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*GENERAL TOPOLOGY*

# Countable Compact Scattered $T_2$ Spaces and Weak Forms of AC

by

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**Summary.** We show that:

(1) It is provable in **ZF** (i.e., Zermelo–Fraenkel set theory minus the Axiom of Choice **AC**) that *every compact scattered* $T_2$ *topological space is zero-dimensional.*
(2) If *every countable union of countable sets of reals is countable*, then *a countable compact* $T_2$ *space is scattered iff it is metrizable.*
(3) If *the real line* $\mathbb{R}$ *can be expressed as a well-ordered union of well-orderable sets*, then *every countable compact zero-dimensional* $T_2$ *space is scattered.*
(4) It is not provable in **ZF**+¬**AC** that *there exists a countable compact* $T_2$ *space which is dense-in-itself.*

**1. Notation and terminology.** Let $(X, \mathcal{T})$ be a topological space.

(i) $X$ is said to be *compact* iff every open cover of $X$ has a finite subcover.
(ii) $X$ is said to be *dense-in-itself* iff it has no isolated points.
(iii) $X$ is said to be *zero-dimensional* iff each of its points has a neighborhood base consisting of clopen (closed and open) sets.
(iv) $X$ is said to be a *Baire space* iff $\overline{\bigcap \mathcal{D}} = X$ for every countable family $\mathcal{D}$ of dense open sets of $X$. (In **ZF**, a compact $T_2$ space is Baire iff it cannot be covered by countably many nowhere dense sets, i.e., sets whose closure has empty interior; see [3]).

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For any topological space $(X, \mathcal{T})$, let

$$\mathrm{Iso}(X) = \{x \in X : x \text{ is isolated in } X\}.$$

By transfinite recursion we define a decreasing sequence $(X_\alpha)_{\alpha \in \mathrm{Ord}}$ of closed subspaces of $X$ as follows:

$$\begin{aligned} X_0 &= X, \\ X_{\alpha+1} &= X_\alpha \setminus \mathrm{Iso}(X_\alpha), \\ X_\alpha &= \bigcap \{X_\beta : \beta < \alpha\} \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

The set $X_\alpha$, $\alpha \in \mathrm{Ord}$, is called the *$\alpha$th Cantor–Bendixson derivative* of $X$.

A topological space $(X, \mathcal{T})$ is called *scattered* iff $\mathrm{Iso}(Y) \neq \emptyset$ for each non-empty subspace $Y$ of $X$. Clearly, $X$ is scattered iff there exists an ordinal $\alpha_0$ such that $X_{\alpha_0} = \emptyset$. If $X$ is scattered, then the ordinal number $\min\{\alpha : X_\alpha = \emptyset\}$ is called the *Cantor–Bendixson rank* of the scattered space $X$ and it is denoted by $|X|_{\mathrm{CB}}$. It is straightforward to see that if $(X, \mathcal{T})$ is a compact scattered space, then $|X|_{\mathrm{CB}}$ is a successor ordinal.

Next we list the topological and set-theoretical statements involved in this paper.

**CCM**: *Every countable compact* $\mathrm{T}_2$ *space is metrizable.*

**CCS**: *Every countable compact* $\mathrm{T}_2$ *space is scattered.*

**CC$_0$M** and **CC$_0$S** stand for **CCM** and **CCS**, respectively, with the additional requirement that the spaces involved are zero-dimensional.

**CAC**($\mathbb{R}$): *Every countable family of non-empty sets of reals has a choice function.*

**CUC**($\mathbb{R}$): *A countable union of countable sets of reals is countable.*

**WO-AC**($\mathbb{R}$): *For every family $\mathcal{A}$ of non-empty sets of reals there exists a function $f$ such that $f(x)$ is a non-empty well-orderable subset of $x$ for all $x \in \mathcal{A}$.*

**2. Introduction and some preliminary results.** In [5, Theorem 2.2] it is shown that the statement:

**CCM**: *Every countable compact* $\mathrm{T}_2$ *space is metrizable*,

is a theorem of **ZF**+**CAC**($\mathbb{R}$). However, **CCM** is not a theorem of **ZF**. In particular, a Cohen forcing model of **ZF** is constructed in [5, Theorem 3.4] in which there exists a countable compact scattered $\mathrm{T}_2$ space which is not second countable, hence not metrizable. Therefore, **CCM**, as well as the statement that *every countable compact scattered* $\mathrm{T}_2$ *space is metrizable*, are not deducible from the **ZF** axioms alone.

