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On the class of functions defined in a halfplane and starlike with respect to the boundary point

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Abstract. The purpose of this paper is to study the class $\mathcal{S}^*_\infty(\mathbb{H})$ of univalent analytic functions defined in the right halfplane $\mathbb{H}$ and starlike w.r.t. the boundary point at infinity. An analytic characterization of functions in $\mathcal{S}^*_\infty(\mathbb{H})$ is presented.

1. Introduction. The aim of this paper is to study the class $\mathcal{S}^*_\infty(\mathbb{H})$ of univalent analytic functions defined in the right halfplane $\mathbb{H}$ and starlike w.r.t. the boundary point at infinity.

The geometric definition of functions in $\mathcal{S}^*_\infty(\mathbb{H})$ is clear: together with every point $w \in f(\mathbb{H})$, the radial halfline with endpoint at w lies in $f(\mathbb{H})$. This property was considered in [4, 6] where properties of functions in $\mathcal{S}^*_\infty(\mathbb{H})$ with the additional (so called) hydrodynamic normalization and a generalization were examined.

In this paper, developing another method based on the Julia Lemma reformulated in $\mathbb{H}$, we are able to improve the results of [6] considerably. The main results are presented in Theorems 3.1–3.3. Instead of the class P used in [6] of functions having positive real part in $\mathbb{H}$ and hydrodynamically normalized we introduce the class $\mathcal{P}(\alpha;\mathbb{H})$ of functions also having positive real part in $\mathbb{H}$ but with the boundary property which follows from the Julia Lemma. It is shown that $\mathcal{P}(\alpha;\mathbb{H})$ is essentially larger than P. The key geometric property of functions in $\mathcal{S}^*_\infty(\mathbb{H})$ is preserving starlikeness in every halfplane contained in $\mathbb{H}$; this is proved in Theorem 3.1. Having this inner geometric behaviour we are able to find an analytic characterization of the class $\mathcal{S}^*_\infty(\mathbb{H})$ in terms of the corresponding class $\mathcal{P}(\alpha;\mathbb{H})$; this is the subject of Theorems 3.2 and 3.3.

Examples of functions show usefulness of these results.

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2. The Julia Lemmas. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$. For each $k > 0$, we define the *oricycle*

$$\mathbb{O}_k = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}.$$

It is a disk in $\mathbb{D}$ whose boundary circle $\partial\mathbb{O}_k$ is tangent to $\mathbb{T}$ at $z = 1$. The center of $\mathbb{O}_k$ is at $(1/(1+k), 0)$ and its radius is $k/(k+1)$.

For $\xi \in \mathbb{D}$ define the Möbius transformation

$$\varphi_\xi(z) = \frac{z-\xi}{1-\overline{\xi}z}, \qquad z \in \mathbb{C} \setminus \{1/\overline{\xi}\}.$$

For every $\xi \in \mathbb{D}$ the function φ_ξ is an analytic automorphism of $\mathbb{D}$. The set of all analytic automorphisms of $\mathbb{D}$ will be denoted by $\mathrm{Aut}(\mathbb{D})$. It is well known that $f \in \mathrm{Aut}(\mathbb{D})$ iff there exist $\xi \in \mathbb{D}$ and $\lambda \in \mathbb{T}$ such that $f(z) = \lambda\varphi_\xi(z), z \in \mathbb{D}$, i.e. every analytic automorphism of $\mathbb{D}$ is the composition of a rotation and a Möbius transformation.

The set of all analytic functions ω in $\mathbb{D}$ such that $|\omega(z)| < 1$ for $z \in \mathbb{D}$ will be denoted by $\mathcal{B}$.

The following lemma ([3]; see also [1, p. 56]) is the basis for our considerations.

LEMMA 2.1 (Julia). *Let $\omega \in \mathcal{B}$. Assume that there exists a sequence (z_n) of points in $\mathbb{D}$ such that*

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} \omega(z_n) = 1, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1-|\omega(z_n)|}{1-|z_n|} = \alpha < \infty. \tag{2.2}$$

Then

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \le \alpha \frac{|1-z|^2}{1-|z|^2}, \qquad z \in \mathbb{D}, \tag{2.3}$$

and hence, for every $k > 0$,

$$\omega(\mathbb{O}_k) \subset \mathbb{O}_{\alpha k}.$$

Equality in (2.3) *for some $z \in \mathbb{D}$ can occur only for $\omega \in \mathrm{Aut}(\mathbb{D})$, i.e., for $\omega(z) = \lambda\varphi_\xi(z)$, $z \in \mathbb{D}$, where $\xi \in \partial\mathbb{O}_{1/\alpha}$ and $\lambda = (1-\overline{\xi})/(1-\xi)$. In particular, equality holds for $\xi = (\alpha-1)/(\alpha+1)$ and $\lambda = 1$.*

REMARK 2.1. Since

$$\frac{1-|\omega(z)|}{1-|z|} \ge \frac{1-|\omega(0)|}{1+|\omega(0)|}, \qquad z \in \mathbb{D},$$

for every $\omega \in \mathcal{B}$, the constant α defined in (2.2) is positive (see [1, p. 43]).

The Julia Lemma can be easily reformulated for the right halfplane. First we introduce some notations. Let $\mathbb{H}_c = \{z \in \mathbb{C} : \operatorname{Re} z > c\}$, $c \ge 0$, and

$\mathbb{H} = \mathbb{H}_0$. Let $\mathcal{A}(\mathbb{H})$ denote the set of all analytic functions in $\mathbb{H}$ and let

$$\mathcal{P}(\mathbb{H}) = \{q \in \mathcal{A}(\mathbb{H}) : \operatorname{Re} q(w) > 0,\ w \in \mathbb{H}\}.$$

LEMMA 2.2. *Let $q \in \mathcal{P}(\mathbb{H})$. Assume that there exists a sequence (w_n) of points in $\mathbb{H}$ such that*

$$\lim_{n\to\infty} w_n = \infty, \qquad \lim_{n\to\infty} q(w_n) = \infty, \tag{2.4}$$

$$\lim_{n\to\infty} \left\{ \frac{|q(w_n)+1| - |q(w_n)-1|}{|w_n+1| - |w_n-1|} \cdot \frac{|w_n+1|}{|q(w_n)+1|} \right\} = \alpha < \infty. \tag{2.5}$$

