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FUNCTIONAL ANALYSIS

Vector Measures, c_0, and (sb) Operators

by

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Summary. Emmanuele showed that if Σ is a σ-algebra of sets, X is a Banach space, and $\mu : \Sigma \to X$ is countably additive with finite variation, then $\mu(\Sigma)$ is a Dunford–Pettis set. An extension of this theorem to the setting of bounded and finitely additive vector measures is established.

A new characterization of strongly bounded operators on abstract continuous function spaces is given. This characterization motivates the study of the set of (sb) operators. This class of maps is used to extend results of P. Saab dealing with unconditionally converging operators. A characterization of the existence of a countably additive, non-strongly bounded representing measure in terms of c_0 is presented. This characterization resolves a question posed in 1970.

1. Introduction. If $\mathcal{R}$ is a ring of sets, X is a (real) Banach space, and $\mu : \mathcal{R} \to X$ is a finitely additive set function, then μ is said to be *strongly additive* if $\mu(A_i) \to 0$ whenever (A_i) is a pairwise disjoint sequence from $\mathcal{R}$. The notion of strong additivity has permeated much of vector measure theory since its introduction by Rickart [24] in 1943. The "Notes and Remarks" section of Chapter 1 of [13] contains an excellent accounting of the main results associated with strong additivity from 1943 through the mid 70's.

We briefly mention two classically important results in which strong additivity (either explicitly or implicitly) played a pivotal role. In [7], Brooks and Jewett established the following generalization of the Vitali–Hahn–Saks theorem.

THEOREM 1.1. *Suppose that $\mathcal{R}$ is a σ-ring and for each n, $\mu_n : \mathcal{R} \to X$ is finitely additive. Let $\mu_n \ll \lambda$, $n = 1, 2, \ldots$, where λ is a nonnegative bounded finitely additive measure. If $\lim \mu_n(R)$ exists for all $R \in \mathcal{R}$, then*

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$\mu_n \ll \lambda$ uniformly in n. If the μ_n are strongly additive, then the boundedness assumption on λ may be omitted.

In [3], Bartle, Dunford, and Schwartz showed that if K is a compact Hausdorff space and $L : C(K) \to X$ is an operator with *representing measure* μ, then L is weakly compact if and only if μ is countably additive. It is not difficult at all to see that a representing measure in this setting is countably additive if and only if it is strongly additive. Thus the Bartle–Dunford–Schwartz theorem can be stated as follows.

THEOREM 1.2. *If $T : C(K) \to X$ is an operator with representing measure μ, then T is weakly compact if and only if μ is strongly additive.*

It is well known that if E and F are Banach spaces, $T : C(K, E) \to F$ is a weakly compact operator with representing measure m, and $\widetilde{m}(A)$ denotes the semivariation ([14, p. 51]) of m on A, then $\widetilde{m}(A_i) \to 0$ whenever (A_i) is any pairwise disjoint sequence of Borel subsets of K. That is, if T is weakly compact, then its representing measure is *strongly bounded*. (The reader may consult [8], [9], [4], [25], [1], or [5] for details concerning representing measures in this setting.) However, examples in [15], [19], and [22] show that a strongly additive representing measure need not be strongly bounded in general. The structure of non-strongly additive and non-strongly bounded measures are studied in this paper.

Our notation and terminology is consistent with that used in Diestel [11] and in Lindenstrauss and Tzafriri [23]. We do note specifically that the canonical unit vector basis of c_0 will be denoted by (e_n) and the canonical basis of ℓ^1 will be denoted by (e_n^*).

2. Non-strongly additive vector measures. Joe Diestel and Barbara Faires did much to reveal the behavior of a non-strongly additive but finitely additive and bounded vector measure defined on an *algebra* of sets in [12] and [13, p. 20]. Their result makes it clear that the classical Banach space c_0 and non-strongly additive vector measures are closely related. Our first result demonstrates that ℓ^1 and *hereditary Dunford–Pettis sets* are also very prominent in the structure of a non-strongly additive vector measure.

