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Concave domains with trivial biholomorphic invariants

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Abstract. It is proved that if F is a convex closed set in $\mathbb{C}^n$, $n \geq 2$, containing at most one $(n-1)$-dimensional complex hyperplane, then the Kobayashi metric and the Lempert function of $\mathbb{C}^n \setminus F$ identically vanish.

Let D be a domain in $\mathbb{C}^n$. Denote by $\mathcal{O}(\mathbb{C}, D)$ and $\mathcal{O}(\Delta, D)$ the spaces of all holomorphic mappings from $\mathbb{C}$ to D and from the unit disc $\Delta \subset \mathbb{C}$ to D, respectively. Let $z, w \in D$ and $X \in \mathbb{C}^n$. The Kobayashi metric and Lempert function are defined by (cf. [1])

$$K_D(z, X) = \inf\{|\alpha|^{-1} : \exists f \in \mathcal{O}(\Delta, D),\ f(0) = z,\ f'(0) = \alpha X\},$$

$$\ell_D(z, w) = \inf\{\tanh^{-1} |\alpha| : \exists f \in \mathcal{O}(\Delta, D),\ f(0) = z,\ f(\alpha) = w\}.$$

These invariants can be characterized as the largest metric and function which decrease under holomorphic mappings and coincide with the Poincaré metric and distance on Δ.

It is well known that if D is a bounded domain in $\mathbb{C}^n$, or a plane domain whose complement contains at least two points, then $K_D(z, X) > 0$ for $X \neq 0$ and $\ell_D(z, w) > 0$ for $z \neq w$. On the other hand, the Kobayashi metric and the Lempert function of a plane domain whose complement contains at most one point identically vanish. Note also that there are domains in $\mathbb{C}^n$ with bounded connected complements and non-vanishing Kobayashi metrics and Lempert functions. For example, if z_0 is a strictly pseudoconvex boundary point of a domain D in $\mathbb{C}^n$, $n \geq 2$, then (cf. [2])

$$\lim_{z \to z_0} \frac{K_D(z, X)}{\|X\|} = \infty$$

uniformly in $X \in \mathbb{C}^n \setminus \{0\}$, and

$$\lim_{z \to z_0} \inf_{w \in D \setminus U} \ell_D(z, w) = \infty$$

for any neighborhood U of z_0.

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A set D in $\mathbb{C}^n$ is called *concave* if its complement $\mathbb{C}^n \setminus D$ is a convex set. The purpose of this note is to characterize the concave domains in $\mathbb{C}^n$, $n \geq 2$, whose Kobayashi metrics and Lempert functions identically vanish.

THEOREM 1. *Let D be a concave domain in $\mathbb{C}^n$, $n \geq 2$. Then the following statements are equivalent*:

(i) *$\mathbb{C}^n \setminus D$ contains at most one $(n-1)$-dimensional complex hyperplane*;
(ii) *for any $z \in D$, $X \in \mathbb{C}^n \setminus \{0\}$ there is an injective $f \in \mathcal{O}(\mathbb{C}, D)$ such that $f(0) = z$, $f'(0) = X$*;
(iii) *for any $z, w \in D$, $z \neq w$ there is an injective $f \in \mathcal{O}(\mathbb{C}, D)$ such that $f(0) = z$, $f(1) = w$*;
(iv) *$K_D \equiv 0$*;
(v) *$\ell_D \equiv 0$.*

Proof. The implications (ii)⇒(iv) and (iii)⇒(v) are trivial.

The implications (iv)⇒(i) and (v)⇒(i) follow from the fact that Δ is the universal covering of $\mathbb{C} \setminus \{0, 1\}$.

Now, we prove that (i)⇒(ii) and (i)⇒(iii).