Then

$$\operatorname{Re} q(w) \ge \frac{1}{\alpha} \operatorname{Re} w, \qquad w \in \mathbb{H}, \tag{2.6}$$

and hence, *for every $c > 0$*,

$$q(\mathbb{H}_c) \subset \mathbb{H}_{c/\alpha}.$$

The result is sharp. Equality in (2.6) *can occur only for*

$$q(w) = \frac{1}{\alpha} w + it, \qquad w \in \mathbb{H},\ t \in \mathbb{R}. \tag{2.7}$$

In particular, *equality holds for*

$$q(w) = \frac{1}{\alpha} w, \qquad w \in \mathbb{H}.$$

Proof. Define

$$\omega(z) = \frac{q\left(\dfrac{1+z}{1-z}\right) - 1}{q\left(\dfrac{1+z}{1-z}\right) + 1}, \qquad z \in \mathbb{D}. \tag{2.8}$$

For the sequence (w_n) of points in $\mathbb{H}$ let

$$z_n = \frac{w_n - 1}{w_n + 1}, \qquad n \in \mathbb{N},$$

be the corresponding sequence of points in $\mathbb{D}$. Since $\lim_{n\to\infty} w_n = \infty$, we have $\lim_{n\to\infty} z_n = 1$. From (2.4) it follows that

$$\lim_{n\to\infty} \omega(z_n) = \lim_{n\to\infty} \frac{q(w_n) - 1}{q(w_n) + 1} = 1$$

so (2.1) holds. In view of (2.5) and (2.8) we see that

$$\lim_{n\to\infty}\frac{1-|\omega(z_n)|}{1-|z_n|}=\lim_{n\to\infty}\frac{1-\left|\dfrac{q(w_n)-1}{q(w_n)+1}\right|}{1-\left|\dfrac{w_n-1}{w_n+1}\right|}\tag{2.9}$$
$$=\lim_{n\to\infty}\left\{\frac{|q(w_n)+1|-|q(w_n)-1|}{|w_n+1|-|w_n-1|}\cdot\frac{|w_n+1|}{|q(w_n)+1|}\right\}=\alpha<\infty.$$

Hence (2.2) holds. On the other hand, by (2.8) we have

$$q(w)=\frac{1+\omega(z)}{1-\omega(z)},\tag{2.10}$$

where $w=(1+z)/(1-z),\ z\in\mathbb{D}$, so

$$\operatorname{Re}q(w)=\frac{1-|\omega(z)|^2}{|1-\omega(z)|^2},\qquad \operatorname{Re}w=\frac{1-|z|^2}{|1-z|^2}.$$

Hence (2.3) yields the assertion.

Suppose now that $q(w_0)\in\partial\mathbb{H}_{c/\alpha}$ for some $w_0\in\partial\mathbb{H}_c$ and $c>0$. Then $z_0=(w_0-1)/(w_0+1)\in\mathbb{D}$ lies on the oricycle $\partial\mathbb{O}_{1/c}$ and $\omega(z_0)\in\partial\mathbb{O}_{\alpha/c}$. By the Julia Lemma this holds only for $\omega(z)=\lambda\varphi_\xi(z),\ z\in\mathbb{D}$, where $\xi\in\partial\mathbb{O}_{1/\alpha}$ and $\lambda=(1-\overline{\xi})/(1-\xi)$. Using (2.10) we find that

$$q(w)=\frac{1+\lambda\varphi_\xi(z)}{1-\lambda\varphi_\xi(z)}=\frac{|1-\xi|^2}{1-|\xi|^2}\,w-i\,\frac{2\operatorname{Im}\xi}{1-|\xi|^2}=\frac{1}{\alpha}w+it,$$

where $t\in\mathbb{R}$.

In particular, $t=0$ for $\xi=(\alpha-1)/(\alpha+1)$ and $\lambda=1$, which yields $q(w)=w/\alpha$.

Remark 2.2. By Remark 2.1 and (2.9), α defined in (2.5) is positive.

Suppose now that the sequence (w_n) consists of only positive real numbers. Then we can formulate the following corollary.

Corollary 2.1. *Let $q\in\mathcal{P}(\mathbb{H})$. Assume that there exists a sequence (x_n) of positive real numbers such that*

$$\lim_{n\to\infty}x_n=\infty,\qquad \lim_{n\to\infty}q(x_n)=\infty,\tag{2.11}$$

$$\lim_{n\to\infty}\left\{(|q(x_n)+1|-|q(x_n)-1|)\,\frac{x_n+1}{|q(x_n)+1|}\right\}=2\alpha<\infty.\tag{2.12}$$

Then $q(\mathbb{H}_c)\subset\mathbb{H}_{c/\alpha}$ for every $c>0$. The result is sharp and (2.7) is the extremal function.

If also the sequence $(q(x_n))$ consists of only positive real numbers then we have

Corollary 2.2. *Let $q\in\mathcal{P}(\mathbb{H})$. Assume that there exists a sequence (x_n) of positive real numbers such that $q(x_n)$ is a positive real number for* *every* $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = \infty$ *and*

$$\lim_{n\to\infty} \frac{q(x_n)}{x_n} = \frac{1}{\alpha} < \infty. \tag{2.13}$$

Then $q(\mathbb{H}_c) \subset \mathbb{H}_{c/\alpha}$ *for every* $c > 0$. *The result is sharp and* (2.7) *is the extremal function.*

Since (x_n) tends to infinity, by (2.13) the same holds for $(q(x_n))$. By Remark 2.2, $\alpha > 0$.

It is possible to consider stronger assumptions instead of (2.4). We can set

$$\lim_{n\to\infty} \frac{q(w_n)}{w_n} = \beta,$$

with the sequence (w_n) tending to infinity, which implies that $\lim_{n\to\infty} q(w_n) = \infty$. Then we have

Corollary 2.3. *Let* $q \in \mathcal{P}(\mathbb{H})$. *Assume that there exists a sequence* (w_n) *of points in* $\mathbb{H}$ *such that*

$$\lim_{n\to\infty} w_n = \infty, \qquad \lim_{n\to\infty} \frac{q(w_n)}{w_n} = \beta \neq 0,$$
$$\frac{1}{|\beta|} \lim_{n\to\infty} \left\{ \frac{|q(w_n)+1| - |q(w_n)-1|}{|w_n+1| - |w_n-1|} \right\} = \alpha < \infty.$$

Then $q(\mathbb{H}_c) \subset \mathbb{H}_{c/\alpha}$ *for every* $c > 0$. *The result is sharp and* (2.7) *is the extremal function.*

Suppose again that the sequence (w_n) consists of only positive real numbers. Then we have

Corollary 2.4. *Let* $q \in \mathcal{P}(\mathbb{H})$. *Assume that there exists a sequence* (x_n) *of positive real numbers such that*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} \frac{q(x_n)}{x_n} = \beta \in \mathbb{C} \setminus \{0\}$$

and

$$\frac{1}{|\beta|} \lim_{n\to\infty} \{|q(x_n)+1| - |q(x_n)-1|\} = 2\alpha < \infty. \tag{2.14}$$

Then $q(\mathbb{H}_c) \subset \mathbb{H}_{c/\alpha}$ *for every* $c > 0$. *The result is sharp and* (2.7) *is the extremal function.*

A special case of Lemma 2.2 was proved in [6]. Let us recall this result.