Recall that a subset A of the Banach space E is a *Dunford–Pettis subset* of E if $T(A)$ is relatively compact in F whenever $T : E \to F$ is a weakly compact operator [2]. Further, a sequence (x_n) in X is a *hereditary Dunford–Pettis sequence* if $\{x_{n_i} : i \in \mathbb{N}\}$ is a Dunford–Pettis subset of $[x_{n_i}]$ for all subsequences (x_{n_i}) of (x_n). Emmanuele [17] showed that if Σ is a σ-algebra, X is a Banach space, and $\mu : \Sigma \to X$ is countably additive and has finite variation, then $\mu(\Sigma)$ is a Dunford–Pettis subset of X. (The survey article [10] by Diestel contains a wealth of information about the Dunford–Pettis property and Dunford–Pettis sets.) The following theorem establishes an extension of Emmanuele's result to the finitely additive case and provides substantial additional information about the nature of a non-strongly additive vector measure defined on a *ring* of sets.

THEOREM 2.1. *If $\mathcal{R}$ is a ring of sets, $m : \mathcal{R} \to X$ is a bounded and finitely additive set function, and (A_i) is a pairwise disjoint sequence from $\mathcal{R}$, then $(m(A_i))$ is a weakly null hereditary Dunford–Pettis set. If (x_i^*) is any bounded sequence in X^* so that $x_i^* m(A_i) = 1$ for all i, then there is a sequence (w_n^*) in $\{x_i^* - x_j^* : i, j \in \mathbb{N}\}$ so that $(w_n^*) \sim (e_n^*)$ and $[w_n^*]$ is complemented in X^*. In fact, if $\|m(A_i)\| \not\to 0$, then there is a subsequence $(m(A_{ij}))$ of $(m(A_i))$ and a sequence (f_j^*) in X^* so that $(m(A_{ij}), f_j^*)$ is biorthogonal in $X \times X^*$, $(m(A_{ij})) \sim (e_j)$, and $(f_j^*) \sim (e_j^*)$.*

Proof. If $m : \mathcal{R} \to X$ is bounded and finitely additive and $x^* \in X^*$, then the scalar measure $x^* m$ has finite variation. Thus $x^* m(A_i) \to 0$ and $(m(A_i))$ is weakly null.

Now suppose that Y is a Banach space, and $T : X \to Y$ is a weakly compact operator. Then $T \circ m : \mathcal{R} \to Y$ is a finitely additive vector measure with relatively weakly compact range. By the corollary on p. 1000 of [6], $T \circ m$ is strongly additive. Therefore $T \circ m(A_i) \to 0$, and $T(\{m(A_i) : i \in \mathbb{N}\})$ is relatively compact. Hence $\{m(A_i) : i \in \mathbb{N}\}$ is a Dunford–Pettis subset of X.

Now suppose that $(m(A_{ij}))$ is a subsequence of $(m(A_i))$, and let $Z = [m(A_{ij})]$. Let $\mathcal{A}$ denote the algebra of subsets of $\mathbb{N}$ consisting of the finite and cofinite sets. Denote the complement of F by $\widetilde{F}$. Define $\nu : \mathcal{A} \to X$ by

$$\nu(F) = \begin{cases} \sum_{j \in F} m(A_{ij}) & \text{if } F \text{ is finite,} \\ -\sum_{j \in \widetilde{F}} m(A_{ij}) & \text{if } F \text{ is cofinite.} \end{cases}$$

It is straightforward to check that ν is bounded and finitely additive on $\mathcal{A}$. By the proof in the preceding paragraph, $\{\nu(\{j\}) : j \in \mathbb{N}\} = \{m(A_{ij}) : j \in \mathbb{N}\}$ is a Dunford–Pettis subset of Z.

Next let (x_i^*) be a bounded sequence in X^* so that $x_i^*(m(A_i)) = 1$ for each i. Since $(m(A_i))$ is a weakly null hereditary Dunford–Pettis set which is not norm null, we appeal to the discussion on pp. 26–28 of [10] and conclude that there is a subsequence $(m(A_{ij}))$ of $(m(A_i))$ so that $(m(A_{ij})) \sim (e_j)$. Certainly

$$\sum_{j=1}^{\infty} |\langle m(A_{ij}), x_{ij}^* \rangle - 1| < \infty.$$

Consequently, the main theorem in [20] applies and produces a sequence (w_n^*) in $\{x_i^* - x_j^* : i, j \in \mathbb{N}\}$ so that $(w_n^*) \sim (e_n^*)$ and $[w_n^*]$ is complemented in X^*.

