Concave domains with trivial biholomorphic invariants

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Abstract. It is proved that if $F$ is a convex closed set in $\mathbb{C}^n$, $n \geq 2$, containing at most one $(n-1)$-dimensional complex hyperplane, then the Kobayashi metric and the Lempert function of $\mathbb{C}^n \setminus F$ identically vanish.

Let $D$ be a domain in $\mathbb{C}^n$. Denote by $\mathcal{O}(\mathbb{C}, D)$ and $\mathcal{O}(\Delta, D)$ the spaces of all holomorphic mappings from $\mathbb{C}$ to $D$ and from the unit disc $\Delta \subset \mathbb{C}$ to $D$, respectively. Let $z, w \in D$ and $X \in \mathbb{C}^n$. The Kobayashi metric and Lempert function are defined by (cf. [1])

$$K_D(z, X) = \inf \{ |\alpha|^{-1} : \exists f \in \mathcal{O}(\Delta, D), \ f(0) = z, \ f'(0) = \alpha X \},$$

$$\ell_D(z, w) = \inf \{ \tanh^{-1} |\alpha| : \exists f \in \mathcal{O}(\Delta, D), \ f(0) = z, \ f(\alpha) = w \}.$$ 

These invariants can be characterized as the largest metric and function which decrease under holomorphic mappings and coincide with the Poincaré metric and distance on $\Delta$.

It is well known that if $D$ is a bounded domain in $\mathbb{C}^n$, or a plane domain whose complement contains at least two points, then $K_D(z, X) > 0$ for $X \neq 0$ and $\ell_D(z, w) > 0$ for $z \neq w$. On the other hand, the Kobayashi metric and the Lempert function of a plane domain whose complement contains at most one point identically vanish. Note also that there are domains in $\mathbb{C}^n$ with bounded connected complements and non-vanishing Kobayashi metrics and Lempert functions. For example, if $z_0$ is a strictly pseudoconvex boundary point of a domain $D$ in $\mathbb{C}^n$, $n \geq 2$, then (cf. [2])

$$\lim_{z \to z_0} \frac{K_D(z, X)}{||X||} = \infty$$

uniformly in $X \in \mathbb{C}^n \setminus \{0\}$, and

$$\lim_{z \to z_0} \inf_{w \in D \setminus U} \ell_D(z, w) = \infty$$

for any neighborhood $U$ of $z_0$.

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A set $D$ in $\mathbb{C}^n$ is called **concave** if its complement $\mathbb{C}^n \setminus D$ is a convex set. The purpose of this note is to characterize the concave domains in $\mathbb{C}^n$, $n \geq 2$, whose Kobayashi metrics and Lempert functions identically vanish.

**Theorem 1.** Let $D$ be a concave domain in $\mathbb{C}^n$, $n \geq 2$. Then the following statements are equivalent:

(i) $\mathbb{C}^n \setminus D$ contains at most one $(n-1)$-dimensional complex hyperplane;

(ii) for any $z \in D$, $X \in \mathbb{C}^n \setminus \{0\}$ there is an injective $f \in \mathcal{O}(\mathbb{C}, D)$ such that $f(0) = z$, $f'(0) = X$;

(iii) for any $z, w \in D$, $z \neq w$ there is an injective $f \in \mathcal{O}(\mathbb{C}, D)$ such that $f(0) = z$, $f(1) = w$;

(iv) $K_D \equiv 0$;

(v) $\ell_D \equiv 0$.

**Proof.** The implications (ii)$\Rightarrow$(iv) and (iii)$\Rightarrow$(v) are trivial.

The implications (iv)$\Rightarrow$(i) and (v)$\Rightarrow$(i) follow from the fact that $\Delta$ is the universal covering of $\mathbb{C} \setminus \{0, 1\}$.

Now, we prove that (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii).