Theorem 2.1 ([6]). *Let* $q \in \mathcal{P}(\mathbb{H})$. *If the following limit exists*:

$$\lim_{w\to\infty} (q(w) - w) = a \neq \infty, \tag{2.15}$$

then $q(\mathbb{H}_c) \subset \mathbb{H}_c$ *for every* $c > 0$.

Remark 2.3. The normalization (2.15) generalizes the so called *hydrodynamic normalization* ($a = 0$). Notice that $\operatorname{Re} a \geq 0$. The method of proof of Theorem 2.1 was based on the maximum principle for harmonic functions.

Now we show that Theorem 2.1 is a special case of Lemma 2.2. In fact we will prove that Theorem 2.1 is a consequence of Corollary 2.1. From (2.15) it follows that

$$\lim_{w\to\infty} q(w) = \infty, \tag{2.16}$$

$$\lim_{w\to\infty} \operatorname{Re}(q(w) - w) = \operatorname{Re} a, \qquad \lim_{w\to\infty} \operatorname{Im}(q(w) - w) = \operatorname{Im} a. \tag{2.17}$$

Since by (2.15),

$$\lim_{w\to\infty} (q(w) - w) = \lim_{w\to\infty} w\left(\frac{q(w)}{w} - 1\right) = a,$$

we have

$$\lim_{w\to\infty} \frac{q(w)}{w} = 1. \tag{2.18}$$

By (2.15) there exists a sequence (x_n) of positive real numbers such that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty}(q(x_n) - x_n) = a$. By (2.16), $\lim_{n\to\infty} q(x_n) = \infty$ so (2.11) is satisfied. Writing $q(x_n) = u_n + iv_n$, by (2.17) we have $\lim_{n\to\infty}(u_n - x_n) = \operatorname{Re} a$ and consequently,

$$\lim_{n\to\infty} u_n = \infty. \tag{2.19}$$

Also

$$\lim_{n\to\infty} v_n = \operatorname{Im} a \tag{2.20}$$

from (2.17). Since by (2.19) and (2.20),

$$\begin{aligned}\lim_{n\to\infty} \left|1 + \frac{1}{q(x_n)}\right|^2 &= \lim_{n\to\infty} \frac{(u_n+1)^2 + v_n^2}{u_n^2 + v_n^2} \\ &= \lim_{n\to\infty} \frac{\left(1 + \dfrac{1}{u_n}\right)^2 + \left(\dfrac{v_n}{u_n}\right)^2}{1 + \left(\dfrac{v_n}{u_n}\right)^2} = 1,\end{aligned}$$

(2.18) yields

$$\lim_{n\to\infty} \frac{x_n + 1}{|q(x_n) + 1|} = \lim_{n\to\infty} \frac{x_n\left(1 + \dfrac{1}{x_n}\right)}{|q(x_n)|\left|1 + \dfrac{1}{q(x_n)}\right|} = 1. \tag{2.21}$$

Using (2.19) and (2.20) we obtain

$$\begin{aligned}
(2.22)\qquad &\lim_{n\to\infty}\{|q(x_n)+1|-|q(x_n)-1|\}\\
&=\lim_{n\to\infty}\{\sqrt{(u_n+1)^2+v_n^2}-\sqrt{(u_n-1)^2+v_n^2}\}\\
&=\lim_{n\to\infty}\left\{\frac{4u_n}{\sqrt{(u_n+1)^2+v_n^2}+\sqrt{(u_n-1)^2+v_n^2}}\right\}\\
&=\lim_{n\to\infty}\left\{\frac{4}{\sqrt{\left(1+\frac{1}{u_n}\right)^2+\left(\frac{v_n}{u_n}\right)^2}+\sqrt{\left(1-\frac{1}{u_n}\right)^2+\left(\frac{v_n}{u_n}\right)^2}}\right\}=2.
\end{aligned}$$

This result together with (2.21) yields (2.12) for $\alpha = 1$. Hence the conclusion of the theorem follows.

EXAMPLES. 1. In fact, the conditions (2.4) and (2.5) are more general than (2.15). As an example, take

$$q(w) = i\log w + w + \pi/2, \qquad w \in \mathbb{H}.$$

It is easy to verify that $\operatorname{Re} q(w) > 0$ for $w \in \mathbb{H}$.

Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n = \infty$. Then

$$\lim_{n\to\infty} q(x_n) = \lim_{n\to\infty}\{x_n + \pi/2 + i\log x_n\} = \infty$$

so (2.11) is true.

Repeating exactly calculations (2.21) and (2.22) with $u_n = x_n + \pi/2$, $v_n = \log x_n$ and using the elementary fact that $\lim_{n\to\infty}(\log x_n)/x_n = 0$ we infer that the limit (2.12) exists with $\alpha = 1$.

On the other hand

$$\lim_{w\to\infty}(q(w) - w) = \lim_{w\to\infty}(i\log w + \pi/2) = \infty$$

so (2.15) does not hold.