In the present paper, we shall show that a strictly weaker principle than **CAC**($\mathbb{R}$), namely **CUC**($\mathbb{R}$), suffices in order to establish that **CCS** implies **CCM**. This is done in Theorem 8. To achieve our goal, we first show in Theorem 7 that *every compact scattered* $\mathrm{T}_2$ *space is zero-dimensional* is provable in **ZF**. The mechanism of this proof is the key to establish Theorem 8. From the above we infer that *every countable compact scattered* $\mathrm{T}_2$ *space is metrizable* does not imply **CAC**($\mathbb{R}$) in **ZF**. (The original Cohen model, model $\mathcal{M}1$ in [2], satisfies **CUC**($\mathbb{R}$) and the negation of **CAC**($\mathbb{R}$).)

In the realm of countable compact $\mathrm{T}_2$ spaces, we prove in Theorem 10 that the weak choice principle **WO-AC**($\mathbb{R}$) implies $\mathbf{CC_0S}$. Thus, under the aforementioned axiom, the notions of "zero-dimensional" and "scattered" coincide for the class of countable compact $\mathrm{T}_2$ spaces. In Theorem 12 it is shown that $\mathbf{CC_0S}$ is strictly weaker than **WO-AC**($\mathbb{R}$) in **ZF**. Furthermore, in the latter theorem we prove that the statement that *there exists a countable compact zero-dimensional* $\mathrm{T}_2$ *space which is dense-in-itself* (i.e., the negation of $\mathbf{CC_0S}$) is not deducible from the axioms of **ZF** set theory.

As might be expected, for countable compact $\mathrm{T}_2$ spaces the notions of "scattered" and "Baire space" are closely related. Arnold Miller's list of interesting problems, posted on his webpage, includes the following question (Problem 13.3 in Section 13 titled "**not AC**"):

*In* **ZF**, *does every countable compact* $T_2$ *space have an isolated point?*

The above problem is referred to as Marianne Morillon's question. Morillon (http://www2.univ-reunion.fr/˜mar/question.html) also poses the following question:

*In* **ZF**, *is every countable compact* $T_2$ *space a Baire space?*

Our first easy result shows that the topological statements of the above two questions are equivalent to **CCS**.

THEOREM 1. *The following statements are equivalent in* **ZF**:

(a) *Every countable compact* $\mathrm{T}_2$ *space is a Baire space.*
(b) *Every countable compact* $\mathrm{T}_2$ *space has at least one isolated point.*
(c) **CCS**.
(d) *In every countable compact* $\mathrm{T}_2$ *space* $(X,\mathcal{T})$ *the set of isolated points is dense. In particular, every countably infinite compact* $\mathrm{T}_2$ *space* $(X,\mathcal{T})$ *has an infinite discrete subspace.*

*Proof.* Let $(X,\mathcal{T})$ be a countable compact $\mathrm{T}_2$ space.

(a)$\Rightarrow$(b). If $X$ has no isolated points then $X$ is not a Baire space (singletons in a dense-in-itself $\mathrm{T}_2$ space are closed nowhere dense sets), contradicting our hypothesis.

(b)⇒(c), (c)⇒(d). These are straightforward.

(d)⇒(a). By hypothesis $X$ has at least one isolated point $x$. Since $x$ is in every dense open subset of $X$ it follows that $X$ is a Baire space as required. ∎

From Theorem 1 it follows that the answer to Morillon's question depends on the answer to the following question:

*Does there exist a model of* **ZF** *in which there is a countable, dense-in-itself, compact Hausdorff space?*

In the next theorem, we give equivalent versions of the above problem.

