=0}^\infty$ by the relations

$$
\begin{aligned}
&\eta_m u_{m+1,t} - \mu\Delta_{u_m} u_{m+1} - \nu\nabla_{u_m}\operatorname{div}_{u_m} u_{m+1} && \\
&\qquad + p'(\eta_m)\nabla_{u_m}\eta_{m+1} = \eta_m f(x_m,t) && \text{in } \Omega^T,\\
&\eta_{m+1,t} + \eta_m \operatorname{div}_{u_m} u_{m+1} = 0 && \text{in } \Omega^T,\\
&u_{m+1} = \alpha(x_m,t) && \text{on } S^T,\\
&u_{m+1}|_{t=0} = v_0, \quad \eta_{m+1}|_{t=0} = \varrho_0 && \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

where $x_m = \xi + \int_0^t u_m(x,t')\,dt'$ and u_0, η_0 are defined by

$$
\begin{aligned}
&\varrho_0 u_{0,t} - \mu\Delta u_0 - \nu\nabla\operatorname{div} u_0 + p'(\varrho_0)\nabla\eta_0 = \varrho_0 f(\xi,t) && \text{in } \Omega^T,\\
&\eta_{0,t} + \varrho_0 \operatorname{div} u_0 = 0 && \text{in } \Omega^T,\\
&u_0 = \alpha(x(\xi,t),t) && \text{on } S^T,\\
&u_0|_{t=0} = v_0, \quad \eta_0|_{t=0} = \varrho_0 && \text{in } \Omega.
\end{aligned}
\tag{3.2}
$$

From Theorem 1.3 we get a solution of (3.2) with the estimate

$$
\begin{aligned}
&\|u_0\|_{W_r^{2,1}(\Omega^T)} + \|\eta_0\|_{V(\Omega^T)}\\
&\le c(T)(\|f\|_{L_r(\Omega^T)} + \|\alpha\|_{W_r^{2-1/r,1-1/(2r)}(S^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\varrho_0\|_{W_r^1(\Omega)}).
\end{aligned}
\tag{3.3}
$$

First we show that $\{u_m,\eta_m\}_{m=0}^\infty$ is uniformly bounded on Ω^T for some $T>0$. By induction we show that

$$X_k = \|u_k\|_{W_r^{2,1}(\Omega^T)} + \|\eta_k\|_{V(\Omega^T)} \le 4M,$$

where

$$M = c(T)(\|f\|_{L_r(\Omega^T)} + \|\alpha\|_{W_r^{2-1/r,1-1/(2r)}(S^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\varrho_0\|_{W_r^1(\Omega)}).$$

For fixed $4M$ we can define T so small that the Jacobian of the transformation (1.4) (with u_m) is bounded.

System (3.1) is examined in the form

$$
\begin{aligned}
&\eta_m u_{m+1,t} - \mu\Delta u_{m+1} - \nu\nabla\operatorname{div} u_{m+1} + p'(\eta_m)\nabla\eta_{m+1} && \\
&\qquad = \eta_m f(x_m,t) + p'(\eta_m)(\nabla - \nabla_{u_m})\eta_{m+1} && \\
&\qquad + \mu(\Delta_{u_m} - \Delta)u_{m+1} + \nu(\nabla_{u_m}\operatorname{div}_{u_m} - \nabla\operatorname{div})u_{m+1} \equiv K && \text{in } \Omega^T,\\
&\eta_{m+1,t} + \eta_m\operatorname{div} u_{m+1} = \eta_m(\operatorname{div} - \operatorname{div}_{u_m})u_{m+1} \equiv L && \text{in } \Omega^T,\\
&u_{m+1} = \alpha(x_m,t), && \text{on } S^T,\\
&u_{m+1}|_{t=0} = v_0, \quad \eta_{m+1}|_{t=0} && \text{in } \Omega.
\end{aligned}
\tag{3.4}
$$

