Asymptotics for quasilinear elliptic non-positone problems

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Abstract. In the recent years, many results have been established on positive solutions for boundary value problems of the form

\[- \text{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = \lambda f(u(x)) \quad \text{in} \ \Omega,\]
\[u(x) = 0 \quad \text{on} \ \partial \Omega,\]

where \(\lambda > 0\), \(\Omega\) is a bounded smooth domain and \(f(s) \geq 0\) for \(s \geq 0\). In this paper, a priori estimates of positive radial solutions are presented when \(N > p > 1\), \(\Omega\) is an \(N\)-ball or an annulus and \(f \in C^1(0,\infty) \cap C^0([0,\infty))\) with \(f(0) < 0\) (non-positone).

1. Introduction. In this paper, we consider the set of positive radial solutions to the following boundary value problem for a quasilinear elliptic P.D.E.:

\[(1.1) \quad \text{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \quad \text{in} \ \Omega,\]
\[(1.2) \quad u = 0 \quad \text{on} \ \partial \Omega,\]

where \(\Omega\) denotes an annulus or a ball in \(\mathbb{R}^N\) (\(N > p > 1\)), and \(\lambda > 0\).

The problem (1.1)–(1.2) arises in the theory of quasiregular and quasi-conformal mappings or in the study of non-Newtonian fluids. In the latter case, the quantity \(p\) is a characteristic of the medium. Media with \(p > 2\) are called dilatant fluids and those with \(p < 2\) are called pseudoplastics (see [1–2]). When \(p \neq 2\), the problem becomes more complicated since certain nice properties inherent to the case \(p = 2\) seem to be invalid or at least difficult to verify. The main differences between \(p = 2\) and \(p \neq 2\) are discussed in [6, 8]. The existence and uniqueness of positive solutions of (1.1)–(1.2) have been studied by many authors, for example, [4–10, 13–21] and the references therein.

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By a positive solution $u$ of (1.1)–(1.2), we mean a function $u \in C^1_0(\Omega)$ with $u > 0$ in $\Omega$ which satisfies

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v = \lambda \int_{\Omega} f(u)v
$$

for any $v \in C_0^\infty(\Omega)$. Thus, these solutions are considered in a weak sense. By a small solution $u_\lambda$ of (1.1)–(1.2), we mean that $\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = 0$ (or $\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = 0$). By a large positive solution $u_\lambda$ of (1.1)–(1.2), we mean that $\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = \infty$ (or $\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \infty$).

When $f$ is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s \to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu$, where $0 < \mu < p - 1$ and $\alpha_1, \alpha_2 > 0$, it has been shown in [6] that there exist at least two positive solutions for (1.1)–(1.2) when $\lambda$ is sufficiently large. If $\liminf_{s \to 0^+} f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it has been proved in [8] that the problem (1.1)–(1.2) has a unique positive solution when $\lambda$ is sufficiently large. If $f(s) > 0$ for $s \geq 0$ and $\limsup_{s \to 0^+} (f(s)/s^{p-2})' < 0$, it has been proved in [9] that the problem (1.1)–(1.2) has a unique small solution when $\lambda$ is sufficiently small. It also has been proved that there exists at least one large positive radial solution of (1.1)–(1.2) for $\Omega$ being an $N$-ball or an annulus when $\lambda$ is sufficiently small. If $f(0) < 0$, related results have been obtained in [7, 20].

A natural question is to determine how $\lambda$ and $d = \max_{\Omega} u(\cdot, \lambda) = \|u(\cdot, \lambda)\|_\infty$ are related. When $p = 2$, $f(0) < 0$ or $f(0) = 0$ and $\Omega$ is a unit ball in $\mathbb{R}^N$, the related results have been obtained by [11, 12]. In [21], the author studied this problem for the case where $\Omega$ is a unit ball in $\mathbb{R}^N$ and $f(0) < 0, p > 1$. In this paper, we further study this problem for $\Omega$ being an $N$-ball ($N > p > 1$) or an annulus and $f(0) < 0$ (non-positone). This extends and complements previous results in the literature [11, 12, 21].

