Periodic solutions of \( n \)th order delay Rayleigh equations

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Abstract. A priori bounds are established for periodic solutions of an \( n \)th order Rayleigh equation with delay. From these bounds, existence theorems for periodic solutions are established by means of Mawhin’s continuation theorem.

In [4], a priori bounds for periodic solutions of the equation

\[
x''(t) + \lambda f(x'(t)) + \lambda g(x(t - \tau(t))) = \lambda p(t), \quad \lambda \in (0, 1),
\]

are established under relatively simple conditions on \( f, g \) and \( p \). Then by means of continuation theorems [1], periodic solutions for the Rayleigh differential equation

\[
x''(t) + f(x'(t)) + g(x(t - \tau(t))) = 0
\]

are obtained.

In this note, we will be concerned with similar equations of the form

\[
x^{(n)}(t) + \lambda f(t; x(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) + \lambda g(t; x(t - \tau_0(t))) = \lambda p(t),
\]

and

\[
x^{(n)}(t) + f(t; x(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) + g(t; x(t - \tau_0(t))) = p(t)
\]

where \( \lambda \in (0, 1), n \geq 2, \tau_0, \ldots, \tau_{n-1} \) and \( p \) are \( T \)-periodic continuous functions defined on \( \mathbb{R} \) with

\[
\int_0^T p(t) \, dt = 0,
\]

\( f \) is continuous on \( \mathbb{R}^n, f(t; 0, \ldots, 0) = 0 \) for \( t \in \mathbb{R} \) and \( f(t + T; x_1, \ldots, x_{n-1}) = f(t; x_1, \ldots, x_{n-1}) \) for \( (t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n \), and \( g \) is continuous on \( \mathbb{R}^2 \)

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such that \( g(t + T, x) = g(t, x) \) for \((t, x) \in \mathbb{R}^2\). To avoid trivial cases, we also assume that the period \( T \) is positive.

We will establish a priori bounds for periodic solutions of equation (2) under several conditions imposed on \( f \) and \( g \). Once these bounds are obtained, existence of periodic solutions for equation (3) can be demonstrated.

We remark that there are a number of studies which are concerned with the existence of periodic solutions of Rayleigh differential equations (see e.g. [2, 3, 5]). But our conditions are novel and relatively simple as compared to many others. For example, in [3], smoothness in addition to boundedness assumptions are needed for the functions in (1) in order to guarantee a periodic solution.

**Theorem 1.** Suppose there are constants \( H \geq 0, D > 0 \) and \( M > 0 \) such that

1. \( |f(t, x_1, \ldots, x_{n-1})| \leq H \) for \((t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n\),
2. \( xg(t, x) > 0 \) and \( |g(t, x)| > H \) for \( t \in \mathbb{R} \) and \( |x| \geq D \), and
3. \( |g(t, x)| \leq M \) for \( t \in \mathbb{R} \) and \( x \leq -D \).

Then there exist \( D_0, \ldots, D_{n-1} > 0 \) such that for any \( T \)-periodic solution \( x = x(t) \) of (2),

\[
|x^{(j)}(t)| \leq D_j, \quad 0 \leq j \leq n - 1, \quad 0 \leq t \leq T.
\]

**Proof.** Let \( x = x(t) \) be a \( T \)-periodic solution of (2). In view of (2), and the periodicity of \( x(t) \),

\[
\int_0^T \{f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) + g(t, x(t - \tau_0(t)))\} \, dt = 0.
\]

Note also that

\[
\int_0^T |f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t)))| \, dt \leq TH.
\]

Thus,

\[
\int_0^T \{g(t, x(t - \tau_0(t))) - H\} \, dt
\leq \int_0^T \{g(t, x(t - \tau_0(t))) - |f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t)))|\} \, dt
\leq \int_0^T \{g(t, x(t - \tau_0(t))) + f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t)))\} \, dt = 0.
\]
Let
\[ G_+(t) = \max\{g(t, x(t - \tau_0(t))) - H, 0\}, \quad t \in \mathbb{R}, \]
\[ G_-(t) = \max\{H - g(t, x(t - \tau_0(t))), 0\}, \quad t \in \mathbb{R}. \]

Then \( G_+ \) and \( G_- \) are nonnegative and continuous on \( \mathbb{R} \),
\[ g(t, x(t - \tau_0(t))) - H = G_+(t) - G_-(t), \quad t \in \mathbb{R}, \]
and in view of (ii) and (iii),
\[ G_-(t) = |G_-(t)| \leq H + M, \quad t \in \mathbb{R}. \]

In view of (6) and (7), we have
\[
\int_0^T G_+(t) \, dt \leq \int_0^T G_-(t) \, dt \leq M_1
\]
where \( M_1 = (H + M)T \). In view of (7), we have
\[
\int_0^T |g(t, x(t - \tau_0(t))) - H| \, dt \leq 2M_1,
\]
which implies
\[ |g(t, x(t - \tau_0(t)))| \leq 2M_1 + TH. \]