Now suppose that $(m(A_{ij}))_{j=1}^{\infty}$ is as before. Let y_j^* denote the coefficient functional (in $[(m(A_{ij}))_{j=1}^{\infty}]^*$) of the Schauder basis element $m(A_{ij})$, and let f_j^* be a Hahn–Banach extension of y_j^* to all of X. Suppose that (f_j^*) has a weakly Cauchy subsequence, say $(f_{j_k}^*)$. Then $(f_{j_{k+1}}^* - f_{j_k}^*) = (u_k^*)$ is weakly null in X^*. Thus, since $(m(A_i))_{i=1}^{\infty}$ is hereditary Dunford–Pettis, $u_k^*(m(A_{i_{j_{k+1}}})) \to 0$ (e.g., see [2] or Theorem 1 of [10]). However, this is a contradiction since $u_k^*(m(A_{i_{j_{k+1}}})) = 1$ for each k. Therefore Rosenthal's ℓ^1-theorem ensures that there is a subsequence $(f_{j_k}^*)$ of (f_j^*) so that $(f_{j_k}^*) \sim (e_k^*)$. ∎

Corollary 2.2 ([16, p. 318]). *If Σ is a σ-algebra and $\mu : \Sigma \to X$ is weakly countably additive, then μ is countably additive.*

Proof. Note first that a weakly countably additive set function $\mu : \Sigma \to X$ is bounded and finitely additive. Further, as was noted earlier, a weakly countably additive set function is countably additive if and only if it is strongly additive. Suppose then that $\varepsilon > 0$ and (A_i) is a pairwise disjoint sequence from Σ so that $\|\mu(A_i)\| > \varepsilon$ for each i. Since $(\mu(A_i))$ is a hereditary Dunford–Pettis sequence which is not norm null, we may assume that $(\mu(A_i)) \sim (e_i)$. This immediately leads to a contradiction since $(\sum_{i=1}^{n} \mu(A_i))_{n=1}^{\infty}$ converges weakly (to $\mu(\bigcup_{i=1}^{\infty} A_i)$) and $(\sum_{i=1}^{n} e_i)_{n=1}^{\infty}$ does not converge weakly. ∎

Corollary 2.3. *If $\mathcal{R}$ is a ring of sets, each of E and F is a Banach space, $L(E,F)$ is the Banach space of all bounded linear operators from E to F, and $m : \mathcal{R} \to L(E,F)$ is a finitely additive vector measure with finite semivariation which is not strongly bounded, then c_0 embeds isomorphically in F.*

Proof. Suppose that (A_i) is a pairwise disjoint sequence from $\mathcal{R}$ and $\varepsilon > 0$ so that $\widetilde{m}(A_i) > \varepsilon$ for each i. Let $(A_{ij})_{j=1}^{n_i}$ be a partition of A_i and let $(x_{ij})_{j=1}^{n_i}$ be norm one vectors in X so that

$$\Big\| \sum_{j=1}^{n_i} m(A_{ij}) x_{ij} \Big\| > \varepsilon$$

for each i. Use the partitions of (A_i) to form a natural partition of $\mathbb{N}$:

$$\{1, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2, n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3, \ldots\}.$$

Let $n_0 = 0$, and let $P_i = \{j \in \mathbb{N} : n_0 + n_1 + \cdots + n_{i-1} < j \le n_0 + \cdots + n_i\}$. If S is a finite subset of $\mathbb{N}$, set

$$\mu(S) = \sum_{i=1}^{\infty} \sum_{j \in P_i \cap S} m(A_{i(j-(n_0+\cdots+n_{i-1}))}) x_{i(j-(n_0+\cdots+n_{i-1}))}.$$

If $\widetilde{S}$ is finite, set $\mu(S) = -\mu(\widetilde{S})$. Then μ is bounded, finitely additive, and not strongly additive on the finite-cofinite algebra of subsets of $\mathbb{N}$. Then $(\sum_{j=1}^{n_i} m(A_{ij})\, x_{ij})_{i=1}^{\infty}$ is a hereditary Dunford–Pettis sequence in F which is not norm null. Thus F contains an isomorphic copy of c_0. ∎

It is well known that there are (reflexive) infinite-dimensional Banach spaces E so that ℓ^∞ embeds isomorphically in $L(E,E)$ (e.g., see [18, p. 267], and [17, Theorem 1]). In fact, if E is any infinite-dimensional Banach space with an unconditional Schauder decomposition [23, p. 47], then the next theorem and corollary show that $L(E,E)$ must contain an isomorphic copy of ℓ^∞. These results also strengthen an implication in Theorem 6 of [18].