Consider a positive radial solution $u$ of (1.1)–(1.2); thus $u = u(r, \lambda)$ satisfies

$$
(r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} f(u) = 0.
$$

If $\Omega$ is an annulus $0 < r_1 \leq r \leq r_2$, we introduce the transformation of variables

$$
s = r^{(p-N)/(p-1)}, \quad u(r) = v(s).
$$

Thus (1.3) becomes

$$
(|v'(s)|^{p-2}v'(s))' + \lambda ((p-1)/(N-p))^{p-1} s^{-p(N-1)/(N-p)} f(v(s)) = 0
$$

and the boundary conditions become

$$
v(s_1) = 0, \quad v(s_2) = 0.
$$
If $\Omega = B_1(0)$, the boundary condition (1.2) becomes
\[ u'(0) = 0, \quad u(1) = 0. \]

2. A priori estimates for $\Omega$ being an annulus. In this section, we consider the set of radially symmetric positive solutions to the equation
\[
\begin{cases}
- \text{div}( |\nabla u|^{p-2} \nabla u) = \lambda f(u) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ denotes an annulus in $\mathbb{R}^N$ ($N > p > 1$) and $\lambda > 0$. Here $f : [0, \infty) \rightarrow \mathbb{R}$ satisfies the following assumptions:

(A) $f \in C^1(0, \infty) \cap C([0, \infty))$, $f(0) < 0$, and there exists $\alpha > 0$ such that $f(s) < 0$ for $0 < s < \alpha$, $f(\alpha) = 0$, $f$ is increasing for $s > \alpha$ and $\lim_{s \rightarrow \infty} f(s) = \infty$.

(B) There are constants $L_0 > 0$ and $p - 1 < q < ((p-1)N + p)/(N-p)$ such that $\lim_{u \rightarrow \infty} f(u)/u^q = L_0$.

Theorem 2.1. Suppose that conditions (A) and (B) hold. Then there exist positive constants $K_1$ and $K_2$ such that for small $\epsilon$,
\[ K_1 < \epsilon \|u(\cdot, \lambda)\|_{C^1_0}^{q-p+1} < K_2, \]
where $\{u(\cdot, \lambda) : \lambda \in (0, \lambda_0)\}$ is an arbitrary positive radially symmetric solution of (1.1)–(1.2). Furthermore, for any sequence $\{\lambda_i\}$ with $\lim_{i \rightarrow \infty} \lambda_i = 0$, there exists a subsequence, still denoted by $\{\lambda_i\}$, a constant $\theta$, and a positive function $w$ such that

(1) $w$ is a solution of the problem
\[ - \text{div}( |\nabla u|^{p-2} \nabla u) = \theta L_0 u^q \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega, \]

(2) $\|u(\cdot, \lambda_i)/u(\cdot, \lambda_i)\|_{C^1_0}$ converges to $w$ in $C^1(\overline{\Omega})$ as $i \rightarrow \infty$.

To obtain Theorem 2.1, the following lemma is established:

Lemma 2.2. Let $f$ satisfy condition (A) and $u_\lambda \in C^1_0(\overline{\Omega})$ be a radially symmetric positive solution of (1.1)–(1.2). Then $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1} = \infty$.

Proof. On the contrary, assume that there exist sequences $\{\lambda_n\}$ and $\{u_n\} \equiv \{u_{\lambda_n}\} \in C^1_0(\overline{\Omega})$ such that $\lambda_n \rightarrow 0$ and $\|u_n\| \leq M$, where $M > 0$ is independent of $n$. Then $\|u_n\|_{C^1} \not\rightarrow 0$ as $n \rightarrow \infty$. Indeed, suppose this does not hold; by the regularity of $- \text{div}( |\nabla|^{p-2} \nabla \cdot )$ (see [6]), there exists $\omega \geq 0$ in $\Omega$ such that $\lambda_n^{1/(p-1)} u_n \rightarrow \omega$ in $C^1(\Omega)$ as $n \rightarrow \infty$. Moreover, $\omega$ satisfies the problem
\[ - \text{div}( |\nabla \omega|^{p-2} \nabla \omega) = f(0) < 0 \quad \text{in } \Omega, \]
\[ \omega = 0 \quad \text{on } \partial \Omega. \]
It follows from the maximum principle that \( \omega < 0 \) in \( \Omega \). This is impossible. Now, since \( u_n \) is uniformly bounded in \( \Omega \) and \( \lambda_n \to 0 \) as \( n \to \infty \), it follows from the regularity of \( -\text{div}(\nabla |\nabla^{\cdot} p-2\nabla^{\cdot}) \) again that there exists \( \overline{\omega} \in C^1_0(\Omega) \) with \( \overline{\omega} \geq 0 \) in \( \Omega \) such that \( u_n \to \overline{\omega} \) in \( C^1(\Omega) \) as \( n \to \infty \) and \( \overline{\omega} \) satisfies

\[
-\text{div}(\nabla \overline{\omega}|\nabla^{\cdot} p-2\nabla^{\cdot}) \equiv 0 \quad \text{in} \quad \Omega, \\
\overline{\omega} = 0 \quad \text{on} \quad \partial \Omega.
\]

