Regularity and Uniqueness of Solutions to Boundary Blow-up Problems for the Complex Monge–Ampère Operator

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Summary. We prove that plurisubharmonic solutions to certain boundary blow-up problems for the complex Monge–Ampère operator are Lipschitz continuous. We also prove that in certain cases these solutions are unique.

1. Introduction. In [3], Cheng and Yau studied the problem

\[
\begin{aligned}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (z) \right) &= f(z) e^{K u(z)}, \quad z \in \Omega, \\
\lim_{z \to z_0} u(z) &= \infty \quad \text{for all } z_0 \in \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary, \( f \) is a smooth strictly positive function and \( K > 0 \) a constant. They showed that there is a unique smooth plurisubharmonic solution to this problem. In this paper we study a similar problem, namely

\[
\begin{aligned}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (z) \right) &= f(z, u(z)), \quad z \in \Omega, \\
\lim_{z \to z_0} u(z) &= \infty \quad \text{for all } z_0 \in \partial \Omega,
\end{aligned}
\]

where the right hand side is a function \( f \in C^\infty(\bar{\Omega} \times \mathbb{R}) \) which is strictly positive, increasing in the second variable and satisfies the following three conditions:

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A: There exist functions $h \in C^\infty(\overline{\Omega})$ and $f_1 \in C^\infty(\mathbb{R})$ and two strictly positive constants $c_1$ and $c_2$ such that

$$\lim_{t \to \infty} \frac{f(z, t)}{f_1(t)} = h(z)$$

uniformly in $\Omega$ and $c_1 f_1(t) \leq f(z, t) \leq c_2 f_1(t)$ for all $(z, t) \in \Omega \times \mathbb{R}$.

B: The function $f_1$ is strictly positive and increasing.

C: The function

$$\Psi_n(a) = \int_a^\infty ((n + 1)F(y))^{-1/(n+1)} \, dy$$

exists for $a > 0$, where $F'(s) = f_1(s)$ and $F(0) = 0$.

Certain aspects of this problem has been studied by the author and Matero in [7].

The following theorem proven by Caffarelli, Kohn, Nirenberg and Spruck in [2] will be useful.

**Theorem 1.1.** Let $\Omega$ be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ be a strictly positive function which is increasing in the second variable. Let $\varphi \in C^\infty(\partial \Omega)$. Then the problem

$$\begin{cases}
\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) = f(z, u) & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega,
\end{cases}$$

(2)

has a unique strictly plurisubharmonic solution $u$. Moreover, $u \in C^\infty(\overline{\Omega})$.

This result is used to construct solutions to Problem (1). A sequence of plurisubharmonic functions $u_N$ which solve Problem (2) on certain pseudo-convex domains $\Omega_N$ is constructed. We construct upper and lower bounds for these solutions and since $\Omega = \bigcup_N \Omega_N$ we can conclude that the sequence $u_N$ converges to a solution for Problem (1) on $\Omega$. This is done in Section 2. In Section 3 the regularity of the solution is studied in some special cases. There it is assumed that the right hand side $f$ depends only on $u$, that is, $f(z, u) = f(u)$, and also satisfies an extra condition. The extra assumption is used to get a priori estimates for the first derivatives of solutions, which lets us conclude that solutions to Problem (1) are Lipschitz under these assumptions. Finally, in Section 4 uniqueness of solutions is studied. Here the right hand side can depend on the $z$-variable but we need to make another extra assumption. This extra assumption together with estimates on the boundary behavior of the solution, which were proved in [7], lets us conclude that solutions to Problem (1) are unique.
We will use the notation

\[ u_j = \frac{\partial u}{\partial z_j}, \quad u_k = \frac{\partial u}{\partial \bar{z}_k}, \quad u_{jk} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}. \]