THEOREM 2. *The following statements are equivalent in* **ZF**:

(i) *There is a countable dense-in-itself compact* $\mathrm{T}_2$ *space.*
(ii) *On every countably infinite set one can define a dense-in-itself compact* $\mathrm{T}_2$ *topology.*
(iii) *Every countable compact* $\mathrm{T}_2$ *space embeds in a countable dense-in-itself compact* $\mathrm{T}_2$ *space.*

*Proof.* (i)⇒(ii). Let $(Y, \mathcal{Q})$ be a countable dense-in-itself compact $\mathrm{T}_2$ space and $X$ a countably infinite set. Let $f : Y \to X$ be a bijection. It can be readily verified that $\mathcal{T} = \{f(O) : O \in \mathcal{Q}\}$ is a dense-in-itself compact $\mathrm{T}_2$ topology on $X$.

(ii)⇒(iii). Fix a countable compact $\mathrm{T}_2$ space $(X, \mathcal{T})$. Let $\mathcal{Q}$ be a dense-in-itself compact $\mathrm{T}_2$ topology on $\omega$. Clearly, the Tikhonov product $(Y, \mathcal{W})$ of $(X, \mathcal{T})$ and $(\omega, \mathcal{Q})$ is a dense-in-itself compact $\mathrm{T}_2$ space and $(X, \mathcal{T})$ embeds in $(Y, \mathcal{W})$.

(iii)⇒(i). This is straightforward. ∎

We will need the following results. We leave the proof of Theorem 5 as an easy exercise for the interested reader.

THEOREM 3 (Good–Tree, [1]). (**ZF**) *Urysohn's metrization theorem: Every regular, second countable topological space is metrizable.*

THEOREM 4 (Keremedis-Tachtsis, [5]). (**ZF**) *Every countable compact* $\mathrm{T}_2$ *space* $(X, \mathcal{T})$ *with a well-orderable base for* $\mathcal{T}$ *is metrizable.*

THEOREM 5. (**ZF**) *Every compact* $\mathrm{T}_2$ *space is* $\mathrm{T}_4$.

THEOREM 6 (Keremedis–Tachtsis, [4]). *The following statements are equivalent in* **ZF**:

(i) **WO**-**AC**($\mathbb{R}$).
(ii) $\mathbb{R}$ *is the union of a well-orderable family of well-orderable sets.*

**3. Main results.** A. J. Ostaszewski ([7, p. 515]) proves that compact scattered $T_2$ topological spaces are zero-dimensional. However, his proof is carried out in the **ZFC** axiom system. We show next that the axiom of choice is not really needed.

THEOREM 7. (**ZF**) *Every compact scattered* $T_2$ *space is zero-dimensional.*

*Proof.* Fix a compact scattered $T_2$ space $(X, \mathcal{T})$ and let $\alpha = |X|_{\mathrm{CB}}$ be its Cantor–Bendixson rank. Then $\alpha = \beta + 1$ for some $\beta \in \mathrm{Ord}$.

For every $i \leq \beta$, let $G_i = \mathrm{Iso}(X_i)$. Clearly, $X = \bigcup_{i \leq \beta} G_i$. We shall show that for every $i \leq \beta$ and every point $x \in G_i$ there exists a subset $\mathcal{V}_x$ of $\mathcal{T}$ such that:

(1) For every $x \in G_i$, $\mathcal{V}_x$ is a neighborhood base at $x$ consisting of clopen subsets of $X$.
(2) For every $x \in G_i$ and every $V \in \mathcal{V}_x$, $V \subset (\bigcup_{j<i} G_j) \cup \{x\}$.

For $i = 0$, if $x \in G_0 = \mathrm{Iso}(X)$, then we set $\mathcal{V}_x = \{\{x\}\}$.