Thus, \( \overline{\omega} \equiv 0 \) in \( \Omega \). This also implies that \( u_n \to 0 \) in \( C^1(\Omega) \) as \( n \to \infty \). But the above argument implies that this is impossible. Hence, we conclude that \( \|u_n\|_\infty \to \infty \) as \( n \to \infty \).

**Lemma 2.3.** Let \( a > 0 \). Then, for any \( \theta \leq 0 \), the equation

\[
(|u'|^{p-2}u') + au(s)\mu = 0 \quad \text{in} \quad (\theta, \infty)
\]

has no bounded positive solution \( u \in C^1(\theta, \infty) \) with \( u'(0) = 0 \). Moreover, the equation

\[
(|u'|^{p-2}u') + au(s)\mu = 0 \quad \text{in} \quad (-\infty, \infty)
\]

has no bounded positive entire solution \( u \in C^1(-\infty, \infty) \) with \( u'(0) = 0 \).

**Proof.** Suppose that such a solution \( u(s) \) exists. Let \( \Phi_p(y) = |y|^{p-2}y \).

Then

\[
(2.2) \quad \Phi_p(u'(s)) = -\int_0^s au(\xi)\mu d\xi \quad \text{for} \quad s \in (0, \infty).
\]

Thus, \( \Phi_p(u'(s_0)) = -k < 0 \) for some \( s_0 > 0 \) where \( k = a\int_0^{s_0} u(\xi)\mu d\xi \). By (2.2), \( \Phi_p(u'(s)) \leq -k \) for \( s > s_0 \), since \( u(s) > 0 \) for \( s > 0 \). Then

\[
(2.3) \quad u'(s) \leq \Phi^{-1}_p(-k) = -k^{1/(p-1)} \quad \text{for} \quad s > s_0.
\]

Integrating (2.3) over \( (s_0, s) \), we obtain \( u(s) \to -\infty \) as \( s \to \infty \), contrary to the assumption that \( u(s) \) is a bounded solution.

**Proof of Theorem 2.1.** By the standard estimates for elliptic equations and condition (B), it follows that

\[
\|u(\cdot, \lambda)\|_{p-1}^p \leq C(\Omega)(\lambda f(u(\cdot, \lambda))) \|_{\infty} \\
= C(\Omega)(\lambda L_0 u(\cdot, \lambda)^q + \{f(u(\cdot, \lambda)) - L_0 u(\cdot, \lambda)^q\}) \|_{\infty}.
\]

That is,

\[
1 \leq C(\Omega)\lambda L_0 \frac{\|u(\cdot, \lambda)^q\|_{\infty}}{\|u(\cdot, \lambda)\|_{p-1}^p} \\
+ C(\Omega)\lambda \left\| \frac{f(u(\cdot, \lambda)) - L_0 u(\cdot, \lambda)^q}{u(\cdot, \lambda)^q + 1} \right\|_{\infty} \frac{\|u(\cdot, \lambda)^q + 1\|_{\infty}}{\|u(\cdot, \lambda)\|_{p-1}^p}.
\]
By (B), there exists a positive constant $K_0$ such that
\[ |(f(u) - L_0 u^q)/(u^q + 1)| < K_0 \quad \text{for } u \in \mathbb{R}^+. \]

Then
\[ 1 \leq C(\Omega) \lambda \|u(\cdot, \lambda)\|^{\frac{q-p+1}{\infty}} + C(\Omega) \lambda K_0 \left\{ \|u(\cdot, \lambda)\|^{\frac{q-p+1}{\infty}} + \frac{1}{\|u(\cdot, \lambda)\|^{\frac{p-1}{\infty}}} \right\}. \]

From $\lim_{\lambda \to 0} \|u(\cdot, \lambda)\| = \infty$, it follows that there exists a positive constant $K_1$ such that, for any $\lambda \in (0, \lambda_0)$, $K_1 < \lambda \|u(\cdot, \lambda)\|^{\frac{q-p+1}{\infty}}$.

