ANNALES
POLONICI MATHEMATICI
LXXVIII.3 (2002)

Periodic solutions of nth order delay Rayleigh equations

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Abstract. A priori bounds are established for periodic solutions of an nth order Rayleigh equation with delay. From these bounds, existence theorems for periodic solutions are established by means of Mawhin's continuation theorem.

In [4], a priori bounds for periodic solutions of the equation

$$x''(t) + \lambda f(x'(t)) + \lambda g(x(t - \tau(t))) = \lambda p(t), \qquad \lambda \in (0, 1), \tag{1}$$

are established under relatively simple conditions on f, g and p. Then by means of continuation theorems [1], periodic solutions for the Rayleigh differential equation

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = 0$$

are obtained.

In this note, we will be concerned with similar equations of the form

$$\begin{aligned} x^{(n)}(t) + \lambda f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) \\ + \lambda g(t, x(t - \tau_0(t))) = \lambda p(t), \end{aligned} \tag{2}$$

and

$$\begin{aligned} x^{(n)}(t) + f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) \\ + g(t, x(t - \tau_0(t))) = p(t) \end{aligned} \tag{3}$$

where $\lambda \in (0, 1)$, $n \geq 2$, $\tau_0, \ldots, \tau_{n-1}$ and p are T-periodic continuous functions defined on $\mathbb{R}$ with

$$\int_0^T p(t)\, dt = 0,$$

f is continuous on $\mathbb{R}^n$, $f(t, 0, \ldots, 0) = 0$ for $t \in \mathbb{R}$ and $f(t+T, x_1, \ldots, x_{n-1}) = f(t, x_1, \ldots, x_{n-1})$ for $(t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$, and g is continuous on $\mathbb{R}^2$

2000 *Mathematics Subject Classification*: 34K10, 34K13.

Key words and phrases: Rayleigh equation, delay, periodic solution.

such that $g(t+T,x)=g(t,x)$ for $(t,x)\in\mathbb{R}^2$. To avoid trivial cases, we also assume that the period T is positive.

We will establish a priori bounds for periodic solutions of equation (2) under several conditions imposed on f and g. Once these bounds are obtained, existence of periodic solutions for equation (3) can be demonstrated.

We remark that there are a number of studies which are concerned with the existence of periodic solutions of Rayleigh differential equations (see e.g. [2, 3, 5]). But our conditions are novel and relatively simple as compared to many others. For example, in [3], smoothness in addition to boundedness assumptions are needed for the functions in (1) in order to guarantee a periodic solution.

THEOREM 1. *Suppose there are constants $H\ge 0$, $D>0$ and $M>0$ such that*

(i) $|f(t,x_1,\ldots,x_{n-1})|\le H$ *for* $(t,x_1,\ldots,x_{n-1})\in\mathbb{R}^n$,
(ii) $xg(t,x)>0$ *and* $|g(t,x)|>H$ *for* $t\in\mathbb{R}$ *and* $|x|\ge D$, *and*
(iii) $|g(t,x)|\le M$ *for* $t\in\mathbb{R}$ *and* $x\le -D$.

Then there exist $D_0,\ldots,D_{n-1}>0$ *such that for any* T*-periodic solution* $x=x(t)$ *of* (2),

$$|x^{(j)}(t)|\le D_j,\qquad 0\le j\le n-1,\ 0\le t\le T.$$

Proof. Let $x=x(t)$ be a T-periodic solution of (2). In view of (2), and the periodicity of $x(t)$,

$$\int_0^T\{f(t,x'(t-\tau_1(t)),\ldots,x^{(n-1)}(t-\tau_{n-1}(t)))+g(t,x(t-\tau_0(t)))\}\,dt=0. \tag{4}$$

Note also that

$$\int_0^T|f(t,x'(t-\tau_1(t)),\ldots,x^{(n-1)}(t-\tau_{n-1}(t)))|\,dt\le TH. \tag{5}$$

Thus,

$$\begin{aligned}
(6)\quad &\int_0^T\{g(t,x(t-\tau_0(t)))-H\}\,dt\\
&\le\int_0^T\{g(t,x(t-\tau_0(t)))-|f(t,x'(t-\tau_1(t)),\ldots,x^{(n-1)}(t-\tau_{n-1}(t)))|\}\,dt\\
&\le\int_0^T\{g(t,x(t-\tau_0(t)))+f(t,x'(t-\tau_1(t)),\ldots,x^{(n-1)}(t-\tau_{n-1}(t)))\}\,dt=0.
\end{aligned}$$

Let

$$G_+(t) = \max\{g(t, x(t-\tau_0(t))) - H, 0\}, \quad t \in \mathbb{R},$$
$$G_-(t) = \max\{H - g(t, x(t-\tau_0(t))), 0\}, \quad t \in \mathbb{R}.$$