For $i \leq \beta$, assume that (1) and (2) hold for all $x \in G_j$, $j < i$, and let $x \in G_i$. We first note that there exists an open neighborhood $V_x$ of $x$ such that $\overline{V}_x \subset (\bigcup_{j<i} G_j) \cup \{x\}$. To see this, consider the following two cases:

(a) $i = \beta$. Then $G_i$ is finite. Let $V_x$ be an open neighborhood of $x$ such that $\overline{V}_x \subset (G_i \setminus \{x\})^{\mathrm{c}}$. Clearly $\overline{V}_x \cap G_i = \{x\}$ and since $X = \bigcup_{j \leq \beta} G_j$ it follows that $V_x$ is as required.
(b) $i < \beta$. Since $X_{i+1}$ is closed and $G_i = \mathrm{Iso}(X_i)$, and $x \notin X_{i+1}$, there is an open neighborhood $O_x$ of $x$ which avoids $X_{i+1}$ and meets $G_i$ only in $x$. Let $V_x$ be an open neighborhood of $x$ such that $\overline{V}_x \subset O_x$. Then $V_x$ is as required.

Clearly, $\partial V_x = \overline{V}_x \setminus V_x \subset \bigcup_{j<i} G_j$, and by the induction hypothesis it follows that $\mathcal{U} = \bigcup_{y \in \partial V_x} \mathcal{V}_y$ is a cover of $\partial V_x$. Let $\mathcal{W}$ be a finite subcover of $\mathcal{U}$. Then $W = \bigcup \mathcal{W}$ is a clopen set including $\partial V_x$ and $F_x = \overline{V}_x \setminus W = V_x \setminus W$ is a clopen neighborhood of $x$ such that $F_x \subset (\bigcup_{j<i} G_j) \cup \{x\}$.

From the above it follows that the collection

$$\mathcal{V}_x = \Big\{ V \subset \Big( \bigcup_{j<i} G_j \Big) \cup \{x\} : V \text{ is a clopen neighborhood of } x \Big\}$$

is non-empty and obviously $\mathcal{V}_x$ is a neighborhood base at $x$. This completes the proof of the theorem. ∎

THEOREM 8. *Under* **CUC**($\mathbb{R}$), **CCS** *iff* **CCM**.

*Proof.* (**CCS** $\Rightarrow$ **CCM**). Let $(X, \mathcal{T})$ be a countable compact scattered $T_2$ space. Let $\alpha = |X|_{\mathrm{CB}}$, $G_i = \mathrm{Iso}(X_i)$ for all $i \in \alpha$, and for each $x \in X$, let $\mathcal{V}_x$ be the clopen neighborhood base at $x$ which was constructed in the proof of Theorem 7. We will show that the clopen base $\mathcal{B} = \bigcup\{\mathcal{V}_x : x \in X\}$ for $\mathcal{T}$ is countable. Then Theorem 3 will imply that $X$ is metrizable.

For every $i \in \alpha$, put $\mathcal{B}_i = \bigcup\{\mathcal{V}_x : x \in \bigcup_{j\le i} G_j\}$.

Claim. *For each $i \in \alpha$, $\mathcal{B}_i$ is a countable set.*

*Proof of Claim.* For $i = 0$, $\mathcal{B}_0 = \{\{x\} : x \in G_0\}$ is clearly countable.

For $i < \alpha$, assume that $\mathcal{B}_j$, $j < i$, is countable. By **CUC**($\mathbb{R}$) it follows that $\bigcup\{\mathcal{B}_j : j < i\}$ is countable. To terminate the induction, it suffices to show that for every $x \in G_i$, $\mathcal{V}_x$ is countable. Fix any $H \in \mathcal{V}_x$. For every $F \in \mathcal{V}_x$ we set

$$\mathcal{U} = \{F \cap H\} \cup \Big\{V \in \bigcup_{j<i} \mathcal{B}_j : V \subset F \setminus H\Big\} \cup \Big\{V \in \bigcup_{j<i} \mathcal{B}_j : V \subset H \setminus F\Big\}.$$