Thus, the left-hand inequality in Theorem 2.1 is established.

To obtain the other half of Theorem 2.1, we show that $T = \lambda \|u\|^{\frac{q-p+1}{\infty}}$ is bounded as $\lambda \to 0$. Let $u_\lambda$ be a positive radial solution of (1.1)–(1.2) satisfying $\|u_\lambda\|^{\infty} \to \infty$ as $\lambda \to 0^+$. Then there exists a positive solution $v_\lambda$ of (1.5)–(1.6) satisfying $\|v_\lambda\| \to \infty$ as $\lambda \to 0^+$. Let $(\lambda_n, v_n)$ be a positive solution of (1.5)–(1.6) with $\lambda = \lambda_n$ satisfying $\lambda_n \to 0^+$ and $\|v_n\| \to \infty$ as $n \to \infty$. Then $w_n = v_n/\|v_n\|\infty$ satisfies
\[ (2.4) \quad -(\Phi_p(w_n(s)))' = \lambda_n \|v_n\|^{\frac{q-p+1}{\infty}} \left( \frac{p-1}{N-p} \right)^p s^{-p(N-1)/(N-p)} f(v_n) \|v_n\|^{q/\infty}, \]

and $w_n(s_1) = w_n(s_2) = 0$, $\|w_n\|\infty = 1$.

Now, we show that $\{\lambda_n \|v_n\|^{\frac{q-p+1}{\infty}}\}$ is bounded. We prove this by a blowing up argument as in [3]. Suppose that $T_n \to \infty$ as $n \to \infty$. Let $\tilde{s}_n \in (s_1, s_2)$ be such that $w_n(\tilde{s}_n) = 1$, $y_n = T_n^{1/p}(s - \tilde{s}_n)$ and $\tilde{w}_n(y_n) = w_n(s)$. Then $\tilde{w}_n(0) = 1$, $\tilde{w}_n'(0) = 0$ and $\tilde{w}_n(y_n)$ satisfies
\[ (2.5) \quad -(\Phi_p(\tilde{w}_n'))' = \left( \frac{p-1}{N-p} \right)^p (y_n T_n^{-1/p} + \tilde{s}_n)^{-p(N-1)/(N-p)} \times f(\|v_n\|^\infty \tilde{w}_n(y_n)) \|v_n\|^{q/\infty}. \]

Since $\tilde{s}_n \in [s_1, s_2]$ and $f(s) \leq \beta_1 + \beta_2 s^q$ and $\|v_n\|\infty \to \infty$ as $n \to \infty$, the right-hand side of (2.5) is uniformly bounded. Thus, there exist subsequences (still denoted by $\{\tilde{s}_n\}$, $\{\tilde{w}_n\}$ and $\{v_n\}$) such that $\tilde{w}_n \to \tilde{w}$ in $C^1_{\text{loc}}(\tilde{s}_n, 0)$ (or $C^1_{\text{loc}}(0, \infty)$) as $n \to \infty$. Here $\theta \leq 0$ is a fixed number since the limit of $\tilde{s}_n$ may be $s_1$ or $s_2$ and $T_n \to \infty$. If $\tilde{s}_n \to s_1$ as $n \to \infty$, we assume that $\lim_{n \to \infty} T_n^{1/p}(s_1 - \tilde{s}_n) = \theta \leq 0$ (or $\theta = -\infty$). Otherwise, we can choose a subsequence of $\{T_n^{1/p}(s_1 - \tilde{s}_n)\}$ whose limit exists (or is $-\infty$). If the limit of $\lim s_n = s_2$, and if we set $y_n = T_n^{1/p}(s_2 - \tilde{s}_n)$, it follows that $\tilde{w}_n \to \tilde{w}$ in $C^1_{\text{loc}}(-\infty, \infty)$ (or $C^1_{\text{loc}}(\tilde{s}_n, \infty)$, $\theta \leq 0$) as $n \to \infty$. Therefore, we assume that $\tilde{w}_n \to \tilde{w}$ in $C^1_{\text{loc}}(\tilde{s}_n, \infty)$ (or $C^1_{\text{loc}}(\tilde{s}_n, \infty)$). Since $\tilde{w} \in C^1(\theta, \infty)$ (or $C^1(-\infty, \infty)$) satisfies $-(\Phi_p(\tilde{w}'))' \geq 0$ in $(\theta, \infty)$ (or $(\infty, \infty)$), and $\tilde{w}(0) = 1$ and $\tilde{w}'(0) = 0$, the strong maximum principle as in Lemma 2.3 of [6] implies that $\tilde{w} > 0$ in $(\theta, \infty)$ (or $(\infty, \infty)$). Thus, for any interval in $(\theta, \infty)$ (or
sequence of \( \text{verges to} \) the set of radially symmetric positive solutions to the equation