2. Construction of solutions. In order to prove existence of a solution of the problem

\[
\begin{align*}
\det (u_{jk}(z)) &= f(z, u(z)), \quad z \in \Omega, \\
\lim_{z \to z_0} u(z) &= \infty \quad \text{for all} \quad z_0 \in \partial \Omega,
\end{align*}
\]

we shall begin by constructing approximate solutions. Let \( \varrho: \Omega \to \mathbb{R} \) be a strictly negative plurisubharmonic function such that \( \varrho \in C^\infty(\overline{\Omega}) \) and \( \lim_{z \to z_0} \varrho(z) = 0 \) for all \( z_0 \in \partial \Omega \). Take a strictly increasing convex function \( g: \mathbb{R}^+ \to \mathbb{R} \) such that \( \lim_{x \to 0^-} g(x) = \infty \). Put \( \varphi(z) = g(\varrho(z)) \). This is a plurisubharmonic function which satisfies \( \lim_{z \to z_0} \varphi(z) = \infty \) for all \( z_0 \in \partial \Omega \).

Let

\[ (\varrho_{jk}) = (\varrho_{jk})^{-1} \]

and

\[ \| d\varrho \|^2 = \varrho_{jk} \varrho_{\bar{k}j}. \]

We see that

\[ \frac{\partial \varphi}{\partial \bar{z}_k} = \varrho_{jk} g'(\varrho) \]

and

\[ \varphi_{jk} = \varrho_{jk} g'(\varrho) + \varrho_{j} \varrho_{k} g''(\varrho). \]

Let \( M_{jk} \) be the minor

\[
\begin{pmatrix}
\varrho_{11} & \cdots & \varrho_{1(k-1)} & \varrho_{1(k+1)} & \cdots & \varrho_{1n} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\varrho_{(j-1)1} & \cdots & \varrho_{(j-1)(k-1)} & \varrho_{(j-1)(k+1)} & \cdots & \varrho_{(j-1)n} \\
\varrho_{(j+1)1} & \cdots & \varrho_{(j+1)(k-1)} & \varrho_{(j+1)(k+1)} & \cdots & \varrho_{(j+1)n} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\varrho_{n1} & \cdots & \varrho_{n(k-1)} & \varrho_{n(k+1)} & \cdots & \varrho_{nn}
\end{pmatrix}
\]

We see that

\[
\det (\varphi_{jk}) = \det (\varrho_{jk} g'(\varrho) + \varrho_{j} \varrho_{k} g''(\varrho))
\]

\[
= g'(\varrho)^n \det (\varrho_{jk}) + g''(\varrho) g'(\varrho)^{n-1} \sum_{j,k=1}^{n} M_{jk} \varrho_{j} \varrho_{k}
\]

\[
= (g'(\varrho)^n + \| d\varrho \|^2 g''(\varrho) g'(\varrho)^{n-1}) \det (\varrho_{jk}).
\]
Since
\[ g'(\varphi(z)) = 1/(g^{-1})'(\varphi(z)) \]
and
\[ g''(\varphi(z)) = -(g^{-1})''(\varphi(z))/(g^{-1})'(\varphi(z))^3, \]
this can be rewritten as
\[ \det(\varphi_{jk}) = \frac{1}{(g^{-1})'(\varphi(z))} \det(\varphi_{jk}) - \frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \|d\varphi\|^2 \det(\varphi_{jk}). \]

We shall show that we can choose \( g \) so that
\[ \frac{1}{(g^{-1})'(\varphi(z))} \leq -\frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \|d\varphi\|^2 \]
near the boundary. This will show that the last term is the important term.

Hopf's lemma, sometimes also referred to as Zaremba's principle, implies that a plurisubharmonic function \( u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \) which satisfies \( u(z) < u(z_0) \) for all \( z \in \Omega \) and a boundary point \( z_0 \) also satisfies \( (\partial u/\partial \nu)(z_0) < 0 \) where \( \nu \) denotes the inward-pointing normal to \( \partial \Omega \). A proof of Hopf's lemma can be found in Taylor's book [9]. Since every boundary point is a global maximum for \( \varphi \) and \( \partial \Omega \) is compact we see that \( \|d\varphi\|^2 > \varepsilon \), for some \( \varepsilon > 0 \), near the boundary.