Then G_+ and G_- are nonnegative and continuous on $\mathbb{R}$,

$$g(t, x(t-\tau_0(t))) - H = G_+(t) - G_-(t), \quad t \in \mathbb{R}, \tag{7}$$

and in view of (ii) and (iii),

$$G_-(t) = |G_-(t)| \le H + M, \quad t \in \mathbb{R}.$$

In view of (6) and (7), we have

$$\int_0^T G_+(t)\,dt \le \int_0^T G_-(t)\,dt \le M_1$$

where $M_1 = (H+M)T$. In view of (7), we have

$$\int_0^T |g(t, x(t-\tau_0(t))) - H|\,dt \le 2M_1,$$

which implies

$$\int_0^T |g(t, x(t-\tau_0(t)))|\,dt \le 2M_1 + TH. \tag{8}$$

By integrating (2), in view of (5) and (8), we see that

$$\begin{aligned}\int_0^T |x^{(n)}(t)|\,dt &\le \int_0^T |f(t, x'(t-\tau_1(t)), \ldots, x^{(n-1)}(t-\tau_{n-1}(t)))|\,dt \\ &\quad + \int_0^T |g(t, x(t-\tau_0(t)))|\,dt + \int_0^T |p(t)|\,dt \\ &\le TH + 2M_1 + TH + T\max_{0\le t\le T}|p(t)|.\end{aligned}$$

Since $x^{(n-2)}(0) = x^{(n-2)}(T)$, there exists $t_1 \in [0,T]$ such that $x^{(n-1)}(t_1) = 0$. Thus

$$|x^{(n-1)}(t)| = \Big|\int_{t_1}^t x^{(n)}(s)\,ds\Big| \le \int_0^T |x^{(n)}(s)|\,ds \le D_{n-1}, \quad t \in [0,T],$$

where $D_{n-1} = TH + 2M_1 + TH + T\max_{0\le t\le T}|p(t)| > 0$. Next we will show that when $n > 2$, we have $|x^{(j)}(t)| \le D_j$ for $1 \le j \le n-2$ and $0 \le t \le T$. Indeed, since $x^{(n-3)}(0) = x^{(n-3)}(T)$, there exists $t_2 \in [0,T]$ such that $x^{(n-2)}(t_2) = 0$. As a consequence,

$$|x^{(n-2)}(t)| = \left| \int_{t_2}^{t} x^{(n-1)}(s)\,ds \right| \le \int_0^T |x^{(n-1)}(s)|\,ds \le TD_{n-1} \equiv D_{n-2}.$$

The rest of the proof follows by induction. To complete our proof, we will show that $|x(t)| \le D_0$, $t \in [0,T]$, for some $D_0 > 0$. Indeed, in view of (4),

$$f(t_3, x'(t_3 - \tau_1(t_3)), \ldots, x^{(n-1)}(t_3 - \tau_{n-1}(t_3))) + g(t_3, x(t_3 - \tau_0(t_3))) = 0$$

for some $t_3 \in [0,T]$. Hence by (i),

$$|g(t_3, x(t_3 - \tau_0(t_3)))| = |f(t_3, x'(t_3 - \tau_1(t_3)), \ldots, x^{(n-1)}(t_3 - \tau_{n-1}(t_3)))| \le H.$$

But then by (ii), $|x(t_3 - \tau_0(t_3))| < D$. Since $x(t)$ is T-periodic, there exists $t_4 \in [0,T]$ such that $|x(t_4)| < D$. Finally,

$$|x(t)| = \left| x(t_4) + \int_{t_4}^{t} x'(s)\,ds \right| \le D + \int_0^T |x'(s)|\,ds \le D + TD_1$$

for $t \in [0,T]$. The proof is complete.

Having the a priori bounds just obtained, we may follow the standard procedures as explained in various places of [1] and the continuation theorem on page 40 of [1] to show the existence of a periodic solution of (3). For completeness, a brief sketch is included.

Let X be the Banach space of all functions $x = x(t) \in C^{(n-1)}(\mathbb{R})$ such that $x(t+T) = x(t)$ for all t, endowed with the norm

$$\|x\| = \sum_{j=0}^{n-1} \max_{0 \le t \le T} |x^{(j)}(t)|.$$

Also let Y be the Banach space of all continuous functions of the form $y = y(t)$ defined on $\mathbb{R}$ such that $y(t+T) = y(t)$ for all t, and endowed with the norm $\|y\|_0 = \max_{0 \le t \le T} |y(t)|$. Now let $L : X \cap C^{(n)}(\mathbb{R}) \to Y$ be the operator defined by $(Lx)(t) = x^{(n)}(t)$ for $t \in \mathbb{R}$, and let $N : X \to Y$ be defined by

$$(Nx)(t) = -f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) - g(t, x(t - \tau_0(t))) + p(t)$$

for $t \in \mathbb{R}$. Let $\operatorname{Im} L$ and $\operatorname{Ker} L$ be respectively the image and kernel of the operator L. Clearly, $\operatorname{Ker} L = \mathbb{R}$. Furthermore, if we define the projections $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y/\operatorname{Im} L$ by

$$(Px)(t) = \frac{1}{T}\int_0^T x(t)\,dt, \qquad (Qy)(t) = \frac{1}{T}\int_0^T y(t)\,dt, \quad t \in \mathbb{R},$$

then $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, L is a Fredholm operator of index zero. The operator N is continuous and maps bounded subsets of X into bounded subsets of Y, thus for any bounded open subset Ω of X, $N(\overline{\Omega})$ is bounded. This shows that $(I-Q)N(\overline{\Omega})$ is bounded. Since the inverse K of $L_{|\operatorname{dom} L\cap\ker P}$ is compact, $K(I-Q)N(\overline{\Omega})$ is relatively compact, and so L-compact on the closure of Ω (see e.g. [1, pp. 166–187]).