\[
f(\|v_n\|_\infty \hat{w}_n) \rightarrow L_0 \hat{w}^q\]

in \( C_{\text{loc}}(\theta, \infty) \) (or \( C_{\text{loc}}(-\infty, \infty) \)) as \( n \rightarrow \infty \). Therefore, \( \hat{w} \) satisfies

\[-(\Phi_p(\hat{w}'))' = L_0 ((p-1)/(N-p))^{q-1} s_*^{p(N-1)/(N-p)} \hat{w}^q\]

in \((\theta, \infty) \) (or \((-\infty, \infty) \)). Here \( s_* = \lim_{n \rightarrow \infty} \hat{s}_n \). This contradicts Lemma 2.3. Thus, \( \{T_n\} \) is bounded. Therefore

\[K_1 < \lambda \|w\|_\infty^{q-p+1} < K_2.\]

Finally, let \( \{\lambda_i\} \) be a sequence with \( \lim_{i \rightarrow \infty} \lambda_i = 0 \) and denote the quantity \( \lambda_i \|v(\cdot, \lambda_i)\|_{q-p+1} \) by \( \theta_i \). Suppose that \( \theta \in [K_1, K_2] \) is any accumulation point of \( \{\theta_i\} \). Thus there exists a subsequence of \( \{\theta_i\} \) (still denoted by \( \{\theta_i\} \) later) which converges to \( \theta \). Let \( w(x, \lambda) = v(x, \lambda)/\|v(\cdot, \lambda)\|_\infty \). Then \( \|w(\cdot, \lambda)\|_\infty = 1 \) and

\[-\text{div}(|\nabla w|^{p-2} \nabla w) = \theta_i \frac{f(v(x, \lambda_i))}{\|v(\cdot, \lambda_i)\|_\infty^q}.\]

Using the same idea as above for (2.4), we find a function \( w(\cdot) \) and a subsequence of \( \{w(\cdot, \lambda_i)\} \) (still denoted by \( \{w(\cdot, \lambda_i)\} \)) such that \( \{w(\cdot, \lambda_i)\} \) converges to \( w \) in \( C^1(s_1, s_2) \) as \( i \rightarrow \infty \). By condition (B), it follows that

\[\lim_{i \rightarrow \infty} \frac{f(\|v(\cdot, \lambda_i)\|_\infty w(x, \lambda_i))}{\|v(\cdot, \lambda_i)\|_\infty^q} = L_0 w^q.\]

Therefore \( w(\cdot) \) is a positive solution of the problem

\[-\text{div}(|\nabla w|^{p-2} \nabla w) = \theta L_0 w^q \quad \text{in} \ \Omega,\]

\[w = 0 \quad \text{on} \ \partial \Omega,\]

and \( \|w(\cdot)\|_\infty = 1 \).

3. A priori estimates for \( \Omega \) being a ball. In this section, consider the set of radially symmetric positive solutions to the equation

\[
\begin{align*}
-\text{div}(|\nabla u|^{p-2} \nabla u) &= \lambda f(u) \quad \text{for} \ x \in \Omega, \\
|u|_{\partial \Omega} &= 0,
\end{align*}
\]

where \( \Omega \) denotes the unit ball in \( \mathbb{R}^N \) \( (N > 1) \), centered at the origin, and \( \lambda > 0 \). Here \( f : [0, \infty) \rightarrow \mathbb{R} \) is assumed to satisfy

\[
(3.3) \quad f(0) < 0 \text{ (non-positone), } f'(u) \geq 0, \text{ and } f(u_0) > 0 \text{ for some } u_0 > 0.
\]

Let \( F \) be defined as \( F(t) = \int_0^t f(s) \, ds \), and let \( \beta \) and \( \theta \ (\beta < \theta) \) be the unique positive zeros of \( f \) and \( F \), respectively.