We are interested in solving
\[
\begin{cases}
\det(u_{jk}^{-1}(z)) = f(z,u(z)), & z \in \Omega, \\
\lim_{z \to z_0} u(z) = \infty & \text{for all } z_0 \in \partial \Omega,
\end{cases}
\]
where \( f \) is strictly positive, increasing in the second variable and satisfies conditions \( A, B \) and \( C \). We deduce what \( g \) should be by solving
\[ -\frac{(g^{-1})''(x)}{(g^{-1})'(x)^{n+2}} = f_1(x). \]

Rewriting this we get
\[ \frac{d}{dx} \left( \frac{1}{(n+1)(g^{-1})'(x)^{n+1}} \right) = f_1(x). \]

Integrating we see that
\[ \frac{1}{(g^{-1})'(x)^{n+1}} = (n+1)F(x). \]

This implies that
\[ g^{-1}(x) = \int ((n+1)F(x))^{-1/(n+1)} \, dx. \]

In particular, we can choose \( g^{-1}(x) = -\Psi_n(x) \). Making this choice we get
\[ (g^{-1})'(x) = ((n+1)F(x))^{-1/(n+1)}. \]
Let us now turn to the question if
\[
\frac{1}{(g^{-1})'(\varphi(z))^n} \leq - \frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \|d\varrho\|_\varrho^2
\]
near the boundary. But this is the same as
\[
\varepsilon^{-1} \leq - \frac{(g^{-1})''(x)}{(g^{-1})'(x)^2} = \frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right)
\]
as \(x\) tends to \(\infty\). Here \(\varepsilon\) is the infimum of \(\|d\varrho\|_\varrho^2\) in some neighborhood of the boundary. Assume that
\[
\frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right) = \frac{d}{dx}((n+1)F(x))^{1/(n+1)} < \varepsilon^{-1}
\]
for large \(x\). We get
\[
((n+1)F(x))^{1/(n+1)} < \varepsilon^{-1}x + C
\]
for large \(x\) but this contradicts the integrability of \(((n+1)F(x))^{-1/(n+1)}\). Hence
\[
\frac{d}{dx}\left(\frac{1}{(g^{-1})'(x)}\right) \geq \varepsilon^{-1}
\]
and we conclude that
\[
\frac{1}{(g^{-1})'(\varphi(z))^n} \leq - \frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \|d\varrho\|_\varrho^2
\]
near the boundary.

Having this at our disposal we can construct plurisubharmonic functions which are approximate solutions to the problem we are interested in. Namely, given \(f\) and \(f_1\) use the method above to choose \(g\). Take a plurisubharmonic function \(\varrho\) which solves
\[
\begin{cases}
\det (\varrho_{jk}(z)) = 1, & z \in \Omega, \\
\lim_{z \to z_0} \varrho(z) = 0 & \text{for all } z_0 \in \partial\Omega.
\end{cases}
\]
By Theorem 1.1 we know that \(\varrho \in C^\infty(\overline{\Omega})\). It is also strictly plurisubharmonic on \(\overline{\Omega}\). Hence \(\|d\varrho\|_\varrho^2 \in C^\infty(\overline{\Omega})\). Put \(\varphi = g \circ \varrho\). We see that \(\lim_{z \to z_0} \varphi(z) = \infty\) for all \(z_0 \in \partial\Omega\) and
\[
\det (\varphi_{jk}(z)) = \frac{1}{(g^{-1})'(\varphi(z))^n} - \frac{(g^{-1})''(\varphi(z))}{(g^{-1})'(\varphi(z))^{n+2}} \|d\varrho\|_\varrho^2 = \kappa(z) f_1(\varphi)
\]
where
\[
0 < C \leq \kappa(z) = \|d\varrho\|_\varrho^2 - \frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)} \leq C'.
\]
The existence of the upper bound \(C'\) is clear. The lower bound is a little trickier. At points near the boundary we know, by Hopf’s lemma, that \(\|d\varrho\|_\varrho^2 > \varepsilon\).
At points where \(||d\varphi||_2^2 \leq \varepsilon\) (these points are not close to the boundary), we see that \(\varphi\) is bounded and hence
\[
-\frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)} \geq \varepsilon'
\]
for some \(\varepsilon' > 0\). We see that there is a lower bound \(C\) so that \(0 < C < \kappa(z)\).