THEOREM 2.4. *If E is an infinite-dimensional Banach space and F is an arbitrary Banach space, then ℓ^∞ embeds isomorphically in $L(E,F)$ if and only if there is a seminormalized sequence (T_n) in $L(E,F)$ so that $\sum |\langle T_n(x), y^*\rangle| < \infty$ for each $x \in E$ and $y^* \in F^*$.*

Proof. Suppose that (T_n) is as in the statement of the theorem. Since $\sum_{n=1}^{\infty} T_n(x)$ is weakly unconditionally convergent for all $x \in E$, $\sup\{\|\sum_{n\in F} T_n\| : F \text{ is a finite subset of } \mathbb{N}\} < \infty$. Let $\mathcal{R}$ denote the ring of all finite subsets of $\mathbb{N}$. Define $\mu : \mathcal{R} \to L(E,F)$ by

$$\mu(A) = \sum_{n\in A} T_n, \quad A \neq \emptyset,$$
$$\mu(\emptyset) = 0.$$

Certainly μ is finitely additive and bounded. Since (T_n) is seminormalized, μ is not strongly additive. Therefore $c_0 \hookrightarrow L(E,F)$, and, by Theorem 1 of [21], $\ell^\infty \hookrightarrow L(E,F)$.

Conversely, suppose that $\phi : \ell^\infty \to L(E,F)$ is an isomorphism, and let $T_n = \phi(e_n)$ for each n. Since ϕ is an isomorphism, (T_n) is seminormalized. Since $\sum e_n$ is weakly unconditionally convergent, $\sum |\langle T_n(x), y^*\rangle| < \infty$ for $x \in E$ and $y^* \in F^*$. ∎

COROLLARY 2.5. *Suppose the infinite-dimensional Banach space E has an unconditional Schauder decomposition $(E_n)_{n=1}^{\infty}$ and $P_n : E \to E_n$ is the projection of E onto E_n. If F is any Banach space for which there is an operator $T : E \to F$ so that $\sum_{n=1}^{\infty} TP_n$ does not converge in norm, then ℓ^∞ embeds isomorphically in $L(E,F)$. In particular, ℓ^∞ embeds isomorphically in $L(E,E)$.*

We conclude this section by noting that 2.3 and the proof of 2.4 show that if E is infinite-dimensional, $m : \mathcal{R} \to L(E,F)$ is finitely additive with finite semivariation, and m is not strongly bounded, then $\ell^\infty \hookrightarrow L(E,F)$.

3. Strongly bounded measures and (sb) operators. The first theorem in this section uses the Dvoretzky–Rogers theorem [11, Chapter VI] and techniques in [15], [19], and [22] to establish the converse of Corollary 2.3 and to answer a question posed in a problem session at a meeting of the American Mathematical Society at the University of Illinois in the fall of 1970. Old examples of Dobrakov [15] and Lewis [19] showed that there exist representing measures which are countably additive and not strongly bounded. However, no characterization of the spaces which support a countably additive and non-strongly bounded representing measure has been given. In this section, we give a complete description of pairs (E, F) of Banach spaces and compact Hausdorff spaces H so that there exists a countably additive representing measure $m : \Sigma \to L(E, F)$ which fails to be strongly bounded. This result highlights a theme particularly well enunciated in Theorem 2.1 of Saab [25].

THEOREM 3.1. *If H is an infinite compact Hausdorff space with Borel sets Σ, X is an infinite-dimensional Banach space, and Y is a Banach space, then there exists a countably additive, non-strongly bounded representing measure $m : \Sigma \to L(X, Y)$ if and only if there exists an isomorphic embedding $T : c_0 \to Y$.*

Proof. Suppose that $T : c_0 \to Y$ is an isomorphic embedding, and let $y_n = T(e_n)$ for each $n \in \mathbb{N}$. Let (t_n) be a sequence of distinct points in H and $\sum x_n^*$ be an unconditionally converging series in X^* such that $\sum_{n=1}^{\infty} \|x_n^*\| = \infty$. Let (N_i) be a pairwise disjoint sequence of finite subsets of $\mathbb{N}$ so that

(i) $\|\sum_{n\in Z\cap N_i} x_n^*\| < 1/2^i$ for all $Z \subset \mathbb{N}$ and for each i,
(ii) $1 \ge \sum_{n\in N_i} \|x_n^*\| > 1/2$ for each i.