2. Every function $q \in \mathcal{P}(\mathbb{H})$ of the form

$$q(w) = \beta w + o(w), \qquad w \in \mathbb{H}, \tag{2.23}$$

with $\beta > 0$ satisfies the assumptions of Corollary 2.4. Let (x_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n = \infty$. Then

$$\lim_{n\to\infty}\frac{q(x_n)}{x_n} = \beta + \lim_{n\to\infty}\frac{o(x_n)}{x_n} = \beta.$$

Let $o(x_n) = s_n + iv_n$ and $q(x_n) = \beta x_n + s_n + iv_n$ for $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{o(x_n)}{x_n} = \lim_{n\to\infty} \left(\frac{s_n}{x_n} + i\frac{v_n}{x_n} \right) = 0$$

we have

$$\lim_{n\to\infty} \frac{s_n}{x_n} = 0, \qquad \lim_{n\to\infty} \frac{v_n}{x_n} = 0.$$

Repeating the calculations in (2.21) and (2.22) with $u_n = \beta x_n + s_n$ and v_n we obtain

$$\lim_{n\to\infty} \frac{x_n + 1}{|q(x_n) + 1|} = \frac{1}{\beta}$$

and

$$\lim_{n\to\infty} \{|q(x_n) + 1| - |q(x_n) - 1|\}$$

$$= \lim_{n\to\infty} \left\{ \frac{4}{\sqrt{\left(1 + \frac{1}{\beta x_n + s_n}\right)^2 + \left(\frac{v_n}{\beta x_n + s_n}\right)^2} + \sqrt{\left(1 - \frac{1}{\beta x_n + s_n}\right)^2 + \left(\frac{v_n}{\beta x_n + s_n}\right)^2}} \right\} = 2$$

since

$$\lim_{n\to\infty} \frac{v_n}{\beta x_n + s_n} = \lim_{n\to\infty} \frac{v_n/x_n}{\beta + s_n/x_n} = 0.$$

Hence (2.12) holds with $\alpha = 1/\beta$.

The class of functions of the form (2.23) is essentially larger than this normalized by (2.15). The function $q(w) = \beta w + w^{\mu},\ w \in \mathbb{H}$, for $\beta > 0$ and $\mu \in [0, 1)$ is of the form (2.23).

3. Starlike functions defined in a halfplane. Classical geometric properties like starlikeness and convexity of analytic functions in the right halfplane were studied in [2, 6].

For $v \in \mathbb{C}$, let $l^+[v] = \{sv : s \in [1, +\infty)\}$ be the radial halfline with endpoint at v.

Let us start with the following definitions.

Definition 3.1. A simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, with $\infty \in \partial\Omega$ is called *starlike with respect to the boundary point at infinity* if the halfline $l^+[v]$ is contained in Ω for every $v \in \Omega$. The set of all such domains will be denoted by $\mathcal{Z}^*_\infty$.

Remark 3.1. Observe that the origin lies outside every domain in $\mathcal{Z}^*_\infty$. Indeed, assume that $0 \in \Omega$ for some $\Omega \in \mathcal{Z}^*_\infty$. Since $\Omega \neq \mathbb{C}$, there exists a finite point $v_0 \in \partial\Omega$ such that $[0, v_0) \subset \Omega$. Hence $l^+[v]$ does not lie in Ω for any point v of the segment $[0, v_0)$.

Definition 3.2. Let $\mathcal{S}^*_\infty(\mathbb{H})$ denote the class of all univalent analytic functions f in $\mathbb{H}$ such that $f(\mathbb{H}) \in \mathcal{Z}^*_\infty$. Functions belonging to $\mathcal{S}^*_\infty(\mathbb{H})$ will be called *starlike w.r.t. the boundary point at infinity.*

It is obvious that for every $f \in \mathcal{S}^*_\infty(\mathbb{H})$ there is some point on the boundary of $\mathbb{H}$ which "corresponds" to infinity on the boundary of $f(\mathbb{H})$. In what follows we will use a kind of boundary normalization for every $f \in \mathcal{S}^*_\infty(\mathbb{H})$ by saying that $\infty \in \partial\mathbb{H}$ corresponds to $\infty \in \partial f(\mathbb{H})$. Since, in general, we cannot extend a function f to $\partial\mathbb{H}$, in order to be precise, we will apply the notion of prime ends to formulate this normalization. Below we construct a prime end $p_\infty(\Omega)$ for every $\Omega \in \mathcal{Z}^*_\infty$ and next using the Prime End Theorem we associate $\infty \in \partial\mathbb{H}$ with $p_\infty(\Omega)$.

Construction of a prime end for a domain starlike w.r.t. the boundary point at infinity. For an arbitrary domain in $\mathcal{Z}^*_\infty$ we introduce a special null chain (C_n).

Let us recall that a *crosscut* of a domain $G \subset \overline{\mathbb{C}}$ is an open Jordan arc C in G such that $\overline{C} = C \cup \{a, b\}$, where $a, b \in \partial G$.

Let $\Omega \in \mathcal{Z}^*_\infty$. Since $\Omega \neq \mathbb{C}$, there exists a finite boundary point v_0 of Ω such that $l^+[v_0] \setminus \{v_0\}$ lies in Ω. For each $t \in (0,\infty)$ set $C(t) = \{v \in \mathbb{C} : |v - v_0| = t|v_0|\}$. It is clear that $\Omega \cap C(t) \neq \emptyset$ for every $t \in (0,\infty)$. By [5, Proposition 2.13, p. 28], for every fixed $t \in (0,\infty)$ there are countably many crosscuts $C_k(t) \subset C(t)$, $k \in \mathbb{N}$, of Ω. We denote by $\Omega_0(t) \subset \Omega$ the component of $\Omega \setminus C(t)$ containing the halfline $l^+[(1+t)v_0] \setminus \{(1+t)v_0\}$, and by $Q(t) \in \bigcup_{k\in\mathbb{N}} C_k(t)$ the crosscut containing the point $(1+t)v_0$. So $Q(t) \subset \partial\Omega_0(t)$. Let now (t_n) be a strictly increasing sequence of points in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and let $(Q(t_n))$ be the corresponding sequence of crosscuts of Ω. It is easy to observe that

(a) $\overline{Q(t_n)} \cap \overline{Q(t_{n+1})} = \emptyset$ for every $n \in \mathbb{N}$.

(b) $\Omega_0(t_{n+1}) \subset \Omega_0(t_n)$ for every $n \in \mathbb{N}$.

(c) $\operatorname{diam}^\# Q(t_n) \to 0$ as $n \to \infty$, where $\operatorname{diam}^\# B$ is the spherical diameter of the set $B \subset \overline{\mathbb{C}}$.

Therefore $(C_n) = (Q(t_n))$ forms a null chain of Ω. Notice also that the null chain (C_n) is independent of the choice of the sequence (t_n). The equivalence class of the null chain (C_n) defines the prime end denoted by $p_\infty(\Omega)$.

We can also derive that infinity is a unique principal point of the prime end $p_\infty(\Omega)$. Therefore, the following proposition holds ([5, p. 39]).

Proposition 3.1. *For every $\Omega \in \mathcal{Z}^*_\infty$ the prime end $p_\infty(\Omega)$ is of the first or of the second kind.*

Let g be an analytic univalent mapping of $\mathbb{D}$ onto Ω. Then there exists a bijective mapping $\widehat{g}$ of $\mathbb{T}$ onto the set $p(\Omega)$ of all prime ends of Ω ([5, p. 30]).