Then $\mathcal{U}$ is a cover of the clopen set $F \cup H$. Let $\mathcal{W}$ be a finite subcover of $\mathcal{U}$ and put

$$\mathcal{W}_1 = \bigcup\{W \in \mathcal{W} : W \subset F \setminus H\}, \quad \mathcal{W}_2 = \bigcup\{W \in \mathcal{W} : W \subset H \setminus F\}.$$

It follows that $F = (H \setminus \bigcup \mathcal{W}_2) \cup (\bigcup \mathcal{W}_1)$. Thus, every set $F \in \mathcal{V}_x$ can be expressed in terms of the set $H$ and a finite subset $\mathcal{W}$ of $\bigcup_{j<i} \mathcal{B}_j$. Since $\bigcup_{j<i} \mathcal{B}_j$ is countable, the set $[\bigcup_{j<i} \mathcal{B}_j]^{<\omega}$ is also countable and consequently $\mathcal{V}_x$ is countable as required.

From the Claim it readily follows that for all $x \in X$, $\mathcal{V}_x$ is countable. Thus, by **CUC**($\mathbb{R}$), the clopen base $\mathcal{B}$ for $\mathcal{T}$ is countable. Hence, $X$ is metrizable as desired.

(**CCM** $\Rightarrow$ **CCS**) This is straightforward. ∎

Theorem 9. (**ZF**)

(i) **CCS**+**CC₀M** *iff* **CCM**.
(ii) **CCM** *iff every countable compact* $\mathrm{T}_2$ *space topologically embeds in the set* $\mathbb{Q}$ *of all rational numbers.*

*Proof.* (i) In view of Theorem 7, this is straightforward.

(ii)($\Rightarrow$) Let $(X, \mathcal{T})$ be a countable compact metrizable space. Then $X$ is scattered and has a countable base $\mathcal{C}$. Let $\{\mathcal{D}_n : n \in \omega\}$ be an enumeration for the set $[\mathcal{C}]^{<\omega}$ of all finite subsets of $\mathcal{C}$. Let also $\mathcal{B}$ be the clopen base for $\mathcal{T}$ defined in the proof of Theorem 8. We show that $\mathcal{B}$ is countable. To this end, define a function $f : \mathcal{B} \to [\mathcal{C}]^{<\omega}$ by setting for each $V \in \mathcal{B}$, $f(V) = \mathcal{D}_{n_V}$, where $n_V = \min\{n \in \omega : V = \bigcup \mathcal{D}_n\}$. Since each $V \in \mathcal{B}$ is a compact set and $\mathcal{C}$ is a base for $\mathcal{T}$, it follows that $f(V)$ is definable for all $V \in \mathcal{B}$. Furthermore, it can be readily verified that $f$ is one-to-one, hence $\mathcal{B}$ is countable. Follow now the proof of Theorem 2 in [6, p. 287], in order to verify that $X$ topologically embeds in $\mathbb{Q}$.

(ii)($\Leftarrow$) This is straightforward. ∎

THEOREM 10. **WO-AC**($\mathbb{R}$) *implies* $\mathbf{CC_0S}$.

*Proof.* Let $(X, \mathcal{T})$ be a countable, zero-dimensional, compact $T_2$ space. In view of Theorem 1 it suffices to show that $\mathrm{Iso}(X) \neq \emptyset$. Assume on the contrary that $X$ is dense-in-itself. We shall reach a contradiction by proving that $|\mathbb{R}| \leq |X|$.

Let $\mathcal{B}$ be a base for $X$ consisting of clopen sets, and by **WO-AC**($\mathbb{R}$) let $A = \{\mathcal{B}_n : n \in \aleph\}$, $\aleph$ a well-ordered cardinal and $\mathcal{B}_n$ an infinite well-orderable set for each $n \in \aleph$, be a family such that $\mathcal{B} = \bigcup A$.