Now notice that \(\varphi_K = K\varphi\) satisfies
\[
\begin{cases}
\det(\varphi_{K,j\bar{k}}(z)) = K^n, & z \in \Omega, \\
\lim_{z \to z_0} \varphi_K(z) = 0 & \text{for all } z_0 \in \partial \Omega.
\end{cases}
\]

The function \(\varphi_K = g \circ \varphi_K\) satisfies \(g^{-1}(\varphi_K) = Kg^{-1}(\varphi)\). Therefore
\[
\det(\varphi_{K,j\bar{k}}(z)) = \det(\varphi_{K,j\bar{k}}(z))\left(||d\varphi_K||_2^2 \frac{(g^{-1})'(\varphi_K)^2}{(g^{-1})''(\varphi_K)}\right)f_1(\varphi_K)
\]
\[
= K^n \left( K||d\varphi||_2^2 - \frac{(g^{-1})'(\varphi_K)^2}{(g^{-1})''(\varphi_K)}\right)f_1(\varphi_K)
\]
\[
= K^{n+1} \left( ||d\varphi||_2^2 - \frac{(g^{-1})'(\varphi)^2}{(g^{-1})''(\varphi)}\right)f_1(\varphi_K) = K^{n+1}K(z)f_1(\varphi_K).
\]

We see that by choosing \(K\) and \(\tilde{K}\) suitably we have \(\tilde{K}^{n+1}K \leq c_1\) and \(c_2 \leq K^{n+1}K\). Let \(\Omega_N = \{z \in \Omega; \varphi_K(z) < N\}\) and \(u_N\) be the solution of
\[
\begin{cases}
\det(u_{N,j\bar{k}}(z)) = f(z, u_N(z)), & z \in \Omega_N, \\
\lim_{z \to z_0} u_N(z) = N & \text{for all } z_0 \in \partial \Omega_N.
\end{cases}
\]
which exists by Theorem 1.1. By Lemma 2.2 in [6] we get \(\varphi_K \leq u_N \leq u_{N+1} \leq \varphi_K\) on \(\Omega_N\). Define \(u(z) = \lim_{N \to \infty} u_N(z)\). We now investigate the regularity of \(u\).

3. A priori estimate of first derivatives of solutions. In this section we assume that \(f(z, u) = f(u)\) is a function satisfying \(B, C\) and the technical condition
\[
\frac{n-1}{n+1} \leq \frac{F(x)f'(x)}{f(x)^2}.
\]
We shall estimate the norm of the gradient of \(u_N\) on compact subsets of \(\Omega\). We do this by studying the functions \(v_N = |\nabla u_N|^2(g^{-1})'(u_N)^2\). Notice that \(|\nabla \varphi_K|^2 = |\nabla \varphi|^2(g^{-1})'(\varphi)^2 \leq \mathcal{C}\) and that \(v_N = |\nabla u_N|^2(g^{-1})'(u_N)^2 \leq |\nabla \varphi|^2(g^{-1})'(\varphi)^2\) on \(\partial \Omega_N\) since \(u_N = \varphi\) on \(\partial \Omega_N\) and \(\varphi \leq u\) on \(\Omega\). We claim that \(\sup(v_N(z); z \in \Omega_N) \leq \sup(v_N(z); z \in \partial \Omega_N) \leq \mathcal{C}\). We shall show that \(v_N\) does not have any interior maximum in \(\Omega_N\) to establish the claim. This calculation was inspired by Bo Guan's work on the regularity of the pluricomplex Green function [4], [5]. Readers interested in the regularity of the pluricomplex Green function should also consult Błocki’s paper [1].
Assume that a local maximum for \( v_N \) is attained at \( p \in \Omega_N \). We know that \( \nabla v_N(p) = 0 \). Choose coordinates near \( p \) so that \( u_{N,jk}(p) = u_{N,j\bar{k}}(p) = 0 \) and \( u_{N,jk}(p) = 0 \) if \( j \neq k \). It is known that such coordinates can be found if \( \nabla u_N(p) \neq 0 \), which is the case at a maximum point of \( v_N \). A proof can be extracted from the calculation on page 130 of [8]. Remember that

\[
v_N = \sum_{l=1}^{n} u_{N,l} u_{N,l}(g^{-1})'(u_N)^2
\]

and hence

\[
v_{N,j} = \sum_{l=1}^{n} (u_{N,l} u_{N,j,l}(g^{-1})'(u_N)^2 + u_{N,jl} u_{N,j,l}(g^{-1})'(u_N)^2 + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,l} u_{N,j,l} u_{N,j}.
\]