If $A \in \Sigma$ and $x \in X$, define

$$m(A)(x) = \sum_{i=1}^{\infty} \Big(\sum_{n\in N_i\cap \widehat{A}} x_n^*(x) \Big) y_i,$$

where $\widehat{A} = \{n : t_n \in A\}$. It is clear that $m(A)$ is a bounded linear operator and m is finitely additive. Further, (i) above and the fact that $(y_i) \sim (e_i)$ ensure that m is countably additive. Also

$$\widetilde{m}(H) \le \sup_i \Big\{ \sum_{n\in N_i} \|x_n^*\| \Big\} \|T\| \le \|T\|.$$

Moreover, if $x \in X$ and $y^* \in Y^*$, then $\langle m(\cdot)x, y^*\rangle : \Sigma \to \mathbb{R}$ is regular since it is a convergent infinite sum of point-mass measures. However, (ii) guarantees that m is not strongly bounded. In fact, if $B_i = \{t_j : j \in N_i\}$, then $\widetilde{m}(B_i) = (\sum_{n\in N_i} \|x_n^*\|)\|y_i\|$.

Conversely, suppose that there is a countably additive representing measure $m : \Sigma \to L(X,Y)$ so that m is not strongly bounded. An appeal to Corollary 2.3 finishes the proof. ∎

We remark that the notion of semivariation $\widetilde{m}(A)$ used in this paper and in Dinculeanu [14] is not equivalent to the semivariation $\|m\|(A)$ which is defined in Diestel and Uhl ([13, p. 2]). Perhaps it is appropriate at this point to note that a representing measure is countably additive if and only if $\|m\|(A_i) \to 0$ for each pairwise disjoint sequence (A_i) from Σ.

In our next theorem, we state a characterization of strongly bounded operators on $C(H,X)$ which does not seem to have been noticed previously. In this theorem, it is helpful to recall that if E and F are Banach spaces, then the algebraic tensor product $E \otimes F$ can be viewed as a subset of $L(E^*, F)$. The completion of $E \otimes F$ with respect to this identification and this operator norm is called the injective tensor product completion—or the least crossnorm completion—of $E \otimes F$ and is denoted by $E \otimes_\lambda F$. In particular, we note that $C(H,X) \cong C(H) \otimes_\lambda X$ (e.g., see Chapter VIII of [13]).

Theorem 3.2. *If $m \leftrightarrow T : C(H,X) \to Y$ is a bounded linear operator, then m is strongly bounded if and only if $T(f_i) \to 0$ whenever (f_i) is a bounded sequence in $C(H) \otimes_\lambda X$ so that $f_i(\varphi) \to 0$ for each $\varphi \in C(H)^*$.*

This theorem follows almost immediately from the following lemmas. If μ is a scalar measure on H and f is Bochner integrable with respect to μ, we denote this integral by (B)-$\int_H f \, d\mu$.

Lemma 3.3. *If $f \in C(H,X)$ and $\varphi \in C(H)^*$, then $f(\varphi) = (\mathrm{B})\text{-}\int_H f \, d\mu$, where μ is the unique member of $\mathrm{rca}(\Sigma)$ which represents φ.*

Proof. Let $f = \sum_{i=1}^n f_i \otimes x_i$, where $f_i \in C(H)$ and $x_i \in X$ for each i. Let $\varphi \in C(H)^*$ and let μ be the unique element in $\mathrm{rca}(\Sigma)$ such that $\varphi(g) = \int_H g \, d\mu$ for every $g \in C(H)$. Then

$$\begin{aligned} f(\varphi) &= \sum_{i=1}^n f_i \otimes x_i(\varphi) = \sum_{i=1}^n \varphi(f_i) x_i \\ &= \sum_{i=1}^n \Big(\int_H f_i \, d\mu \Big) x_i = \sum_{i=1}^n \Big[(B)\text{-} \int_H (f_i \cdot x_i) \, d\mu \Big] \\ &= (B)\text{-} \int_H f \, d\mu. \end{aligned}$$