If $h(w) = (w-1)/(w+1)$, $w \in \overline{\mathbb{H}}$, then $\widehat{f} = \widehat{g} \circ h$ is a bijective mapping of the closed imaginary axis $\partial\mathbb{H} \cup \{\infty\}$ onto $p(\Omega)$. Thus there there is a unique $w_0 \in \partial\mathbb{H} \cup \{\infty\}$ such that $p_\infty(\Omega) = \widehat{f}(w_0)$.

By using Proposition 3.1 and [5, Corollary 2.17, p. 35], we get

PROPOSITION 3.2. *Every function* $f \in \mathcal{S}^*_\infty(\mathbb{H})$ *such that* $p_\infty(f(\mathbb{H})) = \widehat{f}(\infty)$ *has a radial limit*

$$\lim_{\mathbb{R}\ni x\to\infty} f(x) = \infty.$$

To present an analytic characterization of the class $\mathcal{S}^*_\infty(\mathbb{H})$ we will need the following theorem.

THEOREM 3.1. *Let* $f \in \mathcal{A}(\mathbb{H})$ *be univalent in* $\mathbb{H}$. *Then* $f \in \mathcal{S}^*_\infty(\mathbb{H})$ *and* $p_\infty(\mathbb{H}) = \widehat{f}(\infty)$ *if and only if* $f(\mathbb{H}_c) \in \mathcal{Z}^*_\infty$ *for every* $c > 0$.

Proof. Assume that $f \in \mathcal{S}^*_\infty(\mathbb{H})$ and $w_0 = \infty$ corresponds to the prime end $p_\infty(f(\mathbb{H}))$. For each $t > 1$ define

$$q_t(w) = f^{-1}(tf(w)), \qquad w \in \mathbb{H}.$$

Since $f(\mathbb{H})$ is a domain starlike w.r.t. infinity, we have $tf(w) \in f(\mathbb{H})$ for every $t > 1$ and $w \in \mathbb{H}$. The univalence of f shows that the function q_t is well defined for each $t > 1$.

Now fix $t > 1$ and let $v_0 \in \partial f(\mathbb{H})$ be such that $l^+[v_0] \setminus \{v_0\} \subset f(\mathbb{H})$. Consider the sequence $(v_n) = (t^n v_0)$ of points in $l^+[v_0]$ and the corresponding sequence $(w_n) = (f^{-1}(v_n))$ of points in $\mathbb{H}$. Using the same notations as before let $C(t_n) = \{v \in \mathbb{C} : |v - v_0| = (t^n - 1)|v_0|\}$ and let $Q(t_n) \subset C(t_n)$ denote the crosscut of $f(\mathbb{H})$ containing the point v_n. Therefore $(Q(t_n))$ is a null chain representing the prime end $p_\infty(f(\mathbb{H}))$. By the Prime End Theorem [5, p. 30], $(f^{-1}(Q(t_n)))$ is a null chain in $\mathbb{H}$ that separates v_0 and infinity for large n. Since $w_n = f^{-1}(v_n) \in f^{-1}(Q(t_n))$ and $\operatorname{diam}^\# f^{-1}(Q(t_n)) \to 0$ as $n \to \infty$, we conclude that $\lim_{n\to\infty} w_n = \infty$. Observe that

$$q_t(w_n) = f^{-1}(tv_n) = w_{n+1}.$$

Let now

$$a_n = \frac{|q_t(w_n)+1| - |q_t(w_n)-1|}{|w_n+1| - |w_n-1|} \cdot \frac{|w_n+1|}{|q_t(w_n)+1|}, \qquad n \in \mathbb{N}.$$

Hence

$$a_n = \frac{|w_{n+1}+1| - |w_{n+1}-1|}{|w_n+1| - |w_n-1|} \cdot \frac{|w_n+1|}{|w_{n+1}+1|}$$

for all $n \in \mathbb{N}$. Consequently,

$$\lim_{n\to\infty}(a_1a_2\dots a_n) = \lim_{n\to\infty}\left\{\frac{|w_{n+1}+1|-|w_{n+1}-1|}{|w_1+1|-|w_1-1|}\cdot\frac{|w_1+1|}{|w_{n+1}+1|}\right\}$$
$$= \lim_{n\to\infty}\left\{\frac{|w_1+1|}{|w_1+1|-|w_1-1|}\left(1-\frac{\left|1-\dfrac{1}{w_{n+1}}\right|}{\left|1+\dfrac{1}{w_{n+1}}\right|}\right)\right\} = 0.$$

Hence infinitely many a_n must satisfy $0 < a_n \le 1$, so there exists a convergent subsequence (a_{n_k}) such that

$$0 \le \lim_{k\to\infty} a_{n_k} = \alpha(t) \le 1,$$

which means that

$$\lim_{k\to\infty}\left\{\frac{|q_t(w_{n_k})+1|-|q_t(w_{n_k})-1|}{|w_{n_k}+1|-|w_{n_k}-1|}\cdot\frac{|w_{n_k}+1|}{|q_t(w_{n_k})+1|}\right\} = \alpha(t) \le 1$$

for every fixed $t > 1$. In fact, in view of Remark 2.2, $\alpha(t) > 0$ for every $t > 1$.

Hence q_t satisfies the assumptions of Lemma 2.2, and since $\alpha(t) \le 1$ for every $t > 1$, we derive that

$$q_t(\mathbb{H}_c) \subset \mathbb{H}_{c/\alpha(t)} \subset \mathbb{H}_c$$

for every $c > 0$. This yields $tf(\mathbb{H}_c) \subset f(\mathbb{H}_c)$ for every $t > 1$. Therefore $f(\mathbb{H}_c) \in \mathcal{Z}^*_\infty$ for every $c > 0$. Conversely, assume that $f(\mathbb{H}_c) \in \mathcal{Z}^*_\infty$ for every $c > 0$. Since $\infty \in \partial f(\mathbb{H}_c)$ for every $c > 0$ and

$$f(\mathbb{H}) = \bigcup_{c>0} f(\mathbb{H}_c),$$

it follows that $\infty \in \partial f(\mathbb{H})$ and $f(\mathbb{H}) \in \mathcal{Z}^*_\infty$. Observe also that there exists a prime end $p_\infty(f(\mathbb{H}))$ which corresponds to a point $w_0 \in \partial\mathbb{H} \cup \{\infty\}$. We need to show that $w_0 = \infty$.