By Proposition 2.2 we see that $\eta_m, p'(\eta_m) \in C^\alpha(\Omega^T)$, thus we can apply Theorem 1.3 with $T = 1$, $C_* = a$ and $C^* = 2b$, to get

$$(3.5)\quad \|u_{m+1}\|_{W_r^{2,1}(\Omega^T)} + \|\eta_{m+1}\|_{V(\Omega^T)} \le c_0(\|f\|_{L_r(\Omega^T)} + \|\alpha\|_{W_r^{2-1/r,1-1/(2r)}(S^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\varrho_0\|_{W_r^1(\Omega^T)} + \|K\|_{L_r(\Omega^T)} + \|L\|_{W_r^{1,0}(\Omega^T)}).$$

To obtain (3.5) we have used the following estimates:

$$\|\alpha(x_m(\xi,t),t)\|_{L_r(S;W_r^1(0,T))} \le \Big(\int_{S^T} (|\alpha|^r + |\alpha_{x_m}|^r|u_m|^r + |\alpha_t|^r)|\xi_{x_m}|\,dx_m\,dt\Big)^{1/r} \le \phi(a_m)\Big(\int_{S^T} (|\alpha|^r + |\alpha_{x_m}|^r|u_m|^r + |\alpha_t|^r)\,dx_m\,dt\Big)^{1/r},$$

where $\phi(a_m)$ is a positive increasing function such that $\phi(0) > 0$ and $a_m = T^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)}$. To finish estimating the above term we note

$$\Big(\int_{S^T} |\alpha_x|^r|u|^r\,dx\,dt\Big)^{1/r} \le \Big(\int_0^T |\alpha_x|^r_{L_\infty(S)}\,dt\Big)^{1/r}\Big(\sup_t \int_S |u|^r\,dx\Big)^{1/r} \le \Big(\int_0^T \|\alpha\|^r_{W_r^2(S)}\,dt\Big)^{1/r}\Big(\int_S dx\,\Big|\int_0^T u_t\,dt + v_0(x)\Big|^r\Big)^{1/r} \le \|\alpha\|_{W_r^{2,1}(S^T)}(T^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)} + \|v_0\|_{W_r^{2-2/r}(\Omega)}).$$

But

$$K = \eta_m f(x_m,t) + p'(\eta_m)(\nabla - \nabla_{u_m})\eta_{m+1} + \mu(\Delta_{u_m} - \Delta)u_{m+1} - \nu(\nabla\operatorname{div} - \nabla_{u_m}\operatorname{div}_{u_m})u_{m+1} = K_1 + K_2 + K_3 + K_4.$$

Since $f \in L_r(0,T;L_r(\Omega))$ we have

$$\|K_1\|_{L_r(\Omega^T)} = \|\eta_m f(x_m,t)\|_{L_r(\Omega^T)} \le \|\eta_m\|_{L_\infty(\Omega^T)}\|f(x_m(\xi,t),t)\|_{L_r(\Omega^T)} \le \|\eta_m\|_{L_\infty(0,T;W_r^1(\Omega))}\phi(a_m)\|f\|_{L_r(\Omega^T)}$$

and

$$\|\eta_m\|_{L_\infty(0,T;W_r^1(\Omega))} \le \Big\|\int_0^T \eta_{m,t}(t)\,dt + \varrho_0\Big\|_{W_r^1(\Omega)} \le T^{(r-1)/r}\|\eta_m\|_{V(\Omega^T)} + \|\varrho_0\|_{W_r^1(\Omega)}.$$

Because $r > 3$, by Proposition 2.1 and the Hölder inequality we have

$$\Big\| \int_0^t |\nabla u_m(x,t')|\,dt' \Big\|_{L_\infty(\Omega^T)} \le cT^{(r-1)/r} \|u_m\|_{W_r^{2,1}(\Omega^T)},$$

which gives

$$\begin{aligned} \|K_2\|_{L_r(\Omega^T)} &= \|p'(\eta_m)(\nabla - \nabla_{u_m})\eta_{m+1}\|_{L_r(\Omega^T)} \\ &\le cT^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)} \|\eta_{m+1}\|_{V(\Omega^T)}, \end{aligned}$$