In this section, the following theorem is proved:
Theorem 3.1. Let \( u \) be a radially symmetric positive solution of (3.1)–(3.2) with \( u(0) = d \) and suppose \( f \) satisfies (3.3). Then for large \( \lambda \),

\[
\left( \frac{p}{p-1} \right)^{p-1} (N-1) \leq \lambda f(d) \frac{d}{dp-1}.
\]

\[
\leq \frac{2N}{dp-1} \left( \frac{p}{p-1} \right)^{p-1} \left( \int_0^d \frac{ds}{f(s)^{1/(p-1)}} \right)^{p-1}.
\]

**Remark.** If \( f(u) \leq M \) for all \( u \), or if \( f(u) = u^\alpha - 1 \) where \( 0 < \alpha < p-1 \), then \( f(d) d^{-(p-1)} (\int_0^d f(s)^{-1/(p-1)} ds)^{p-1} \) is finite.

Note that radially symmetric positive solutions of (3.1)–(3.2) are strictly decreasing in \( r \) for \( r \in (0, 1) \) where \( r = \|x\| \). Thus, they satisfy

\[
(\Phi_p(u'))' + \frac{N-1}{r} \Phi_p(u') + \lambda f(u) = 0 \quad \text{in } (0, 1),
\]

\[
u(0) = d, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(r) < 0 \quad \text{in } (0, 1).
\]

where \( \Phi_p(s) = |s|^{p-2} s, \ p > 1 \).

If \( u \) is a solution of (3.5)–(3.6), then multiplying (3.5) by \( r^{N-1} \) and integrating from 0 to \( t \) gives

\[
\int_0^t (r^{N-1} \Phi_p(u'))' dr = \int_0^t \lambda r^{N-1} f(u) dr.
\]

Since \( u \) is decreasing and \( f \) is increasing, it follows that

\[
-t^{N-1} \Phi_p(u') = \lambda \int_0^t r^{N-1} f(u) dr \geq \lambda f(u(t)) \int_0^t r^{N-1} dr = \frac{\lambda t^{N-1} f(u)}{N}.
\]

Hence

\[
(-u')^{p-1} \geq \frac{\lambda t f(u)}{N}.
\]

Next, multiplying (3.5) by \( u' \) and integrating over \([0, 1]\) yields

\[
\frac{p-1}{p} |u'(1)|^p + \frac{1}{p} \int_0^1 \frac{N-1}{r} |u'|^p dr = \lambda F(d).
\]

Note that this implies

\[
d > \theta.
\]

To prove Theorem 3.1, we need the following lemma:

**Lemma 3.2 (see [19]).** Let \( u \) be a radially symmetric positive solution of (3.1)–(3.2). Then there exists \( M > 0 \) such that for large \( \lambda \),

\[
|u'(1)| > \lambda^{1/(p-1)} M.
\]
The proof of Theorem 3.1 is based upon a modification of the method of Iaia [12].

Proof of Theorem 3.1. First, Hölder’s inequality gives

\[ d = u(0) - u(1) = -\int_0^1 u'(t) \, dt = \int_0^1 \frac{-u'}{t^{1/p}} \, t^{1/p} \, dt \]

\[ \leq \left( \int_0^1 \frac{|u'|^p}{t} \, dt \right)^{1/p} \left( \int_0^1 t^{1/(p-1)} \, dt \right)^{(p-1)/p}. \]

Next, it follows from (3.8) that

\[ dp \leq \left( \frac{p-1}{p} \right)^{p-1} \left( \int_0^1 \frac{|u'|^p}{t} \, dt \right)^{p-1} \frac{\lambda F(d)}{N-1}. \]

Thus

\[ \frac{\lambda F(d)}{dp} \geq \left( \frac{p}{p-1} \right)^{p-1} (N-1). \]

Finally, since \( f' \geq 0, \)

(3.10) \[ F(d) = \int_0^d f(s) \, ds = df(d) - \int_0^d sf'(s) \, ds \leq df(d). \]

This proves the left-hand inequality of (3.4).