To this end, fix $c > 0$ and suppose that $w_0 \ne \infty$. Let $v_0 \in f(\mathbb{H})$ be such that $l^+[v_0] \setminus \{v_0\} \subset f(\mathbb{H})$.

Let $C(t) = \{v \in \mathbb{C} : |v - v_0| = t|v_0|\}$, $t > 0$. Repeating the construction of a null chain for the domain $f(\mathbb{H})$ and using the same notation let $Q(t)$, $t > 0$, denote the crosscut of $f(\mathbb{H})$ containing the point $(1+t)v_0$. Choosing a strictly increasing sequence (t_n) of points in $(1, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ let $(Q(t_n))$ be the corresponding sequence of crosscuts of $f(\mathbb{H})$ which represents the prime end $p_\infty(f(\mathbb{H}))$ corresponding in a unique way to $w_0 \in \partial\mathbb{H} \cup \{\infty\}$. By the Prime End Theorem, $(f^{-1}(Q(t_n)))$ is a null chain that separates in $\mathbb{H}$ the points w_0 and v_0 for large n. Since $v_0 \ne \infty$ and $\operatorname{diam}^{\#} f^{-1}(Q(t_n)) \to 0$ as $n \to \infty$ we see that

$$f^{-1}(Q(t_n)) \cap \mathbb{H}_c = \emptyset \tag{3.1}$$

for large n.

On the other hand, $f(\mathbb{H}_c)$ is in $\mathcal{Z}^*_\infty$, which implies that $Q(t_n)\cap f(\mathbb{H}_c) \neq \emptyset$ for large $n \in \mathbb{N}$. This contradicts (3.1) and finally proves that $w_0 = \infty$, so $p_\infty(f(\mathbb{H})) = \widehat{f}(\infty)$. The proof of the theorem is finished.

In the proof of the next theorem we need the lemma below.

Lemma 3.1. *If $f \in \mathcal{A}(\mathbb{H})$ is univalent in $\mathbb{H}$ and $f(w) \neq 0$ for $w \in \mathbb{H}$, then*

$$\left|\frac{f'(x)}{f(x)}\right| \leq \frac{2}{x}, \qquad x > 0. \tag{3.2}$$

Proof. Since $f \in \mathcal{A}(\mathbb{H})$ is univalent in $\mathbb{H}$, the function

$$g(z) = f\left(\frac{1+z}{1-z}\right) = f(w), \qquad z \in \mathbb{D},$$

where $w = (1+z)/(1-z)$, is analytic univalent in $\mathbb{D}$ and maps it onto the domain $\Omega = f(\mathbb{H})$. Hence

$$\frac{|g'(z)|}{\operatorname{dist}(g(z), \partial g(\mathbb{D}))} \leq \frac{4}{1-|z|^2}$$

for $z \in \mathbb{D}$ (see [5, p. 92]). For any $\upsilon \notin \Omega$, the above shows that

$$\left|\frac{g'(z)}{g(z)-\upsilon}\right| \leq \frac{4}{1-|z|^2}$$

for $z \in \mathbb{D}$. Hence

$$\left|\frac{f'(w)}{f(w)-\upsilon}\right| \leq \frac{8}{|w+1|^2 - |w-1|^2}$$

for $w \in \mathbb{H}$. Therefore for $\upsilon = 0$ and $w = x > 0$ we obtain (3.2).

To present an analytic characterization of the class $\mathcal{S}^*_\infty(\mathbb{H})$ it is useful to define the following class of functions.

Definition 3.3. For $\alpha > 0$ let $\mathcal{P}(\alpha; \mathbb{H})$ be the subclass of all functions $q \in \mathcal{P}(\mathbb{H})$ satisfying (2.4) and (2.5).

Theorem 3.2. *If $f \in \mathcal{S}^*_\infty(\mathbb{H})$ and $p_\infty(f(\mathbb{H})) = \widehat{f}(\infty)$, then there exists $\alpha \in (0,1]$ and a function $q \in \mathcal{P}(\alpha; \mathbb{H})$ such that*

$$\frac{f'(w)}{f(w)} = \frac{2}{q(w)}, \qquad w \in \mathbb{H}.$$

Proof. Define

$$q(w) = 2\,\frac{f(w)}{f'(w)}, \qquad w \in \mathbb{H}.$$

Since $f'(w) \neq 0$ in $\mathbb{H}$, q is analytic in $\mathbb{H}$.

We will prove that q is in $\mathcal{P}(\alpha; \mathbb{H})$ for some $\alpha \in (0,1]$.

1. First we show that $\operatorname{Re} q(w) > 0$ for $w \in \mathbb{H}$, i.e.

$$\operatorname{Re}\left\{\frac{f'(w)}{f(w)}\right\} > 0, \qquad w \in \mathbb{H}.$$

Remark 3.1 yields $0 \notin f(\mathbb{H})$. Therefore the function

$$\gamma_c = \partial\mathbb{H}_c \ni w \mapsto \arg f(w)$$

is well defined on the analytic arc γ_c. We use the following parametrization:

$$\gamma_c : w = \gamma(y) = c + iy, \qquad y \in \mathbb{R}. \tag{3.3}$$

From Theorem 3.1 it follows that $f(\mathbb{H}_c) \in \mathcal{Z}^*_\infty$ for every $c > 0$. This means geometrically that the function

$$\mathbb{R} \ni y \mapsto \arg f(\gamma(y)) \tag{3.4}$$

is monotonic on the analytic arc γ_c for every $c > 0$. Since f is a conformal mapping, it preserves the orientation of γ_c. We have

$$\begin{aligned}\frac{d}{dy}\arg f(\gamma(y)) &= \frac{d}{dy}\operatorname{Im}\log f(\gamma(y)) = \operatorname{Im}\left\{\gamma'(y)\,\frac{f'(\gamma(y))}{f(\gamma(y))}\right\}\\ &= \operatorname{Re}\left\{\frac{f'(\gamma(y))}{f(\gamma(y))}\right\} \geq 0\end{aligned} \tag{3.5}$$

for $y \in \mathbb{R}$. By the above

$$\operatorname{Re}\left\{\frac{f'(w)}{f(w)}\right\} \geq 0 \tag{3.6}$$

for $w \in \mathbb{H}$.

Assume now that equality holds in (3.6) for some $w_0 \in \mathbb{H}$. By the maximum principle for harmonic functions it holds in the whole halfplane $\mathbb{H}$, which implies that there exists $b \in \mathbb{R} \setminus \{0\}$ so that

$$\frac{f'(w)}{f(w)} \equiv bi, \qquad w \in \mathbb{H}.$$

But the solution

$$f(w) = f_0(w) = a\exp(ibw), \qquad w \in \mathbb{H},\ a \in \mathbb{C} \setminus \{0\},$$

of the last equation is not univalent in $\mathbb{H}$. So $f_0 \notin \mathcal{S}^*_\infty(\mathbb{H})$ and hence the strict inequality holds in (3.6).