If $F := \mathbb{C}^n \setminus D$ contains exactly one complex hyperplane, we may assume that $D = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1}$ and $z = (1, 0, \ldots, 0)$. Let $X = (X_1, \ldots, X_n)$, $X \neq 0$, and $w = (w_1, \ldots, w_n)$. For any integer $j \in \{2, \ldots, n\}$, set

$$f_j(\eta) = \begin{cases} X_j \eta & \text{if } X_j \neq 0, \\ \exp(X_1 \eta) - 1 - X_1 \eta & \text{if } X_j = 0, \end{cases}$$

and

$$\widetilde{f}_j(\eta) = \begin{cases} w_j \eta & \text{if } w_j \neq 0, \\ w_1^\eta - 1 + (1 - w_1)\eta & \text{if } w_j = 0. \end{cases}$$

Then $f(\eta) = (\exp(X_1 \eta), f_2(\eta), \ldots, f_n(\eta))$ and $\widetilde{f}(\eta) = (w_1^\eta, \widetilde{f}_2(\eta), \ldots, \widetilde{f}_n(\eta))$ are injective holomorphic mappings from $\mathbb{C}$ to D such that $f(0) = z$, $f'(0) = X$ and $\widetilde{f}(0) = z$, $\widetilde{f}(1) = w$.

Assume now that F contains no $(n-1)$-dimensional complex hyperplanes and let $z \in D$. Since F coincides with its hull with respect to the real-valued linear functions on $\mathbb{C}^n$, there are two polynomials $\ell_1, \ell_2 : \mathbb{C}^n \to \mathbb{C}$ of degree 1 such that $\ell_1 - \ell_1(0)$ and $\ell_2 - \ell_2(0)$ are linearly independent and

$$\operatorname{Re}(\ell_1(z)) > 0 = \max_F \operatorname{Re}(\ell_1) = \max_F \operatorname{Re}(\ell_2).$$

Replacing ℓ_2 by $\ell_1 + \varepsilon \ell_2$, where $\varepsilon > 0$ is small enough, we may assume that

$$\operatorname{Re}(\ell_2(z)) > 0 \geq \max_F \operatorname{Re}(\ell_2).$$

So, if $D \ni z = (z_1, \ldots, z_n)$ and $\mathbb{C}^n \setminus \{0\} \ni X = (X_1, \ldots, X_n)$, after a translation and a linear change of coordinates, we may assume that $\operatorname{Re}(z_1) > 0$, $\operatorname{Re}(z_2) > 0$ and

$$F \subset G := \{\zeta \in \mathbb{C}^n : \operatorname{Re}(\zeta_1) \leq 0,\ \operatorname{Re}(\zeta_2) \leq 0\}.$$

If $X_1 = X_2 = 0$, then the mapping $f(\eta) = z + \eta X$ has the properties required in (ii). Otherwise, we may assume that $X_2 \neq 0$ and for $\lambda > 0$, set

$$\varphi(t) = \begin{cases} \dfrac{z_2 X_1}{X_2 \lambda (1 - \exp(z_2\lambda))} & \text{for } t \in [0, \lambda], \\ \dfrac{z_2 X_1}{X_2 \lambda \exp(z_2\lambda)(\exp(z_2\lambda) - 1)} & \text{for } t \in (\lambda, 2\lambda], \end{cases}$$

$$f_j(\eta) = z_j + \eta X_j \quad \text{for } j = 2, \ldots, n,$$

$$f_1(\eta) = z_1 + \int_0^{2\lambda} \varphi(t) \exp(t f_2(\eta))\, dt.$$

Then $f = (f_1, \ldots, f_n)$ is an injective holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}^n$ and $f(0) = z$, $f'(0) = X$. Note that if $\mathrm{Re}(f_2(\eta)) \le 0$, then

$$|f_1(\eta) - z_1| \le \int_0^{2\lambda} |\varphi(t)|\, dt.$$

Since the last integral tends to 0 as $\lambda \to \infty$, it follows that $f \in \mathcal{O}(\mathbb{C}, D)$ for any $\lambda \gg 1$, which completes the proof of (i)⇒(ii).