Evaluating this at \( p \) yields

\[
v_{N,j} = \sum_{l=1}^{n} (u_{N,l} u_{N,j,l}(g^{-1})'(u_N)^2 + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,l} u_{N,j,l} u_{N,j} = u_{N,j} u_{N,j,l}(g^{-1})'(u_N)^2 + \sum_{l=1}^{n} 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,l} u_{N,l} u_{N,j,l} u_{N,j} = u_{N,j} (g^{-1})'(u_N)^2 \left( u_{N,j,l} + \frac{2(g^{-1})''(u_N)}{(g^{-1})'(u_N)} |\nabla u_N|^2 \right) = 0.
\]

At the relevant local maximum point we have \( |\nabla u_N| > 0 \) and therefore

\[
\prod_{j=1}^{n} \left( u_{N,j} + \frac{2(g^{-1})''(u_N)}{(g^{-1})'(u_N)} |\nabla u_N|^2 \right) = 0.
\]

Thus we have

\[
|\nabla u_N|^2 = -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)} u_{N,j}\bar{j}
\]

for some \( j \). We see that

\[
|\nabla u_N|^2 \leq -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)} \sum_{j=1}^{n} u_{N,j}\bar{j}
\]

with equality if and only if \( u_{N,j}\bar{j} = 0 \) for all but one \( j \). Hence

\[
|\nabla u_N|^2 \leq -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)} \sum_{j=1}^{n} u_{N,j}\bar{j}
\]

because otherwise \( \det(u_{N,j\bar{k}}) = 0 \). Remembering that

\[
(g^{-1})'(x) = ((n + 1)F(x))^{-1/(n+1)}
\]
we get
\[ |\nabla u_N|^2 < \frac{(n + 1)F(u_N)}{2f(u_N)} \sum_{j=1}^{n} u_{N,j\bar{k}}. \]

So far we have only used the fact that \( p \) is a critical point. Now we shall use the fact that it is a local maximum point. We have \( \log \det (u_{N,j\bar{k}}) = \log f(u) \). Differentiating we see that

\[ \frac{\partial}{\partial z_j} \log \det (u_{N,k\bar{l}}) = \sum_{k,l=1}^{n} \frac{M_{k\bar{l}}}{\det (u_{N,k\bar{l}})} u_{N,k\bar{l}j} = \sum_{k,l=1}^{n} u_{N,k\bar{l}} u_{N,k\bar{l}j} = \sum_{l=1}^{n} u_{N}^{l} u_{N,l\bar{j}} \]

and hence we get the relation

\[ \sum_{l=1}^{n} u_{N}^{l} u_{N,l\bar{j}} = \frac{f'(u_N)}{f(u_N)} u_{N,j}. \]

We also have

\[ \sum_{l=1}^{n} u_{N}^{l} u_{N,l\bar{l}} = \frac{f''(u_N)}{f(u_N)} u_{N,\bar{j}}. \]

If we differentiate \( v_N \) twice we get

\[ v_{N,j\bar{k}} = (g^{-1})'(u_N)^2 \sum_{l=1}^{n} (u_{N,j\bar{k}} u_{N,\bar{l}} + u_{N,j\bar{l}} u_{N,\bar{k}} + u_{N,l} u_{N,j\bar{k}\bar{l}}) \]
\[ + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,j\bar{l}} \]
\[ + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,j} \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}} \]
\[ + 2(g^{-1})'(u_N)(g^{-1})''(u_N) u_{N,j\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}} \]
\[ + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2) u_{N,j} u_{N,\bar{k}} \sum_{l=1}^{n} u_{N,l} u_{N,\bar{l}}. \]