2. Now we prove that the function q satisfies the conditions (2.11) and (2.12) of Corollary 2.1. Let (x_n) be an arbitrary sequence of positive real numbers such that $\lim_{n\to\infty} x_n = \infty$. Since, by Remark 3.1, $0 \notin f(\mathbb{H})$, applying Lemma 3.1 we have

$$|q(x_n)| = 2\left|\frac{f(x_n)}{f'(x_n)}\right| \geq x_n,$$

which implies that $\lim_{n\to\infty} q(x_n) = \infty$. Moreover Lemma 3.1 yields

$$\frac{x_n}{|q(x_n)|} = \frac{x_n}{2}\left|\frac{f'(x_n)}{f(x_n)}\right| \leq 1$$

for $n \in \mathbb{N}$. Take a convergent subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} \frac{x_{n_k}}{|q(x_{n_k})|} = \alpha_1 \leq 1.$$

Hence

$$\lim_{k\to\infty} \frac{x_{n_k}+1}{|q(x_{n_k})+1|} = \lim_{k\to\infty} \frac{x_{n_k}\left(1+\dfrac{1}{x_{n_k}}\right)}{|q(x_{n_k})|\left|1+\dfrac{1}{q(x_{n_k})}\right|} = \alpha_1.$$

It is easy to check that for $x > 0$,

$$|q(x)+1| - |q(x)-1| \leq 2.$$

In particular, the last inequality holds for $x = x_{n_k}$ so there exists a subsequence $(x_{n_{k_l}})$ such that

$$\lim_{l\to\infty} \{|q(x_{n_{k_l}})+1| - |q(x_{n_{k_l}})-1|\} = 2\alpha_2 \leq 2.$$

Consequently,

$$\lim_{l\to\infty} \left\{(|q(x_{n_{k_l}})+1| - |q(x_{n_{k_l}})-1|)\,\frac{x_{n_{k_l}}+1}{|q(x_{n_{k_l}})+1|}\right\} = 2\alpha_1\alpha_2 = 2\alpha,$$

where $\alpha \in [0,1]$. By Remark 2.2, $\alpha \in (0,1]$. In this way the function q satisfies the assumptions of Corollary 2.1.

This ends the proof of the theorem.

THEOREM 3.3. *Let $f \in \mathcal{A}(\mathbb{H})$. Assume that there exists an $\alpha \in (0,1]$ and a function $q \in \mathcal{P}(\mathbb{H})$ with $\lim_{w\to\infty} q(w) = \infty$ such that*

$$\lim_{w\to\infty} \frac{q(w)}{w} = \frac{1}{\alpha}. \tag{3.7}$$

If

$$\frac{f'(w)}{f(w)} = \frac{2}{q(w)}, \qquad w \in \mathbb{H}, \tag{3.8}$$

then $f \in \mathcal{S}^*_\infty(\mathbb{H})$ and $p_\infty(f(\mathbb{H})) = \widehat{f}(\infty)$.

Proof. Notice that from (3.8) it follows that f is locally univalent in $\mathbb{H}$. Also $f(w) \neq 0$ for $w \in \mathbb{H}$.

Now we prove that f is univalent in $\mathbb{H}$.

Consider again the function (3.3) defined on the analytic arcs $\gamma_c = \partial\mathbb{H}_c$ for each $c > 0$, parametrized by (3.4). Repeating the calculations (3.5) we see that the condition (3.8) implies

$$\frac{d}{dy}\arg f(\gamma(y)) > 0, \qquad y \in \mathbb{R},$$

i.e. the function (3.3) is strictly increasing on each γ_c. By monotonicity the limits

$$\psi_c = \lim_{y\to+\infty} \arg f(\gamma(y)), \qquad \varphi_c = \lim_{y\to-\infty} \arg f(\gamma(y))$$

(possibly infinite) both exist and $\psi_c - \varphi_c > 0$ (see [4, Theorem 1, p. 17]). We will prove that $\psi_c - \varphi_c \le 2\pi$ for every $c > 0$.

For every $c > 0$ and $R > c$ consider two arcs in $\mathbb{H}$:

$$\gamma_{c,R} = \{c + iy : y \in [-\sqrt{R^2 - c^2}, \sqrt{R^2 - c^2}]\},$$
$$\sigma_{c,R} = \{w \in \mathbb{C} : |w| = R\} \cap \overline{\mathbb{H}}_c.$$

We will use the following parametrization:

$$\sigma_{c,R} : w = \sigma(t) = Re^{it}, \qquad t \in [-t(c,R), t(c,R)],$$

where $t(c,R) = \arctan(\sqrt{R^2 - c^2}/c)$. The arc $\gamma_{c,R}$ will be parametrized by (3.3) where $y \in [-\sqrt{R^2 - c^2}, \sqrt{R^2 - c^2}]\}$.

(a) Assume first that $\alpha \in (0,1)$. Fix $c > 0$. Then by (3.7) there exists $R_1 > c$ such that $|w/q(w)| \le \alpha < 1$ for $w \in \mathbb{H} \setminus \mathbb{D}(0, R_1)$. Take $R > R_1$. Hence $\mathbb{H} \setminus \mathbb{D}(0,R) \subset \mathbb{H} \setminus \mathbb{D}(0, R_1)$.