By integrating (2), in view of (5) and (8), we see that
\[
\int_0^T |x^{(n)}(t)| \, dt \leq \int_0^T |f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t)))| \, dt
\]
\[ + \int_0^T |g(t, x(t - \tau_0(t)))| \, dt + \int_0^T |p(t)| \, dt
\]
\[ \leq TH + 2M_1 + TH + T \max_{0 \leq t \leq T} |p(t)|. \]

Since \( x^{(n-2)}(0) = x^{(n-2)}(T) \), there exists \( t_1 \in [0, T] \) such that \( x^{(n-1)}(t_1) = 0 \). Thus
\[ |x^{(n-1)}(t)| = \left| \int_{t_1}^t x^{(n)}(s) \, ds \right| \leq \int_{t_1}^T |x^{(n)}(s)| \, ds \leq D_{n-1}, \quad t \in [0, T], \]
where \( D_{n-1} = TH + 2M_1 + TH + T \max_{0 \leq t \leq T} |p(t)| > 0 \). Next we will show that when \( n > 2 \), we have \( |x^{(j)}(t)| \leq D_j \) for \( 1 \leq j \leq n - 2 \) and \( 0 \leq t \leq T \). Indeed, since \( x^{(n-3)}(0) = x^{(n-3)}(T) \), there exists \( t_2 \in [0, T] \) such
that $x^{(n-2)}(t_2) = 0$. As a consequence,

$$|x^{(n-2)}(t)| = \left| \int_{t_2}^{t} x^{(n-1)}(s)\, ds \right| \leq \int_{0}^{T} |x^{(n-1)}(s)|\, ds \leq TD_{n-1} = D_{n-2}.$$ 

The rest of the proof follows by induction. To complete our proof, we will show that $|x(t)| \leq D_0$, $t \in [0, T]$, for some $D_0 > 0$. Indeed, in view of (4),

$$f(t_3, x'(t_3 - \tau_1(t_3)), \ldots, x^{(n-1)}(t_3 - \tau_{n-1}(t_3))) + g(t_3, x(t_3 - \tau_0(t_3))) = 0$$

for some $t_3 \in [0, T]$. Hence by (i),

$$|g(t_3, x(t_3 - \tau_0(t_3)))| = |f(t_3, x'(t_3 - \tau_1(t_3)), \ldots, x^{(n-1)}(t_3 - \tau_{n-1}(t_3)))| \leq H.$$ 

But then by (ii), $|x(t_3 - \tau_0(t_3))| < D$. Since $x(t)$ is $T$-periodic, there exists $t_4 \in [0, T]$ such that $|x(t_4)| < D$. Finally,

$$|x(t)| = |x(t_4) + \int_{t_4}^{t} x'(s)\, ds| \leq D + \int_{0}^{T} |x'(t_4)|\, ds \leq D + TD_1$$

for $t \in [0, T]$. The proof is complete.

Having the a priori bounds just obtained, we may follow the standard procedures as explained in various places of [1] and the continuation theorem on page 40 of [1] to show the existence of a periodic solution of (3). For completeness, a brief sketch is included.

Let $X$ be the Banach space of all functions $x = x(t) \in C^{n-1}(\mathbb{R})$ such that $x(t + T) = x(t)$ for all $t$, endowed with the norm

$$\|x\| = \max_{0 \leq t \leq T} |x(j)(t)|.$$ 

Also let $Y$ be the Banach space of all continuous functions of the form $y = y(t)$ defined on $\mathbb{R}$ such that $y(t + T) = y(t)$ for all $t$, and endowed with the norm $\|y\|_0 = \max_{0 \leq t \leq T} |y(t)|$. Now let $L : X \cap C^{n}(\mathbb{R}) \to Y$ be the operator defined by $(Lx)(t) = x^{(n)}(t)$ for $t \in \mathbb{R}$, and let $N : X \to Y$ be defined by

$$(N x)(t) = -f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) - g(t, x(t - \tau_0(t))) + p(t)$$

for $t \in \mathbb{R}$. Let $\text{Im} L$ and $\text{Ker} L$ be respectively the image and kernel of the operator $L$. Clearly, $\text{Ker} L = \mathbb{R}$. Furthermore, if we define the projections $P : X \to \text{Ker} L$ and $Q : Y \to Y/\text{Im} L$ by

$$(Px)(t) = \frac{1}{T} \int_{0}^{T} x(t)\, dt, \quad (Qy)(t) = \frac{1}{T} \int_{0}^{T} y(t)\, dt,$$

then $\text{Ker} L = \text{Im} P$ and $\text{Ker} Q = \text{Im} L$. Furthermore, $L$ is a Fredholm operator of index zero. The operator $N$ is continuous and maps bounded subsets
of $X$ into bounded subsets of $Y$, thus for any bounded open subset $\Omega$ of $X$, $N(\partial \Omega)$ is bounded. This shows that $(I - Q)N(\partial \Omega)$ is bounded. Since the inverse $K$ of $L|_{\text{dom } L \cap \ker P}$ is compact, $K(I - Q)N(\partial \Omega)$ is relatively compact, and so $L$-compact on the closure of $\Omega$ (see e.g. [1, pp. 166–187]).