Here we have used the fact that the Hessian of \( u_N \) is diagonal to simplify the expression. Since \( p \) is assumed to be a local maximum point we know that

\[ \sum_{j, k=1}^{n} u_{N,j\bar{k}} u_{N,j\bar{k}} \leq 0. \]
Therefore
\[ \sum_{j,k=1}^{n} u_{N,jk}^j v_{N,jk} = \sum_{j=1}^{n} u_{N,jj}^j v_{N,jj} \]

\[ = (g^{-1})'(u_N)^2 \left( \sum_{j,l=1}^{n} (u_{N,lj}^j u_{N,lj}^j + u_{N,lj}^j u_{N,lj}^j) + \sum_{j=1}^{n} u_{N,jj}^j \right) \]

\[ + (4 + 2n)(g^{-1})'(u_N)(g^{-1})''(u_N) \sum_{j=1}^{n} u_{N,jj}^j u_{N,j} \]

\[ + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2) \sum_{j,l=1}^{n} u_{N,lj}^j u_{N,j} u_{N,lj}^j \]

\[ = 2(g^{-1})'(u_N)^2 \frac{f'(u_N)}{f(u_N)} |\nabla u_N|^2 + (g^{-1})'(u_N)^2 \sum_{j=1}^{n} u_{N,jj}^j \]

\[ + (4 + 2n)(g^{-1})'(u_N)(g^{-1})''(u_N) |\nabla u_N|^2 \]

\[ + (2(g^{-1})'(u_N)(g^{-1})'''(u_N) + 2(g^{-1})''(u_N)^2) |\nabla u_N|^2 \sum_{j=1}^{n} u_{N,jj}^j u_{N,j} \leq 0 \]

at \( p \). We need to analyze \( \sum_{j=1}^{n} u_{N,jj}^j u_{N,jj} \). At \( p \) we have

\[ u_{N,jj} = -2 \frac{(g^{-1})''(u_N)}{(g^{-1})'(u_N)} |\nabla u_N|^2 \]

if \( u_{N,j} \neq 0 \). Therefore

\[ \sum_{j=1}^{n} u_{N,jj}^j u_{N,jj} = \sum_{j=1}^{n} u_{N,jj}^j \frac{u_{N,jj}}{u_{N,jj}} = -\frac{(g^{-1})'(u_N)}{2(g^{-1})''(u_N)}. \]

Using this gives the inequality

\[ \sum_{j=1}^{n} u_{N,jj} \leq |\nabla u_N|^2 \left( \frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} - \frac{2f'(u_N)}{f(u_N)} - \frac{3 + 2n}(g^{-1})''(u_N) \right). \]

We have

\[ (g^{-1})'(x) = ((n + 1)F(x))^{-1/(n+1)}, \]

\[ (g^{-1})''(x) = -f(x)((n + 1)F(x))^{-1-1/(n+1)}, \]

\[ (g^{-1})'''(x) = -f'(x)((n + 1)F(x))^{-1-1/(n+1)} \]

\[ + (n + 2)f(x)^2((n + 1)F(x))^{-2-1/(n+1)}. \]
Hence
\[
\frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} - \frac{2f'(u_N)}{f(u_N)} - \frac{(3 + 2n)(g^{-1})''(u_N)}{(g^{-1})'(u_N)} = \frac{(g^{-1})'''(u_N)}{(g^{-1})''(u_N)} \frac{2f'(u_N)}{f(u_N)} + \frac{(3 + 2n)f(u_N)}{(n + 1)F(u_N)} = -\frac{f'(u_N)}{f(u_N)} + \frac{f(u_N)}{F(u_N)}.
\]

Combining the two inequalities
\[
|\nabla u_N|^2 < \frac{(n + 1)F(u_N)}{2f(u_N)} \sum_{j=1}^{n} u_{N,j}\]
and
\[
\sum_{j=1}^{n} u_{N,j} \leq |\nabla u_N|^2 \left( \frac{f(u_N)}{F(u_N)} - \frac{f'(u_N)}{f(u_N)} \right)
\]
yields
\[
|\nabla u_N|^2 < \frac{n + 1}{2} \frac{F(u_N)}{f(u_N)} \sum_{j=1}^{n} u_{N,j} \leq \frac{n + 1}{2} \left( 1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2} \right)|\nabla u_N|^2,
\]
which gives a contradiction if
\[
\frac{n + 1}{2} \left( 1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2} \right) \leq 1.
\]