Let now $z, w \in D$ and $z \neq w$. As above, we may assume that $\mathrm{Re}(z_2) > 0$, $\mathrm{Re}(w_1) > 0$ and $F \subset G$. If $z_1 = w_1$ or $z_2 = w_2$, then the mapping $f(\eta) = z + \eta(w - z)$ has the properties required in (iii). Otherwise, we may assume that $w_2 \neq z_2$ and, for $m \in \mathbb{N}$, set

$$\lambda = \frac{(2m-1)\pi}{|z_2 - w_2|},$$

$$\varphi(t) = \begin{cases} \dfrac{z_2(z_1 - w_1)\exp(w_2\lambda)}{(\exp(z_2\lambda) - 1)(\exp(w_2\lambda) - \exp(z_2\lambda))} & \text{for } t \in [0, \lambda], \\ \dfrac{z_2(z_1 - w_1)}{(\exp(z_2\lambda) - 1)(\exp(z_2\lambda) - \exp(w_2\lambda))} & \text{for } t \in (\lambda, 2\lambda], \end{cases}$$

$$f_j(\eta) = z_j + \eta(w_j - z_j) \quad \text{for } j = 2, \ldots, n,$$

$$f_1(\eta) = w_1 + \int_0^{2\lambda} \varphi(t) \exp(t f_2(\eta))\, dt.$$

It follows as above that for any $\lambda \gg 1$, $f = (f_1, \ldots, f_n)$ is an injective holomorphic mapping from $\mathbb{C}$ to D with $f(0) = z$, $f(1) = w$. Taking m large enough completes the proof of (i)⇒(iii). ■

Theorem 1 implies the following

COROLLARY 2. *Let F be the Cartesian product of n closed subsets $F_1, \ldots, F_n$ of $\mathbb{C}$ $(n \ge 2)$. Assume that $F_1 \neq \mathbb{C}$ and $F_n \neq \mathbb{C}$. Then*

(i) *for any* $z \in D := \mathbb{C}^n \setminus F$ *and any* $X \in \mathbb{C}^n$ *there is an* $f \in \mathcal{O}(\mathbb{C}, D)$ *such that* $f(0) = z$ *and* $f'(0) = X$;

(ii) *for any* $z \in D_1 = (\mathbb{C} \setminus F_1) \times \mathbb{C}^{n-1}$ *and any* $w \in D_n = \mathbb{C}^{n-1} \times (\mathbb{C} \setminus F_n)$ *there is a* $g \in \mathcal{O}(\mathbb{C}, D)$ *such that* $g(0) = z$ *and* $g(1) = w$.

In particular, D *is a domain in* $\mathbb{C}^n$, $K_D \equiv 0$, *and* $\ell_D = 0$ *on* $D_1 \times D_n$.

Proof. Let $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$, $\Delta_* = \{\eta \in \mathbb{C} : 0 < |\eta| < 1\}$ and $H = \{\eta \in \mathbb{C} : \mathrm{Re}(\eta) \geq 0\}$. Without loss of generality, we may suppose in (i) that $z_1 \notin F_1$. After a translation and a linear change of coordinates, we may assume that $z \in G_1 := \Delta_* \times \mathbb{C}^{n-2} \times \mathbb{C}_*$, $w \in G_n := \mathbb{C}_* \times \mathbb{C}^{n-2} \times \Delta_*$ and $G_1 \subset D_1$, $G_n \subset D_n$. Since $\mathbb{C}^n \setminus (H \times \mathbb{C}^{n-2} \times H)$ is a covering of $G_1 \cup G_n$, Corollary 2 follows from Theorem 1. ∎

Remark. The authors do not know if part (ii) of Corollary 1 still holds for any two different points $z, w \in D$. (*Added in proof*: Cf. N. Nikolov, *Entire curves in complements of cartesian products in* $\mathbb{C}^n$, Univ. Iag. Acta Math., to appear.)

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