By induction on the length of sequences in ${}^{<\mathbb{N}}2 = \bigcup\{{}^n2 : n \in \mathbb{N}\}$ we shall construct a family $\{B_s : s \in {}^{<\mathbb{N}}2\}$ such that:

(1) For all $s \in {}^{<\mathbb{N}}2$, $B_s$ is a compact, zero-dimensional, dense-in-itself subspace of $X$.
(2) For all $s \in {}^{<\mathbb{N}}2$ and $t \in 2$, $B_{s^\frown t} \subset B_s$, where for $s = (s_1, \ldots, s_n)$ and $t \in 2$, $s^\frown t = (s_1, \ldots, s_n, t)$, $n \in \mathbb{N}$, i.e. the concatenation of the sequence $s$ with the element $t$ of 2.
(3) For all $s \in {}^{<\mathbb{N}}2$, $B_{s^\frown 0} \cap B_{s^\frown 1} = \emptyset$.

For the first step of the induction we argue as follows: Without loss of generality assume that $X \in \mathcal{B}_0$ and let $\Gamma_0$ be the Boolean subalgebra of $\mathcal{P}(X)$ which is generated by $\mathcal{B}_0$. Since $\mathcal{B}_0$ is well-orderable, it follows that so is $\Gamma_0$. Let $S_0 = \mathrm{Ult}(\Gamma_0)$ be the *Stone space* of $\Gamma_0$, i.e., the space of ultrafilters of $\Gamma_0$ having the collection $\mathcal{U} = \{U_p = \{\mathcal{F} \in S_0 : p \in \mathcal{F}\} : p \in \Gamma_0\}$ as a base for its topology. Then $S_0$ is a compact zero-dimensional $T_2$ space (compactness follows without using any form of choice since $\Gamma_0$ is well-orderable).

We assert that $S_0$ is a countable set. To see this, first let $\mathcal{A} = \{\bigcap \mathcal{F} : \mathcal{F} \in S_0\}$. By compactness of $X$, it follows that each member of $\mathcal{A}$ is a non-empty set. Moreover, $\mathcal{A}$ is pairwise disjoint. Indeed, let $\mathcal{F}, \mathcal{G} \in S_0$ be such that $\mathcal{F} \neq \mathcal{G}$. Let $V \in \mathcal{F} \setminus \mathcal{G}$ and assume that there exists an element $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. Then $V^{\mathrm{c}} \in \mathcal{G}$ for $\mathcal{G}$ is an ultrafilter of $\Gamma_0$, and $x \in V \cap V^{\mathrm{c}} = \emptyset$, a contradiction. Thus, $\mathcal{A}$ is countable and the function $f : S_0 \to \mathcal{A}$ defined by $f(\mathcal{F}) = \bigcap \mathcal{F}$ for all $\mathcal{F} \in S_0$ is one-to-one. It follows that $S_0$ is a countable set and can be effectively enumerated.

By Theorem 4 we deduce that $S_0$ is a metrizable space, moreover $\overline{\mathrm{Iso}(S_0)} = S_0$. As $|S_0| \geq 2$ (if $V \in \Gamma_0$, let $\mathcal{F}$ and $\mathcal{G}$ be the ultrafilters of $\Gamma_0$ generated by the filters $F = \{U \in \Gamma_0 : V \subset U\}$ and $G = \{U \in \Gamma_0 : V^{\mathrm{c}} \subset U\}$ respectively; the construction of $\mathcal{F}$ and $\mathcal{G}$ can be done without employing any choice principle since $\Gamma_0$ is well-orderable) and $\overline{\mathrm{Iso}(S_0)} = S_0$, it follows that $S_0$ has at least two isolated points, say $\mathcal{F}$ and $\mathcal{G}$. Assume that $\mathcal{F}$ is less than $\mathcal{G}$ according to some prescribed enumeration of $S_0$. It is well known that the atoms of a Boolean algebra correspond to the isolated points of its Stone space. Thus, there exist unique disjoint elements $p_{\mathcal{F}}, q_{\mathcal{G}} \in \Gamma_0$ such that $U_{p_{\mathcal{F}}} = \{\mathcal{F}\}$ and $U_{q_{\mathcal{G}}} = \{\mathcal{G}\}$. Now, $p_{\mathcal{F}}$ and $q_{\mathcal{G}}$, being clopen subsets of $(X, \mathcal{T})$, are compact, zero-dimensional, dense-in-itself subspaces of $X$. Put $B_{\langle 0\rangle} = p_{\mathcal{F}}$ and $B_{\langle 1\rangle} = q_{\mathcal{G}}$.