We see that, on the assumption
\[
\frac{n - 1}{n + 1} \leq \frac{F(u_N)f'(u_N)}{f(u_N)^2},
\]
the function \(|\nabla u_N|^2 (g^{-1})'(u_N)^2\) attains its maximum on the boundary and hence we have
\[
|\nabla u_N|^2 (g^{-1})'(u_N)^2 \leq C
\]
on \(\Omega_N\). Since any compact set \(K \subseteq \Omega\) is contained in \(\Omega_N\) for sufficiently large \(N\) we have proven that
\[
\sup(|\nabla u_N(z)|^2 (g^{-1})'(u_N(z)); z \in K) < C
\]
for all \(N\) large enough. Hence
\[
|\nabla u_N(z)|^2 \leq C g'(u_N(z))^2
\]
in \(K\) and since \(u_N(z) \leq \varphi(z) \leq C\) in \(K\) we see that \(\|u_N\|_{C^1(K)} \leq C\). Since the sequence of \(u_N\)'s converges uniformly on compacts we can conclude that \(u\) is Lipschitz. We state this in a theorem.

**Theorem 3.1.** Let \(\Omega\) be a bounded strongly pseudoconvex domain in \(\mathbb{C}^n\) with smooth boundary. Suppose that \(f\) satisfies B, C and
\[
\frac{n - 1}{n + 1} \leq \frac{f'(x)F(x)}{f(x)^2}.
\]
Then the problem
\[
\begin{aligned}
(dd^c u)^n &= f(u(z)), \quad z \in \Omega \\
\lim_{z \to z_0} u(z) &= \infty \quad \text{for all } z_0 \in \partial \Omega,
\end{aligned}
\]
has a solution \( u \) that is Lipschitz.

**Remark 3.2.** Note that \( f(u) = e^{Ku} \), \( K > 0 \), and \( f(u) = u^\gamma \) (suitably modified for \( u < 1 \)) where \( \gamma \geq (n - 1)/2 \), satisfies all the conditions in the theorem.

4. **Uniqueness.** We shall now establish a uniqueness result. Uniqueness for boundary blow-up problems is not as straightforward as for the Dirichlet problem. This is because the comparison principles in [6] and [7] are not formulated with the situation in mind where both plurisubharmonic functions tend to \( \infty \) as we approach the boundary.

We need the following definition and theorem from [7].

**Definition 4.1.** Assume that \( \Omega = \{ z \in \mathbb{C}^n; \varrho(z) < 0 \} \) where \( \varrho \in C^\infty(\overline{\Omega}) \). For \( z_0 \in \partial \Omega \) suppose that \( |\nabla \varrho(z_0)| = 1 \). Let \( \Pi(z_0) \) be the product of the eigenvalues of the form
\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k}(z_0) \, dz_j \wedge d\overline{z}_k
\]
restricted to the vector space \( \{ w \in \mathbb{C}^n; \sum_{j=1}^{n} \frac{\partial \varrho}{\partial z_j}(z_0)w_j = 0 \} \).

**Theorem 4.2.** Let \( \Omega \) be a bounded, strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Let \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) be a strictly positive function which is increasing in the second variable and satisfies assumptions A, B and C. For boundary points \( z_0 \in \partial \Omega \) let \( \Pi(z_0) \) be the number described in Definition 4.1. Then any solution \( u \) to Problem (1) satisfies
\[
\lim_{z \to z_0} \frac{\Psi_n(u(z))}{d_\Omega(z)} = 4^{1/(n+1)}h(z_0)^{1/(n+1)}\Pi(z_0)^{-1/(n+1)}
\]
for any \( z_0 \in \partial \Omega \).

We can now prove the following proposition.