The fact that $f(\varphi) = (B)\text{-}\int_H f \, d\mu$ for each $f \in C(H,Y)$ readily follows. ∎

Lemma 3.4. *If (f_i) is a bounded sequence in $C(H,X)$, then $f_i(h) \to 0$ for every $h \in H$ if and only if $f_i(\varphi) \to 0$ for every $\varphi \in C(H)^*$.*

Proof. Let (f_i) be a bounded sequence in $C(H,X)$. Suppose $f_i(h) \to 0$ for every $h \in H$. Let $\varphi \in C(H)^*$, and let μ be the unique element in $\mathrm{rca}(\Sigma)$ such that $\varphi(g) = \int_H g\,d\mu$ for every $g \in C(H)$. It then follows from the vector-valued Dominated Convergence Theorem that $f_i(\varphi) = \int_H f_i\,d\mu \to 0$ for each $\varphi \in C(H)^*$.

For the reverse direction, observe that for each $h \in H$ one can define $\widehat{h} \in C(H)^*$ by $\widehat{h}(f) \equiv f(h)$ for every $f \in C(H)$. It is readily seen that if $f \in C(H,Y)$ then $f(\widehat{h}) = f(h)$. Hence if $f_i(\varphi) \to 0$ for every $\varphi \in C(H)^*$, then $f_i(h) \to 0$ for every $h \in H$. ∎

In Theorem 2.8 of [1], the authors showed that a representing measure $m \leftrightarrow T$ is strongly bounded if and only if $T(f_i) \to 0$ for each bounded sequence (f_i) such that $f_i(h) \to 0$ for each $h \in H$. Theorem 3.2 now follows easily from the preceding lemmas.

We remark that Theorem 3.2 makes it plain that an operator

$$T : C(H) \otimes_\lambda X \ (\subseteq L(C(H)^*, X)) \to Y$$

is strongly bounded precisely when T maps sequences which converge in the strong operator topology ([16, pp. 475–476]) into norm convergent sequences. In general, if Z is a Banach space and $T : X \otimes_\lambda Y \to Z$ is a bounded linear operator, we say that T is (sb) if $T(\mu_i) \to 0$ whenever (μ_i) is any bounded sequence in $X \otimes_\lambda Y$ $(\subseteq L(X^*, Y))$ so that $\mu_i(x^*) \to 0$ for each $x^* \in X^*$. The following example is somewhat surprising considering the fact that weakly compact—even weakly completely continuous and unconditionally converging—operators on $C(H,X)$ are strongly bounded [8].

Example 3.5. Let X be an infinite-dimensional reflexive space, Y be any Banach space and let $y_0^* \in Y^*$ be such that $y_0^* \neq 0$. Define $S : Y \otimes_\lambda X \to X$ by $S(\mu) = \mu(y_0^*)$. Then S is linear, bounded by $\|y_0^*\|$, and S is weakly compact since X is reflexive. (It is easily checked that S is (sb).)

Let $I : X \otimes_\lambda Y \to Y \otimes_\lambda X$ be the natural isometry, and $T : X \otimes_\lambda Y \to X$ be defined by $T = S \circ I$. Certainly T is weakly compact. To see that T is not (sb), choose $y_0 \in Y$ such that $y_0^*(y_0) \neq 0$ and choose (x_n) in X such that (x_n) is weakly null but not norm null. Consider the sequence $(x_n \otimes y_0)_n$ in $X \otimes_\lambda Y$. It is bounded since $\|x_n \otimes y_0\|_{X \otimes_\lambda Y} \leq \|x_n\|\,\|y_0\|$ and (x_n) is bounded in X. Also, if $x^* \in X^*$ then $(x_n \otimes y_0)(x^*) = x^*(x_n)y_0 \xrightarrow{\|\cdot\|} 0$. However, $T(x_n \otimes y_0) = (y_0 \otimes x_n)(y_0^*) = y_0^*(y_0)x_n \not\to 0$ in norm.

However, as the next theorem demonstrates, completely continuous operators on $X \otimes_\lambda Y$ are (sb).