Let $D, D_0, \ldots, D_{n-1}$ be as in Theorem 1, and let $\Omega$ be the subset of $X$ consisting of the functions of the form $x = x(t)$ such that $\|x\| < \overline{D}$, where $\overline{D}$ is a fixed number which satisfies $\overline{D} > \max\{D_0, D_1, \ldots, D_{n-1}\} + D$. For any $\lambda \in (0, 1)$ and any $x = x(t)$ in the domain of $L$ which also belongs to $\partial \Omega$, we must have $Lx \neq \lambda Nx$. For otherwise in view of $\|x\| < \overline{D}$, $x$ belongs to the interior of $\Omega$, contrary to the assumption that $x \in \partial \Omega$. Next, note that a function $x = x(t) \in \ker L \cap \partial \Omega$ must be the constant function $x(t) \equiv \overline{D}$ or $x(t) \equiv -\overline{D}$. Hence

$$
(QN)(x) = \frac{1}{T} \int_0^T \left[ -f(t, x'(t - \tau_1(t)), \ldots, x^{(n-1)}(t - \tau_{n-1}(t))) \right] dt
$$

$$
+ \frac{1}{T} \int_0^T [-g(t, x(t - \tau_0(t)) + p(t)] dt
$$

$$
= \frac{1}{T} \int_0^T \left[ -f(t, 0, \ldots, 0) - g(t, x(t - \tau_0(t))) \right] dt
$$

$$
= -\frac{1}{T} \int_0^T g(t, x(t - \tau_0(t))) \ dt = -\frac{1}{T} \int_0^T g(t, x) \ dt \neq 0.
$$

Finally, consider the mapping

$$
H(x, s) = sx + (1 - s) \frac{1}{T} \int_0^T g(t, x) \ dt, \quad 0 \leq s \leq 1.
$$

Since for every $s \in [0, 1]$ and $x \in \ker L \cap \partial \Omega$, we have

$$
xH(x, s) = sx^2 + (1 - s)x \frac{1}{T} \int_0^T g(t, x) \ dt > 0,
$$

$H(x, s)$ is an admissible homotopy. This shows that

$$
\deg \{QN x, \Omega \cap \ker L, 0\} = \deg \left\{ -\frac{1}{T} \int_0^T g(t, x) \ dt, \Omega \cap \ker L, 0 \right\}
$$

$$
= \deg \{-x, \Omega \cap \ker L, 0\}
$$

$$
= \deg \{-x, \Omega \cap \mathbb{R}, 0\} \neq 0.
$$

We have thus verified all the assumptions of the continuation theorem [1, p. 40]. Under the assumptions of Theorem 1, equation (3) thus has a $T$-periodic solution.
THEOREM 2. Suppose the assumptions of Theorem 1 hold. Then equation (3) has a $T$-periodic solution.

As an example, consider the equation

$$x'''(t) + \exp\{-\sin^2 t - (x'(t - \cos t))^2 - (x''(t - \sin t))^2\}
+ (1 + \cos^2 t) \arctan(x(t - \sin t))
= \sin t + \exp(-\sin^2 t).$$

Take

$$f(t, x_1, x_2) = \exp(-\sin^2 t - x_1^2 - x_2^2) - \exp(-\sin^2 t),$$
$$g(t, x) = (1 + \cos^2 t) \arctan x,$$
$$\tau_0(t) = \sin t, \quad \tau_1(t) = \cos t, \quad \tau_2(t) = \sin t, \text{ and } p(t) = \sin t \text{ and } T = 2\pi.$$ 

It is then easy to verify that all the assumptions of Theorem 1 are satisfied with $H = 1$, $D > \pi/4$ and $M = \pi$. Hence this equation has a $2\pi$-periodic solution.

We remark that by symmetric arguments, we can establish the following existence theorem.

THEOREM 3. Suppose there are constants $H \geq 0$, $D > 0$ and $M > 0$ such that (i) $|f(t, x_1, \ldots, x_{n-1})| \leq H$ for $(t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$, (ii) $xg(t, x) > 0$ and $|g(t, x)| > H$ for $t \in \mathbb{R}$ and $|x| \geq D$, and (iii) $|g(t, x)| \leq M$ for $t \in \mathbb{R}$ and $x \geq D$. Then (3) has a $T$-periodic solution.

References