If $F := \mathbb{C}^n \setminus D$ contains exactly one complex hyperplane, we may assume that $D = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1}$ and $z = (1, 0, \ldots, 0)$. Let $X = (X_1, \ldots, X_n)$, $X \neq 0$, and $w = (w_1, \ldots, w_n)$. For any integer $j \in \{2, \ldots, n\}$, set

$$f_j(\eta) = \begin{cases} X_j \eta & \text{if } X_j \neq 0, \\
\exp(X_1 \eta) - 1 - X_1 \eta & \text{if } X_j = 0, 
\end{cases}$$

and

$$\tilde{f}_j(\eta) = \begin{cases} w_j \eta & \text{if } w_j \neq 0, \\
w_1 \eta - 1 + (1 - w_1) \eta & \text{if } w_j = 0. 
\end{cases}$$

Then $f(\eta) = (\exp(X_1 \eta), f_2(\eta), \ldots, f_n(\eta))$ and $\tilde{f}(\eta) = (w_1, \tilde{f}_2(\eta), \ldots, \tilde{f}_n(\eta))$ are injective holomorphic mappings from $\mathbb{C}$ to $D$ such that $f(0) = z$, $f'(0) = X$ and $\tilde{f}(0) = z$, $\tilde{f}(1) = w$.

Assume now that $F$ contains no $(n-1)$-dimensional complex hyperplanes and let $z \in D$. Since $F$ coincides with its hull with respect to the real-valued linear functions on $\mathbb{C}^n$, there are two polynomials $\ell_1, \ell_2 : \mathbb{C}^n \to \mathbb{C}$ of degree 1 such that $\ell_1 - \ell_1(0)$ and $\ell_2 - \ell_2(0)$ are linearly independent and

$$\Re(\ell_1(z)) > 0 = \max_F \Re(\ell_1) = \max_F \Re(\ell_2).$$

Replacing $\ell_2$ by $\ell_1 + \varepsilon \ell_2$, where $\varepsilon > 0$ is small enough, we may assume that

$$\Re(\ell_2(z)) > 0 \geq \max_F \Re(\ell_2).$$

So, if $D \ni z = (z_1, \ldots, z_n)$ and $\mathbb{C}^n \setminus \{0\} \ni X = (X_1, \ldots, X_n)$, after a translation and a linear change of coordinates, we may assume that $\Re(z_1) > 0$, $\Re(z_2) > 0$ and

$$F \subset G := \{\zeta \in \mathbb{C}^n : \Re(\zeta_1) \leq 0, \Re(\zeta_2) \leq 0\}.$$
If $X_1 = X_2 = 0$, then the mapping $f(\eta) = z + \eta X$ has the properties required in (ii). Otherwise, we may assume that $X_2 \neq 0$ and for $\lambda > 0$, set

$$\varphi(t) = \begin{cases} 
\frac{z_2 X_1}{X_2 \lambda (1 - \exp(z_2 \lambda))} & \text{for } t \in [0, \lambda], \\
\frac{z_2 X_1}{X_2 \lambda \exp(z_2 \lambda)(\exp(z_2 \lambda) - 1)} & \text{for } t \in (\lambda, 2\lambda],
\end{cases}$$

$$f_j(\eta) = z_j + \eta X_j \quad \text{for } j = 2, \ldots, n,$$

$$f_1(\eta) = z_1 + \int_0^{2\lambda} \varphi(t) \exp(tf_2(\eta)) \, dt.$$ 

Then $f = (f_1, \ldots, f_n)$ is an injective holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}^n$ and $f(0) = z$, $f'(0) = X$. Note that if $\text{Re}(f_2(\eta)) \leq 0$, then

$$|f_1(\eta) - z_1| \leq \int_0^{2\lambda} |\varphi(t)| \, dt.$$ 

Since the last integral tends to 0 as $\lambda \to \infty$, it follows that $f \in O(\mathbb{C}, D)$ for any $\lambda \gg 1$, which completes the proof of (i)$\implies$(ii).