Let $D, D_0, \ldots, D_{n-1}$ be as in Theorem 1, and let Ω be the subset of X consisting of the functions of the form $x = x(t)$ such that $\|x\| < \overline{D}$, where $\overline{D}$ is a fixed number which satisfies $\overline{D} > \max\{D_0, D_1, \ldots, D_{n-1}\} + D$. For any $\lambda \in (0,1)$ and any $x = x(t)$ in the domain of L which also belongs to $\partial\Omega$, we must have $Lx \neq \lambda Nx$. For otherwise in view of $\|x\| < \overline{D}$, x belongs to the interior of Ω, contrary to the assumption that $x \in \partial\Omega$. Next, note that a function $x = x(t) \in \operatorname{Ker} L \cap \partial\Omega$ must be the constant function $x(t) \equiv \overline{D}$ or $x(t) \equiv -\overline{D}$. Hence

$$\begin{aligned}
(QN)(x) &= \frac{1}{T}\int_0^T [-f(t, x'(t-\tau_1(t)), \ldots, x^{(n-1)}(t-\tau_{n-1}(t)))]\,dt \\
&\quad + \frac{1}{T}\int_0^T [-g(t, x(t-\tau_0(t)) + p(t)]\,dt \\
&= \frac{1}{T}\int_0^T [-f(t, 0, \ldots, 0) - g(t, x(t-\tau_0(t)))]\,dt \\
&= -\frac{1}{T}\int_0^T g(t, x(t-\tau_0(t)))\,dt = -\frac{1}{T}\int_0^T g(t,x)\,dt \neq 0.
\end{aligned}$$

Finally, consider the mapping

$$H(x,s) = sx + (1-s)\,\frac{1}{T}\int_0^T g(t,x)\,dt, \qquad 0 \le s \le 1.$$

Since for every $s \in [0,1]$ and $x \in \operatorname{Ker} L \cap \partial\Omega$, we have

$$xH(x,s) = sx^2 + (1-s)x\,\frac{1}{T}\int_0^T g(t,x)\,dt > 0,$$

$H(x,s)$ is an admissible homotopy. This shows that

$$\begin{aligned}
\deg\{QNx, \Omega\cap\operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T g(t,x)\,dt, \Omega\cap\operatorname{Ker} L, 0\right\} \\
&= \deg\{-x, \Omega\cap\operatorname{Ker} L, 0\} \\
&= \deg\{-x, \Omega\cap\mathbb{R}, 0\} \neq 0.
\end{aligned}$$

We have thus verified all the assumptions of the continuation theorem [1, p. 40]. Under the assumptions of Theorem 1, equation (3) thus has a T-periodic solution.

THEOREM 2. *Suppose the assumptions of Theorem* 1 *hold. Then equation* (3) *has a* T*-periodic solution.*

As an example, consider the equation

$$\begin{aligned} x'''(t) + \exp\{-\sin^2 t - (x'(t-\cos t))^2 - (x''(t-\sin t))^2\} \\ + (1+\cos^2 t)\arctan(x(t-\sin t))) \\ = \sin t + \exp(-\sin^2 t). \end{aligned}$$

Take

$$\begin{aligned} f(t,x_1,x_2) &= \exp(-\sin^2 t - x_1^2 - x_2^2) - \exp(-\sin^2 t), \\ g(t,x) &= (1+\cos^2 t)\arctan x, \end{aligned}$$

$\tau_0(t) = \sin t$, $\tau_1(t) = \cos t$, $\tau_2(t) = \sin t$, and $p(t) = \sin t$ and $T = 2\pi$. It is then easy to verify that all the assumptions of Theorem 1 are satisfied with $H = 1$, $D > \pi/4$ and $M = \pi$. Hence this equation has a 2π-periodic solution.

We remark that by symmetric arguments, we can establish the following existence theorem.

THEOREM 3. *Suppose there are constants* $H \geq 0$, $D > 0$ *and* $M > 0$ *such that* (i) $|f(t,x_1,\ldots,x_{n-1})| \leq H$ *for* $(t,x_1,\ldots,x_{n-1}) \in \mathbb{R}^n$, (ii) $xg(t,x) > 0$ *and* $|g(t,x)| > H$ *for* $t \in \mathbb{R}$ *and* $|x| \geq D$, *and* (iii) $|g(t,x)| \leq M$ *for* $t \in \mathbb{R}$ *and* $x \geq D$. *Then* (3) *has a* T*-periodic solution.*

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Reçu par la Rédaction le 15.1.2001
Révisé le 20.12.2001 (1234)