$$\begin{aligned} \|K_3\|_{L_r(\Omega^T)} &= \|\mu(\Delta_{u_m} - \Delta)u_{m+1}\|_{L_r(\Omega^T)} \\ &\le \|\mu \operatorname{div}(\nabla - \nabla_{u_m})u_{m+1}\|_{L_r(\Omega^T)} + \|\mu(\operatorname{div} - \operatorname{div}_{u_m})\nabla_{u_m} u_{m+1}\|_{L_r(\Omega^T)} \\ &\le cT^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)}(1 + T^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)})\|u_{m+1}\|_{W_r^{2,1}(\Omega^T)} \end{aligned}$$

and the same for K_4:

$$\begin{aligned} \|K_4\|_{L_r(\Omega^T)} &\le cT^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)} \\ &\quad \times (1 + T^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)})\|u_{m+1}\|_{W_r^{2,1}(\Omega^T)}, \\ \|L\|_{W_r^{1,0}(\Omega^T)} &\le \|\eta_m(\operatorname{div} - \operatorname{div}_{u_m})u_{m+1}\|_{W_r^{1,0}(\Omega^T)} \\ &\le cT^{(r-1)/r}\|u_m\|_{W_r^{2,1}(\Omega^T)}\|\eta_m\|_{V(\Omega^T)}\|u_{m+1}\|_{W_r^{2,1}(\Omega^T)}. \end{aligned}$$

Then by (3.5), and the estimates on K and L, we obtain

$$X_{m+1} \le c_1 T^{(r-1)/r} X_m^2 X_{m+1} + c_2 T^{1/r} X_m M + c_3 T^{2(r-1)/r} X_m^2 X_{m+1} + M.$$

Since $r > 3$ we have

$$X_{m+1} \le c_4 T^{1/r}(X_m + X_m^2)X_{m+1} + c_5 T^{1/r} X_m M + M.$$

But we assume that $X_m \le 4M$ (X_0 satisfies (3.3)). Taking T so small that

$$c_4 T^{1/r}((4M) + (4M)^2) < 1/2,$$

we get

$$X_{m+1} \le 2c_5 T^{1/r} X_m M + 2M$$

and if $c_5 T^{1/r} 4M < 1$ we get $X_{m+1} \le 4M$. By induction we have proved that

$$X_k \le 4M \tag{3.6}$$

for all $k \in \mathbb{N}$. Finally, we choose T^* so small that (1.7) is satisfied for all η_k.

Let $U_m = u_{m+1} - u_m$ and $E_m = \eta_{m+1} - \eta_m$. Then from (3.4) we obtain

$$\begin{aligned} &\eta_m U_{m,t} - \mu\Delta U_m - \nu\nabla \operatorname{div} U_m + p'(\eta_m)\nabla E_m = M && \text{in } \Omega^T, \\ &E_{m,t} + \eta_m \operatorname{div} U_m = N && \text{in } \Omega^T, \\ &U_m = \alpha(x_m,t) - \alpha(x_{m-1},t) && \text{on } S^T, \\ &U_m|_{t=0} = 0, \quad E_m|_{t=0} = 0 && \text{on } \Omega, \end{aligned} \tag{3.7}$$