Assume now that we have constructed sets $B_s$ for all dyadic sequences of length $n$. Let $s$ be any such sequence and let $S_n$ be the Stone space of the Boolean subalgebra $\Gamma_n$ of $\mathcal{P}(B_s)$ which is generated by $\{O \cap B_s : O \in \mathcal{B}_n\}$. As in the first step of the induction we may prove that $S_n$ is a countable (with an effective enumeration) compact metrizable space, hence we may choose two isolated points of $S_n$, say $\mathcal{F}$ and $\mathcal{G}$. Assume that $\mathcal{F}$ is less than $\mathcal{G}$ according to some prescribed enumeration of $S_n$. Let $p_{\mathcal{F}}$ and $q_{\mathcal{G}}$ be the unique atoms of $\Gamma_n$ such that $U_{p_{\mathcal{F}}} = \{\mathcal{F}\}$ and $U_{q_{\mathcal{G}}} = \{\mathcal{G}\}$. Put $B_{s^\frown 0} = p_{\mathcal{F}}$ and $B_{s^\frown 1} = q_{\mathcal{G}}$. The induction is terminated.

For every $g \in 2^{\mathbb{N}}$, let $G_g = \bigcap\{B_{g|_n} : n \in \mathbb{N}\}$. Since $\{B_{g|_n} : n \in \mathbb{N}\}$ is a descending family of non-empty closed subsets of $X$, it follows that $G_g \neq \emptyset$ and we may choose $u_g = \min(G_g)$ where the minimum is taken with respect to some prescribed enumeration of $X$. By condition (3) it follows that for all $s, t \in 2^{\mathbb{N}}$, if $s \neq t$, then for some $n \in \mathbb{N}$, $B_{s|_n} \cap B_{t|_n} = \emptyset$. Therefore, the function $F : 2^{\mathbb{N}} \to X$ defined by $F(g) = u_g$ for all $g \in 2^{\mathbb{N}}$ is injective, hence $|\mathbb{R}| \le |X|$. This is a contradiction, thus $X$ has at least one isolated point as desired. ∎

From the proof of Theorem 10 it is apparent that all the Boolean algebras involved in the inductive construction have the same cardinality as their metrizable Stone spaces. In particular, they are both countable. Motivated by this observation we prove the following result.

Theorem 11. *Assume that for every Boolean algebra $(B, +, \cdot)$, $|B| \le |P(B)|$, where $P(B)$ is the set of all prime ideals of $B$. Then every compact zero-dimensional* $\mathrm{T}_2$ *space $(X, \mathcal{T})$ has a base of size at most $|X|$. In particular, every countable compact zero-dimensional* $\mathrm{T}_2$ *space is metrizable.*

*Proof.* Fix a compact zero-dimensional $\mathrm{T}_2$ space $(X, \mathcal{T})$ and let $B$ be the Boolean algebra of all clopen subsets of $X$ under union and intersection. Clearly $B$ has no free ultrafilters (i.e., ultrafilters $\mathcal{F} \subset B$ such that $\bigcap \mathcal{F} = \emptyset$). Thus, $|X| = |P(B)|$ and our assumption implies that $|B| \le |X|$. Since $B$ is a base for $X$ the desired result follows. ∎

Theorem 12.