Since $\gamma_{c,R} \cup \sigma_{c,R}$ is a closed curve lying in $\mathbb{H}$, we have

$$\begin{aligned}(3.9)\qquad \arg f(\gamma(\sqrt{R^2-c^2})) - \arg f(\gamma(-\sqrt{R^2-c^2})) &= \Delta_{\gamma_{c,R}} \arg f(w)\\ &= \operatorname{Im}\int_{\gamma_{c,R}} \frac{f'(w)}{f(w)}\,dw = \operatorname{Im}\int_{\gamma_{c,R}} \frac{2}{q(w)}\,dw = \operatorname{Im}\int_{\sigma_{c,R}} \frac{2}{q(w)}\,dw\\ &= \operatorname{Im}\int_{-t(c,R)}^{t(c,R)} \frac{2iRe^{it}}{q(Re^{it})}\,dt \le 2\int_{-t(c,R)}^{t(c,R)} \frac{R}{|q(Re^{it})|}\,dt\\ &\le 4t(c,R)\alpha \le 2\pi\alpha < 2\pi.\end{aligned}$$

If now $R \to \infty$, then $t(c,R) \to \pi/2$, which implies

$$\begin{aligned}0 &< \psi_c - \varphi_c\\ &= \lim_{R\to\infty} (\arg f(\sqrt{R^2-c^2}) - \arg f(-\sqrt{R^2-c^2})) \le 2\pi.\end{aligned}$$

This means that $f(\mathbb{H}_c) \subset V_c$, where V_c is the sector with vertex at the origin and with opening angle $\psi_c - \varphi_c$. Since $f(\mathbb{H}_{c_2}) \subset f(\mathbb{H}_{c_1})$ for $0 < c_1 < c_2$, we have $V_{c_2} \subset V_{c_1}$. Hence $V = \bigcup_{c>0} V_c$ is a sector with vertex at the origin and with opening angle $\psi - \varphi = \sup\{\psi_c - \varphi_c : c > 0\}$. Obviously,

$0 < \psi - \varphi \le 2\pi$. As

$$f(\mathbb{H}) = \bigcup_{c>0} f(\mathbb{H}_c) \subset \bigcup_{c>0} V_c = V,$$

there exists $\varphi_0 \in \mathbb{R}$ such that

$$\begin{aligned} \varphi_0 < \arg f(w) &= \operatorname{Im} \log f(w) = \operatorname{Im} g(w) \\ &< \varphi_0 + \psi - \varphi \le \varphi_0 + 2\pi, \qquad w \in \mathbb{H}, \end{aligned} \tag{3.10}$$

where $g(w) = \log f(w),\ w \in \mathbb{H}$, is well defined in $\mathbb{H}$. But (3.8) yields

$$\operatorname{Re}\left\{\frac{f'(w)}{f(w)}\right\} = \operatorname{Re}\{g'(w)\} > 0, \qquad w \in \mathbb{H}.$$

Now [5, Proposition 1.10, p. 16] yields the univalence of g in $\mathbb{H}$. Since $f(w) = \exp(g(w))$, $w \in \mathbb{H}$, using (3.10) we deduce finally that f is univalent in $\mathbb{H}$.

(b) Let now $\alpha = 1$. Fix $c > 0$. For $n \in \mathbb{N}$ there exists $R_1(n) > c$ such that $|w/q(w)| \le 1 + 1/n$ for $w \in \mathbb{H} \setminus \mathbb{D}(0, R_1(n))$. For each $n \in \mathbb{N}$ take $R(n) > R_1(n)$ in such a way that $\lim_{n\to\infty} R(n) = \infty$. Clearly, $\mathbb{H} \setminus \mathbb{D}(0, R(n)) \subset \mathbb{H} \setminus \mathbb{D}(0, R_1(n))$. Let $\gamma_{c,R(n)}$ and $\sigma_{c,R(n)}$ be defined as in part (a). Since $\gamma_{c,R(n)} \cup \sigma_{c,R(n)}$ is a closed curve lying in $\mathbb{H}$, we have, as in (3.9), for each $n \in \mathbb{N}$,

$$\begin{aligned} \Delta_{\gamma_{c,R(n)}} \arg f(w) &\le 2 \int_{-t(c,R(n))}^{t(c,R(n))} \left| \frac{R(n)e^{it}}{q(R(n)e^{it})} \right| dt \\ &\le 4\left(1 + \frac{1}{n}\right) t(c, R(n))\alpha \le 2\left(1 + \frac{1}{n}\right)\pi, \end{aligned}$$

and we complete the proof as before.

Examples. Now we present some examples of functions. The first two examples are geometrically obvious. The first function maps univalently $\mathbb{H}$ onto a slit plane, the second one maps $\mathbb{H}$ onto a wedge. The third example does not exhibit an evident geometrical property. Therefore Theorem 3.3 offers a useful analytical method to check if the function is an element of $\mathcal{S}^*_\infty(\mathbb{H})$.

1. $f(w) = \sqrt{w^2 + a^2},\ w \in \mathbb{H},\ a \ge 0$. Then

$$q(w) = 2\,\frac{f(w)}{f'(w)} = 2\,\frac{w^2 + a^2}{w} = 2\left(w + \frac{a^2}{w}\right), \qquad w \in \mathbb{H}.$$

Consequently, $q \in \mathcal{A}(\mathbb{H})$, $\operatorname{Re} q(w) > 0$ for $w \in \mathbb{H}$, and $\lim_{w\to\infty} q(w)/w = 2$ so (3.7) is satisfied with $\alpha = 1/2$. By Theorem 3.3, $f \in \mathcal{S}^*_\infty(\mathbb{H})$.

2. $f(w) = w^\mu,\ w \in \mathbb{H},\ \mu \in (0, 2]$. Then

$$q(w) = \frac{2}{\mu}\, w, \qquad w \in \mathbb{H}.$$

Consequently, $q \in \mathcal{A}(\mathbb{H})$, $\operatorname{Re} q(w) > 0$ for $w \in \mathbb{H}$, and $\lim_{w\to\infty} q(w)/w = 2/\mu$ so (3.7) is satisfied with $\alpha = \mu/2$. By Theorem 3.3, $f \in \mathcal{S}^*_\infty(\mathbb{H})$ for each $\mu \in (0,2]$.

3. $f(w) = (w+1)^\beta \exp\left(\frac{1-w}{1+w}\right)$, $w \in \mathbb{H}$, $\beta > 0$. Then

$$q(w) = 2\,\frac{(w+1)^2}{\beta w + \beta - 2}, \qquad w \in \mathbb{H}.$$

For $\beta \in (0,2)$ the function q has a pole at $w = (2-\beta)/\beta$ so $q \notin \mathcal{A}(\mathbb{H})$.

For $\beta \geq 2$, $q \in \mathcal{A}(\mathbb{H})$. Moreover, for $w \in \mathbb{H}$, i.e. for $w = x + iy$ with $x > 0$, we have

$$\operatorname{Re} q(w) = 2\,\frac{(x+1)^2(\beta x + \beta - 2) + (2 + \beta(x+1))y^2}{(\beta(x+1) - 2)^2 + \beta^2 y^2} > 0.$$

Since $\lim_{w\to\infty} q(w)/w = 2/\beta$, we conclude that (3.7) is satisfied only for $\beta = 2$, i.e. for $\alpha = 1$. By Theorem 3.3, $f \in \mathcal{S}^*_\infty(\mathbb{H})$ for $\beta = 2$.

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