where

$$
\begin{aligned}
M = & -E_{m-1}u_{m,t} - (p'(\eta_m) - p'(\eta_{m-1}))\nabla\eta_m + E_{m-1}f(x_m,t)\\
& + (f(x_m,t) - f(x_{m-1},t))\eta_{m-1}\\
& + (p'(\eta_m) - p'(\eta_{m-1}))(\nabla - \nabla_{u_m})\eta_{m+1}\\
& + p'(\eta_m)(\nabla - \nabla_{u_m})E_m + p'(\eta_m)(\nabla_{u_m} - \nabla_{u_{m-1}})\eta_m\\
& + \mu(\Delta_{u_m} - \Delta)U_m + \mu\,\mathrm{div}_{u_m}(\nabla_{u_m} - \nabla_{u_{m-1}})u_m\\
& + \mu(\mathrm{div}_{u_m} - \mathrm{div}_{u_{m-1}})\nabla_{u_{m-1}}u_m + \nu(\nabla_{u_m}\mathrm{div}_{u_m} - \nabla\,\mathrm{div})U_m\\
& + \nu\nabla_{u_m}(\mathrm{div}_{u_m} - \mathrm{div}_{u_{m-1}})u_m + \nu(\nabla_{u_m} - \nabla_{u_{m-1}})\,\mathrm{div}_{u_{m-1}}u_m,\\
N = {} & E_{m-1}(\mathrm{div} - \mathrm{div}_{u_m})u_{m+1} + \eta_{m-1}(\mathrm{div} - \mathrm{div}_{u_m})U_m\\
& + \eta_{m-1}(\mathrm{div}_{u_{m-1}} - \mathrm{div}_{u_m})u_m - E_{m-1}\,\mathrm{div}\,u_m.
\end{aligned}
$$

By Theorem 1.3 we get the estimate on U_m and E_m:

$$
\begin{aligned}
&(3.8)\qquad \|U_m\|_{W_r^{2,1}(\Omega^T)} + \|E_m\|_{V(\Omega^T)}\\
&\le c(\|M\|_{L_r(\Omega^T)} + \|N\|_{W_r^{1,0}(\Omega^T)} + \|\alpha(x_m,t) - \alpha(x_{m-1},t)\|_{W_r^{2-1/r,1-1/(2r)}(S^T)}).
\end{aligned}
$$

We have to estimate all the terms of the r.h.s. of (3.8). First we consider $M = \sum_{k=1}^{13} M_k$. By Proposition 2.2 and since $E_{m-1}|_{t=0} = 0$ we have

$$
\|M_1\|_{L_r(\Omega^T)} = \|E_{m-1}u_{m,t}\|_{L_r(\Omega^T)} \le cT^{\alpha}\|E_{m-1}\|_{V(\Omega^T)},
$$

and for $p'(\cdot) \in C^2$,

$$
\begin{aligned}
\|M_2\|_{L_r(\Omega^T)} &= \|(p'(\eta_m) - p'(\eta_{m-1}))\nabla\eta_m\|_{L_r(\Omega^T)} \le cT^{\alpha}\|E_{m-1}\|_{V(\Omega^T)},\\
\|M_3\|_{L_r(\Omega^T)} &= \|E_{m-1}f(x_m,t)\|_{L_r(\Omega^T)} \le cT^{\alpha}\|f\|_{L_r(\Omega^T)}\|E_{m-1}\|_{V(\Omega^T)}.
\end{aligned}
$$

Since $f \in L_r(0,T;W^1_\infty(\Omega))$ we have

$$
\begin{aligned}
\|M_4\|_{L_r(\Omega^T)} &= \|(f(x_m,t) - f(x_{m-1},t))\eta_{m-1}\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)},\\
\|M_5\|_{L_r(\Omega^T)} &= \|(p(\eta_m) - p'(\eta_{m-1}))(\nabla - \nabla_{u_m})\eta_{m+1}\|_{L_r(\Omega^T)}\\
&\le cT^{\alpha+(r-1)/r}\|E_{m-1}\|_{V(\Omega^T)},\\
\|M_6\|_{L_r(\Omega^T)} &= \|p'(\eta_m)(\nabla - \nabla_{u_m})E_m\|_{L_r(\Omega^T)} \le cT^{(r-1)/r}\|E_m\|_{V(\Omega^T)},\\
\|M_7\|_{L_r(\Omega^T)} &= \|p'(\eta_m)(\nabla_{u_m} - \nabla_{u_{m-1}})\eta_m\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)}.
\end{aligned}
$$