**Proposition 4.3.** Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary and assume that \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) is a strictly positive function, increasing in the second variable and satisfying A, B and C. Assume also that
\[ \Psi_n(t)/\Psi_n'(t) \]
is bounded for large \( t \). If \( u \) and \( v \) are plurisubharmonic solutions of Problem (1) then \( u \equiv v \).
Remark 4.4. The assumption that $\Psi_n(t)/\Psi'_n(t)$ is bounded for large $t$ is fulfilled when $f_1$ has exponential growth but not when it has only polynomial growth.

Proof of Proposition 4.3. Assume that we have two distinct plurisubharmonic solutions $u$ and $v$ of Problem (1). Assume for the moment that we know that $\lim_{z \to z_0} (u(z) - v(z)) = 0$ for all $z_0 \in \partial \Omega$. We shall return to this claim later to finish the proof. Assume that $\sup(u(z) - v(z); z \in \Omega) = K > 0$. Then there is a $p \in \Omega$ such that $u(p) - v(p) = K$. At $p$ we have $\det (u_{j\kappa}(p)) \leq \det (v_{j\kappa}(p))$. However, since $u(p) > v(p)$ we see that

$$\det (u_{j\kappa}(p)) = f(p, u(p)) > f(p, v(p)) = \det (v_{j\kappa}(p)),$$

which is a contradiction. Hence $u(z) - v(z) \leq 0$ in $\Omega$. Arguing in the same way we also see that $v(z) - u(z) \leq 0$ in $\Omega$. This proves uniqueness.

It remains to prove our claim that $\lim_{z \to z_0} (u(z) - v(z)) = 0$. We know that for all $z_0 \in \partial \Omega$ we have

$$\lim_{z \to z_0} \frac{\Psi_n(u(z))}{d_\Omega(z)} = \lim_{z \to z_0} \frac{\Psi_n(v(z))}{d_\Omega(z)} = C(z_0)$$

where $C(z_0)$ is the constant given in Theorem 4.2. Given $\varepsilon > 0$, for $z$ close to $z_0$ we have

$$(C(z_0) - \varepsilon)d_\Omega(z) \leq \Psi_n(u(z)) \leq (C(z_0) + \varepsilon)d_\Omega(z)$$

and

$$(C(z_0) - \varepsilon)d_\Omega(z) \leq \Psi_n(v(z)) \leq (C(z_0) + \varepsilon)d_\Omega(z).$$

This gives

$$\Psi_n^{-1}((C(z_0) + \varepsilon)d_\Omega(z)) \leq u(z) \leq \Psi_n^{-1}((C(z_0) - \varepsilon)d_\Omega(z))$$

and

$$-\Psi_n^{-1}((C(z_0) - \varepsilon)d_\Omega(z)) \leq -v(z) \leq -\Psi_n^{-1}((C(z_0) + \varepsilon)d_\Omega(z)).$$

We get

$$u(z) - v(z) \leq \Psi_n^{-1}((C(z_0) - \varepsilon)d_\Omega(z)) - \Psi_n^{-1}((C(z_0) + \varepsilon)d_\Omega(z)) = -2\varepsilon d_\Omega(z)(\Psi_n^{-1})'(\eta(z))$$

for some $\eta(z) \in [(C(z_0) - \varepsilon)d_\Omega(z), (C(z_0) + \varepsilon)d_\Omega(z)]$ by the mean-value theorem. Hence

$$u(z) - v(z) \leq -2\varepsilon d_\Omega(z)(\Psi_n^{-1})'(\eta(z)) = -2\varepsilon d_\Omega(z) \frac{\eta(z)}{\Psi_n(\eta(z))} \frac{1}{\Psi'_n(\Psi_n^{-1}(\eta(z)))}$$

$$= -\frac{2\varepsilon d_\Omega(z)}{\eta(z)} \frac{\Psi_n(\Psi_n^{-1}(\eta(z)))}{\Psi'_n(\Psi_n^{-1}(\eta(z)))} \leq -\frac{2\varepsilon}{(C(z_0) - \varepsilon) \Psi'_n(\Psi_n^{-1}(\eta(z)))}.$$
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