Let now $z, w \in D$ and $z \neq w$. As above, we may assume that $\text{Re}(z_2) > 0$, $\text{Re}(w_1) > 0$ and $F \subset G$. If $z_1 = w_1$ or $z_2 = w_2$, then the mapping $f(\eta) = z + \eta (w - z)$ has the properties required in (iii). Otherwise, we may assume that $w_2 \neq z_2$ and, for $m \in \mathbb{N}$, set

$$\lambda = \frac{(2m - 1)\pi}{|z_2 - w_2|},$$

$$\varphi(t) = \begin{cases} 
\frac{z_2(z_1 - w_1)}{(\exp(z_2 \lambda) - 1)(\exp(w_2 \lambda) - \exp(z_2 \lambda))} & \text{for } t \in [0, \lambda], \\
\frac{z_2(z_1 - w_1)}{(\exp(z_2 \lambda) - 1)(\exp(z_2 \lambda) - \exp(w_2 \lambda))} & \text{for } t \in (\lambda, 2\lambda],
\end{cases}$$

$$f_j(\eta) = z_j + \eta (w_j - z_j) \quad \text{for } j = 2, \ldots, n,$$

$$f_1(\eta) = w_1 + \int_0^{2\lambda} \varphi(t) \exp(tf_2(\eta)) \, dt.$$ 

It follows as above that for any $\lambda \gg 1$, $f = (f_1, \ldots, f_n)$ is an injective holomorphic mapping from $\mathbb{C}$ to $D$ with $f(0) = z$, $f(1) = w$. Taking $m$ large enough completes the proof of (i)$\implies$(iii).

Theorem 1 implies the following

**Corollary 2.** Let $F$ be the Cartesian product of $n$ closed subsets $F_1, \ldots, F_n$ of $\mathbb{C}$ $(n \geq 2)$. Assume that $F_1 \neq \mathbb{C}$ and $F_n \neq \mathbb{C}$. Then
(i) for any \( z \in D := \mathbb{C}^n \setminus F \) and any \( X \in \mathbb{C}^n \) there is an \( f \in \mathcal{O}(\mathbb{C}, D) \) such that \( f(0) = z \) and \( f'(0) = X \);

(ii) for any \( z \in D_1 = (\mathbb{C} \setminus F_1) \times \mathbb{C}^{n-1} \) and any \( w \in D_n = \mathbb{C}^{n-1} \times (\mathbb{C} \setminus F_n) \) there is a \( g \in \mathcal{O}(\mathbb{C}, D) \) such that \( g(0) = z \) and \( g(1) = w \).

In particular, \( D \) is a domain in \( \mathbb{C}^n \), \( K_D \equiv 0 \), and \( \ell_D = 0 \) on \( D_1 \times D_n \).

Proof. Let \( \mathcal{C}_* = \mathbb{C} \setminus \{0\}, \Delta_* = \{\eta \in \mathbb{C} : 0 < |\eta| < 1\} \) and \( H = \{\eta \in \mathbb{C} : \text{Re}(\eta) \geq 0\} \). Without loss of generality, we may suppose in (i) that \( z_1 \notin F_1 \).

After a translation and a linear change of coordinates, we may assume that \( z \in G_1 := \Delta_* \times \mathbb{C}^{n-2} \times \mathcal{C}_*, w \in G_n := \mathcal{C}_* \times \mathbb{C}^{n-2} \times \Delta_* \) and \( G_1 \subset D_1, G_n \subset D_n \). Since \( \mathbb{C}^n \setminus (H \times \mathbb{C}^{n-2} \times H) \) is a covering of \( G_1 \cup G_n \), Corollary 2 follows from Theorem 1.

Remark. The authors do not know if part (ii) of Corollary 1 still holds for any two different points \( z, w \in D \). (Added in proof: Cf. N. Nikolov, Entire curves in complements of cartesian products in \( \mathbb{C}^n \), Univ. Iag. Acta Math., to appear.)

References


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