As in the case of K_3 we estimate

$$\begin{aligned}
\|M_8\|_{L_r(\Omega^T)} &= \|\mu(\Delta_{u_m}-\Delta)U_m\|_{L_r(\Omega^T)} \le cT^{(r-1)/r}\|U_m\|_{W_r^{2,1}(\Omega^T)},\\
\|M_9\|_{L_r(\Omega^T)} &= +\|\mu \operatorname{div}_{u_m}(\nabla_{u_m}-\nabla_{u_{m-1}})u_m\|_{L_r(\Omega^T)},\\
\|M_{10}\|_{L_r(\Omega^T)} &= \|\mu(\operatorname{div}_{u_m}-\operatorname{div}_{u_{m-1}})\nabla_{u_{m-1}}u_m\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)},\\
\|M_{11}\|_{L_r(\Omega^T)} &= \|\nu(\nabla_{u_m}\operatorname{div}_{u_m}-\nabla\operatorname{div})U_m\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_m\|_{W_r^{2,1}(\Omega^T)},\\
\|M_{12}\|_{L_r(\Omega^T)} &= \|\nu\nabla_{u_m}(\operatorname{div}_{u_m}-\operatorname{div}_{u_{m-1}})u_m\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)},\\
\|M_{13}\|_{L_r(\Omega^T)} &= \|\nu(\nabla_{u_m}-\nabla_{u_{m-1}})\operatorname{div}_{u_{m-1}}u_m\|_{L_r(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)}.
\end{aligned}$$

Now we estimate $N=\sum_{k=1}^4 N_k$:

$$\begin{aligned}
\|N_1\|_{W_r^{1,0}(\Omega^T)} &= \|E_{m-1}(\operatorname{div}-\operatorname{div}_{u_m})u_{m+1}\|_{W_r^{1,0}(\Omega^T)}\\
&\le cT^{(r-1)/r}\|E_{m-1}\|_{V(\Omega^T)},\\
\|N_2\|_{W_r^{1,0}(\Omega^T)} &= \|\eta_{m-1}(\operatorname{div}-\operatorname{div}_{u_m})U_m\|_{W_r^{1,0}(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_m\|_{W_r^{2,1}(\Omega^T)},\\
\|N_3\|_{W_r^{1,0}(\Omega^T)} &= \|\eta_{m-1}(\operatorname{div}_{u_{m-1}}-\operatorname{div}_{u_m})u_m\|_{W_r^{1,0}(\Omega^T)}\\
&\le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)},\\
\|N_4\|_{W_r^{1,0}(\Omega^T)} &= \|E_{m-1}\operatorname{div}u_m\|_{W_r^{1,0}(\Omega^T)}\\
&\le cT^{\alpha}\|E_{m-1}\|_{V(\Omega^T)}+\|\nabla E_{m-1}\operatorname{div}u_m\|_{L_r(\Omega^T)}.
\end{aligned}$$

To estimate the last term we note that $\operatorname{div}u_m\in W_r^{1,1/2}(\Omega^T)$ and

$$\begin{aligned}
\|\operatorname{div}u_m\|_{L_\infty(\Omega^T)} &\le c\sup_{t\le T}\|\operatorname{div}u_m\|_{W_r^{1-2/r}(\Omega^T)}\\
&\le \widetilde{c}(\|u_m\|_{W_r^{2,1}(\Omega^T)}+\|u_0\|_{W_r^{2-2/r}(\Omega)}),
\end{aligned}$$

where $\widetilde{c}$ does not depend on T. Hence we get

$$\begin{aligned}
&\|\nabla E_{m-1}\operatorname{div}u_m\|_{L_r(\Omega^T)}\\
&\quad\le c\|\operatorname{div}u_m\|_{L_\infty(\Omega^T)}\Big\|\int_0^t|\nabla E_{m-1,t}|\,dt'\Big\|_{L_r(\Omega^T)}\le cT^{(r-1)/r}\|E_{m-1}\|_{V(\Omega^T)}.
\end{aligned}$$

Thus

$$\|N_4\|_{W_r^{1,0}(\Omega^T)}\le c(T^{\alpha}+T^{(r-1)/r})\|E_{m-1}\|_{V(\Omega^T)}.$$