In order to establish the right-hand inequality of (3.4), from (3.7) we get

\[ -u'(t) \geq \left( \frac{\lambda t f(u)}{N} \right)^{1/(p-1)}. \]

Let \( q_\lambda \in (0, 1) \) be such that \( u(q_\lambda) = \theta. \) Then \( u(t) \geq \theta > \beta \) on \([0, q_\lambda].\) Thus \( f(u(t)) \geq f(\theta) > f(\beta) = 0 \) on \([0, q_\lambda].\) So, on \([0, q_\lambda] \) we have

\[ \int_0^{q_\lambda} \frac{-u'}{f(u)^{1/(p-1)}} \, dt \geq \int_0^{q_\lambda} \left( \frac{\lambda t}{N} \right)^{1/(p-1)} \, dt = \left( \frac{\lambda}{N} \right)^{1/(p-1)} \left( \frac{p-1}{p} \right) q_\lambda^{p/(p-1)}. \]

Changing variables in the first integral via \( s = u(t) \) gives

\[ \int_\theta^d \frac{ds}{f(s)^{1/(p-1)}} \geq \left( \frac{\lambda}{N} \right)^{1/(p-1)} \left( \frac{p-1}{p} \right) q_\lambda^{p/(p-1)}. \]

Thus,

(3.11) \[ \frac{f(d)^{1/(p-1)}}{d} \int_\theta^d \frac{ds}{f(s)^{1/(p-1)}} \geq \left( \frac{\lambda f(d)}{N^{1/(p-1)} d} \right)^{1/(p-1)} \left( \frac{p-1}{p} \right) q_\lambda^{p/(p-1)}. \]

Therefore, the proof of Theorem 3.1 will be completed once the following lemma is established.
Lemma 3.3. \( \lim_{\lambda \to \infty} q_\lambda = 1. \)

From this lemma, for large \( \lambda \), we have \( q_\lambda^p \geq 1/2 \). Substituting this into (3.11), one can deduce

\[
\frac{\lambda f(d)}{d^{p-1}} \leq \frac{2N f(d)}{d^{p-1}} \left( \frac{p-1}{p} \right)^{p-1} \left( \int_0^d \frac{ds}{f(s)^{1/(p-1)}} \right)^{p-1},
\]

which completes the proof of Theorem 3.1.

Proof of Lemma 3.3. Multiplying (3.5) by \( u_0 \) and integrating from \( t \) to 1 gives

\[
\int_t^1 \left[ u'(\Phi_p(u'))' + \frac{N-1}{r} |u'|^p \right] dr = \int_t^1 (-\lambda f(u)u') dr.
\]

Thus

\[
\frac{p-1}{p} [|u'|^p(1) - |u'|^p(t)] + \int_t^1 \frac{N-1}{r} |u'|^p dr = -\lambda [F(u(1)) - F(u(t))].
\]

Since \( F(u(1)) = F(0) = 0 \), it follows that

\[
\frac{p-1}{p} [|u'|^p(1) - |u'|^p(t)] \leq \lambda F(u(t)).
\]

Now, for \( q_\lambda \leq t \leq 1 \), it follows that \( \theta = u(q_\lambda) \geq u(t) \geq u(1) = 0 \), and then \( F(u(t)) \leq 0 \). Hence,

\[
(3.12) \quad |u'|^p(t) \leq |u'|^p(1) \quad \text{for} \ t \in [q_\lambda, 1]. \]

Now Lemma 3.2 shows that there exists a \( c > 0 \) independent of \( \lambda \) such that

\[
-u'(1) \geq c\lambda^{1/(p-1)} \quad \text{for large} \ \lambda.
\]

Consequently, it follows from (3.12) that

\[
(-u'(t))^p \geq (-u'(1))^p \geq c^p \lambda^{p/(p-1)} \quad \text{for} \ t \in [q_\lambda, 1].
\]

Integrating on \([q_\lambda, 1]\) gives

\[
\theta = u(q_\lambda) = -\int_{q_\lambda}^1 u'(t) dt \geq \int_{q_\lambda}^1 c\lambda^{1/(p-1)} dt = c\lambda^{1/(p-1)}(1 - q_\lambda).
\]

Thus

\[
0 \leq 1 - q_\lambda \leq \frac{\theta}{c\lambda^{1/(p-1)}}.
\]

As \( \lambda \to \infty \) the right-hand side of the above expression tends to zero; hence \( \lim_{\lambda \to \infty} q_\lambda = 1 \) and this completes the proof of the lemma.
References


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