(i) *The negation of* $\mathbf{CC_0S}$, *i.e. the statement "there exists a countable compact zero-dimensional* $\mathrm{T}_2$ *space which is dense-in-itself", is not provable in* $\mathbf{ZF} + \neg\mathbf{AC}$.
(ii) $\mathbf{CC_0S}$ *is strictly weaker than* $\mathbf{CAC}(\mathbb{R})$ *in* $\mathbf{ZF}$.
(iii) *None of the statements* $\mathbf{CCS}$, $\mathbf{CCM}$, $\mathbf{CC_0S}$, *and* $\mathbf{CC_0M}$ *implies* $\mathbf{WO}$-$\mathbf{AC}(\mathbb{R})$ *in* $\mathbf{ZF}$.
(iv) *In* $\mathbf{ZF}$, $\mathbf{CC_0S}$ *does not imply* $\mathbf{CUC}(\mathbb{R})$.

*Proof.* (i) By Theorem 10 we know that $\neg\mathbf{CC_0S}$ implies $\neg\mathbf{WO\text{-}AC}(\mathbb{R})$, i.e., $\mathbb{R}$ cannot be written as a well-orderable union of well-orderable sets. Since $\neg\mathbf{WO\text{-}AC}(\mathbb{R})$ fails in the Feferman–Lévy forcing model (model $\mathcal{M}9$ in [2]; in it, $\mathbb{R}$ can be expressed as a countable union of countable sets) it follows that $\neg\mathbf{CC_0S}$ is not provable in **ZF**.

(ii) From the proof of (i) and from Theorem 10 it follows that $\mathbf{CC_0S}$ holds in the model $\mathcal{M}9$. Since in $\mathcal{M}9$, $\mathbb{R}$ is a countable union of countable sets and the axiom $\mathbf{CAC}(\mathbb{R})$ implies $\mathbf{CUC}(\mathbb{R})$, it follows that $\mathbf{CAC}(\mathbb{R})$ fails in $\mathcal{M}9$.

(iii) In Feferman's forcing model (model $\mathcal{M}2$ in [2]) $\mathbf{CAC}(\mathbb{R})$ holds, hence each of the topological statements listed in (iii) holds true in this model ($\mathbf{CAC}(\mathbb{R})$ implies each of these statements, see [5]). On the other hand, as the axiom of choice for well-ordered families of non-empty sets holds in $\mathcal{M}2$ and $\mathbb{R}$ is not well-orderable in that model (see [2]), it follows that $\mathbf{WO\text{-}AC}(\mathbb{R})$ fails in $\mathcal{M}2$.

(iv) $\mathbf{CC_0S}$ holds in $\mathcal{M}9$, whereas $\mathbf{CUC}(\mathbb{R})$ fails in that model (see the proof of (i)). ∎

**4. Summary.** In the following table, if the entry in the row labeled 'A' and column labeled 'B' is:

(1) "?", then it is unknown whether A $\to$ B in **ZF**;
(2) "$\to$", then A $\to$ B in **ZF**;
(3) "$\not\to$", then A $\not\to$ B in **ZF**.

| | $\mathbf{CAC}(\mathbb{R})$ | $\mathbf{WO-AC}(\mathbb{R})$ | **CCS** | **CCM** | $\mathbf{CC_0M}$ | $\mathbf{CC_0S}$ |
|---|---|---|---|---|---|---|
| $\mathbf{CAC}(\mathbb{R})$ | $\to$ | $\not\to$ | $\to$ | $\to$ | $\to$ | $\to$ |
| $\mathbf{WO-AC}(\mathbb{R})$ | $\not\to$ | $\to$ | ? | ? | ? | $\to$ |
| **CCS** | ? | $\not\to$ | $\to$ | ? | ? | $\to$ |
| **CCM** | ? | $\not\to$ | $\to$ | $\to$ | $\to$ | $\to$ |
| $\mathbf{CC_0M}$ | ? | $\not\to$ | ? | ? | $\to$ | $\to$ |
| $\mathbf{CC_0S}$ | $\not\to$ | $\not\to$ | ? | ? | ? | $\to$ |

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