Now we estimate the r.h.s. of $(3.7)_3$. Since $\alpha \in C^3$ we have

$$\|\alpha(x_m,t)-\alpha(x_{m-1},t)\|_{W_r^{2-2/r,1-1/(2r)}(S^T)} \le cT^{(r-1)/r}\|U_{m-1}\|_{W_r^{2,1}(\Omega^T)}.$$

Let

$$Y_k = \|U_k\|_{W_r^{2,1}(\Omega^T)} + \|E_k\|_{V(\Omega^T)}.$$

Summing all estimates for the r.h.s. of (3.8) we obtain

$$Y_m \le c_6T^\alpha Y_{m-1} + c_7T^{(r-1)/r}Y_{m-1} + c_8Y_m. \tag{3.9}$$

Taking T so small that $c_8T^{(r-1)/r} < 1/2$, from (3.9) we get

$$Y_m \le c_9(T^\alpha + T^{(r-1)/r})Y_{m-1}.$$

If $c_9(T^\alpha + T^{(r-1)/r}) < 1$ we obtain $Y_k \to 0$ as $k \to \infty$.

Thus we have obtained (as $k \to \infty$)

$$\begin{aligned} &u_k \to u^* && \text{in } W_r^{2,1}(\Omega^T),\\ &\eta_k \to \eta^* && \text{in } V(\Omega^T),\\ &p'(\eta_k) \to p'(\eta^*) && \text{in } C^\alpha(\Omega^T). \end{aligned}$$

We conclude that (u^*,η^*) is a unique solution of (1.5) and the estimates (1.6) and (1.7) come from (3.6). We have obtained the local-in-time existence of solutions.

4. Almost global existence. First we show the boundedness of the L_2-norm of solutions of (1.9).

LEMMA 4.1. *If* $\|\sigma\|_{L_\infty(\Omega\times[0,t])}$ *is sufficiently small then*

$$\|v(\cdot,t)\|_{L_2(\Omega)} + \|\sigma(\cdot,t)\|_{L_2(\Omega)} \le A(\|v_0\|_{L_2(\Omega)} + \|\varrho_0-\overline{\varrho}\|_{L_2(\Omega)}), \tag{4.1}$$

where A *is a positive constant.*

Proof. Multiplying $(1.1)_1$ by v and integrating over Ω we get

$$\frac{d}{dt}\int_\Omega\left[\frac12\varrho v^2 + \frac{a}{\kappa-1}\varrho^\kappa - \varrho\varphi\right]dx + \int_\Omega[\mu|\nabla v|^2 + \nu|\operatorname{div} v|^2]\,dx = 0. \tag{4.2}$$

Next we examine

$$I(\sigma) = \int_\Omega\left(\frac{a}{\kappa-1}(\overline{\varrho}+\sigma)^\kappa - (\overline{\varrho}+\sigma)\varphi\right)dx - \int_\Omega\left(\frac{a}{\kappa-1}\overline{\varrho}^\kappa - \overline{\varrho}\varphi\right)dx.$$

It is easy to see that if $\|\sigma\|_{L_\infty}$ is sufficiently small then

$$a_1\|\sigma\|_{L_2(\Omega)} \le I(\sigma) \le a_2\|\sigma\|_{L_2(\Omega)}. \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$\frac{d}{dt}\left[\int_\Omega\frac12\varrho v^2\,dx + I(\sigma)\right] \le 0,$$

which gives (4.1) for σ sufficiently small.