Theorem 3.6. *If $T : X \otimes_\lambda Y \to Z$ is completely continuous, then T is (sb).*

Proof. Suppose $T : X \otimes_\lambda Y \to Z$ is completely continuous and let (φ_i) be a bounded sequence in $X \otimes_\lambda Y$ such that $\varphi_i(x^*) \to 0_Y$ for every $x^* \in X^*$. Let $K = (B_{X^*}, w^*)$. Then we may consider $X \otimes_\lambda Y$ to be a closed subspace of $C(K) \otimes_\lambda Y \cong C(K, Y)$. As $\varphi_i(x^*) \xrightarrow{\|\cdot\|} 0_Y$ for every $x^* \in B_{X^*}$, $\varphi_i(x^*) \xrightarrow{w} 0_Y$ for every $x^* \in B_{X^*}$. It follows from Lemma 3.2 in [1] that (φ_i) is weakly null in $C(K, Y)$, and thus in $X \otimes_\lambda Y$. Since T is completely continuous, $T(\varphi_i) \xrightarrow{\|\cdot\|} 0_Y$. Therefore T is (sb). ■

At this time we point out that the collection of (sb) operators is a closed linear subspace of $L(X \otimes_\lambda Y, Z)$. It is also clear that if $T : X \otimes_\lambda Y \to Z$ is (sb) and $S : Z \to W$ then $S \circ T$ is (sb). However in Example 3.5 we defined operators $S : Y \otimes_\lambda X \to X$ and $I : X \otimes_\lambda Y \to Y \otimes_\lambda X$ such that S is (sb), but $T = S \circ I$ is not (sb). This example also indicates that the definition of an (sb) operator is dependent on the first factor in the tensor product.

Paulette Saab [25] showed that if $m \leftrightarrow T : C(H, X) \to Y$ is a strongly bounded operator, then (1) T is unconditionally converging if c_0 does not embed in X and (2) T is completely continuous if X is a Schur space. The next theorem generalizes each of these results and continues the theme of Corollary 2.3 and Theorem 3.1.

THEOREM 3.7.

(i) *The Banach space Y does not contain c_0 if and only if for each pair (X, Z) of Banach spaces it follows that every operator $T : X \otimes_\lambda Y \to Z$ which is (sb) is also unconditionally converging.*

(ii) *The Banach space Y is a Schur space if and only if for each pair (X, Z) of Banach spaces it follows that every operator $T : X \otimes_\lambda Y \to Z$ which is (sb) is also completely continuous.*

Proof. (i) Suppose $T : X \otimes_\lambda Y \to Z$ is (sb) and further suppose that c_0 does not embed into Y. To see that T is unconditionally converging, it suffices to show that whenever $\sum_{i=1}^{\infty} \mu_i$ is a weakly unconditionally converging series in $X \otimes_\lambda Y$, then $T(\mu_i) \to 0$. Let $\sum_{i=1}^{\infty} \mu_i$ be a weakly unconditionally converging series in $X \otimes_\lambda Y$ and let $x^* \in X^*$. Then (μ_i) is bounded in $X \otimes_\lambda Y$, and $\sum_{i=1}^{\infty} \mu_i(x^*)$ is weakly unconditionally converging in Y. Since c_0 does not embed into Y, $\sum_{i=1}^{\infty} \mu_i(x^*)$ is unconditionally converging. Thus $\mu_i(x^*) \xrightarrow{\|\cdot\|} 0$ for each $x^* \in X^*$, and since T is (sb), $T(\mu_i) \xrightarrow{\|\cdot\|} 0$. Thus T is unconditionally converging.

Conversely, if each (sb) operator is unconditionally converging, then every strongly bounded operator $m \leftrightarrow T : C(K, Y) \to Z$ is unconditionally converging. Theorem 2.1 of [25] shows that Y does not contain c_0.

(ii) Again assume that $T : X \otimes_\lambda Y \to Z$ is (sb) and further suppose that Y has the Schur property. Let (μ_i) be a weakly null sequence in $X \otimes_\lambda Y$

and let $x^* \in X^*$. Then $(\mu_i(x^*))$ is weakly null in Y. Since Y has the Schur property, $\mu_i(x^*) \xrightarrow{\|\cdot\|} 0$. It follows that $T(\mu_i) \xrightarrow{\|\cdot\|} 0$ since T is (sb). Therefore T is completely continuous.

The converse follows exactly as in part (i). ■

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