Proof of Theorem 1.2. We examine (1.10) in the form

$$
\begin{aligned}
&\eta u_t - \mu\Delta u - \nu\nabla u + \gamma\nabla\chi = P,\\
&\chi_t + \eta \operatorname{div} u = Q,\\
&u|_{S^T} = 0,\\
&u|_{t=0} = v_0, \qquad \chi|_{t=0} = \sigma_0,
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
&P = \chi f - \chi\nabla_u\gamma - \mu(\Delta - \Delta_u)u - \nu(\nabla \operatorname{div} u - \nabla_u \operatorname{div}_u)u + \gamma(\nabla_u - \nabla)\chi,\\
&Q = \eta(\operatorname{div} - \operatorname{div}_u)u - \overline{\eta}_t.
\end{aligned}
$$

By Theorem 1.3 we get the following estimate for solutions of (4.4):

$$
\begin{aligned}
\|u\|_{W_r^{2,1}(\Omega\times[0,T])} + \|\eta\|_{V_r(\Omega\times[0,T])} &\le c(T, 4\overline{\varrho}^*, \overline{\varrho}_*/4)\\
\times [\|P\|_{L_r(\Omega\times[0,T])} + \|Q\|_{W_r^{1,0}(\Omega\times[0,T])} &+ \|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\sigma_0\|_{W_r^1(\Omega)}].
\end{aligned} \tag{4.5}
$$

From interpolation theorems we get

$$
\|\chi f\|_{L_r(\Omega\times[0,T])} \le \varepsilon\|\chi\|_{V_r(\Omega\times[0,T])} + c(\varepsilon)\|\sigma\|_{L_2(\Omega\times[0,T])}. \tag{4.6}
$$

Since

$$
\|\chi\|_{L_\infty(0,T;W_r^1(\Omega))} \le c(\|\chi\|_{V_r(\Omega\times[0,T])} + \|\sigma_0\|_{W_r^1(\Omega)}),
$$

we estimate

$$
\begin{aligned}
&\|\chi\nabla_u\gamma\|_{L_r(\Omega\times[0,T])}\\
&\quad\le cT^{1/r}\|\sigma_0\|_{W_r^1(\Omega)} + cT^{1/r}\varepsilon\|\chi\|_{V_r(\Omega\times[0,T])} + T^{1/r}c(\varepsilon)\|\sigma\|_{L_\infty(0,T;L_2(\Omega))}.
\end{aligned} \tag{4.7}
$$

The rest is estimate by

$$
\begin{aligned}
&\|(\Delta - \Delta_u)u\|_{L_r(\Omega\times[0,T])} \le cT^{(r-1)/r}\|u\|^2_{W_r^{2,1}(\Omega\times[0,T])},\\
&\|(\nabla \operatorname{div} u - \nabla_u \operatorname{div}_u)u\|_{L_r(\Omega\times[0,T])} \le cT^{(r-1)/r}\|u\|^2_{W_r^{2,1}(\Omega\times[0,T])},\\
&\|\gamma(\nabla_u - \nabla)\chi\|_{L_r(\Omega\times[0,T])} \le cT^{(r-1)/r}\|u\|_{W_r^{2,1}(\Omega\times[0,T])}\|\chi\|_{V_r(\Omega\times[0,T])},\\
&\|\eta(\operatorname{div} - \operatorname{div}_u)u\|_{W_r^{1,0}(\Omega\times[0,T])} \le cT^{(r-1)/r}\|u\|^2_{W_r^{2,1}(\Omega\times[0,T])}.
\end{aligned} \tag{4.8}
$$

Set

$$
\alpha = \|u\|_{W_r^{2,1}(\Omega\times[0,T])} + \|\chi\|_{V_r(\Omega\times[0,T])}.
$$

Taking ε sufficiently small, from (4.5)–(4.8) we get

$$
\alpha \le A\alpha^2 + \beta,
$$

where

$$
A = cT^{(r-1)/r}, \qquad \beta = c(\varepsilon, T)(\|v_0\|_{W_r^{2-2/r}(\Omega)} + \|\sigma_0\|_{W_r^1(\Omega)}),
$$

and if the initial data are small enough (so that $1-4A\beta>0$) then

$$\alpha \le \frac{1-\sqrt{1-4A\beta}}{2A} \le 2\beta. \tag{4.9}$$

Estimate (4.9) gives the boundedness from Theorem 1.2. To end the proof of Theorem 1.2 we have to note that since we have (4.9), by Theorem 1.1 we can continue the solutions onto the whole interval $[0,T]$. Theorem 1.2 is proved.

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