BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 54, No. 1, 2006

GENERAL TOPOLOGY

A Basic Fixed Point Theorem

by

Lech PASICKI

Presented by Czesław BESSAGA

Summary. The paper contains a fixed point theorem for stable mappings in metric discus spaces (Theorem 10). A consequence is Theorem 11 which is a far-reaching extension of the fundamental result of Browder, Göhde and Kirk for non-expansive mappings.

DEFINITION 1. A metric space (X, d) is a *discus space* if there exists a mapping $\varrho : [0, \infty) \times (0, \infty) \to [0, \infty)$ such that

(1) $$\varrho(\beta, r) < \varrho(0, r) = r, \quad \beta, r > 0,$$

(2) $$\varrho(\cdot, r) \text{ is nonincreasing}, \quad r > 0,$$

(3) $$\varrho(\delta, \cdot) \text{ is upper semicontinuous}, \quad \delta \geq 0,$$

(4) for each $x, y \in X$, $r, \varepsilon > 0$ there exists a $z \in X$ such that
$$B(x, r) \cap B(y, r) \subset B(z, \varrho(d(x, y), r) + \varepsilon).$$

EXAMPLE 2. Let $(Y, (\cdot, \cdot))$ be an inner product space with $|x| = \sqrt{(x, x)}$. We have $|x + h|^2 + |x - h|^2 = 2(|x|^2 + |h|^2)$ and hence $|h|^2 = (|x + h|^2 + |x - h|^2 - 2|x|^2)/2$. Now for $y = -x$, $|x + h| = |x - h| = r$, $\delta = 2|x|$ we take $z = 0$ and thus ϱ satisfying (4) is given by

$$\varrho(\delta, r) = |h| = \begin{cases} \sqrt{r^2 - \delta^2/4}, & \delta \in [0, 2r], \\ 0, & \delta > 2r. \end{cases}$$

It is clear that each nonempty convex set $X \subset Y$ is a discus space with the same ϱ.

Now let us consider a more general case.

2000 *Mathematics Subject Classification*: Primary 54H25.

Key words and phrases: discus space, uniformly convex space, stable mapping, non-expansive mapping.

A Banach space is *uniformly convex* (see [1, p. 34]) if there exists an increasing surjection $\varphi : [0,2] \to [0,1]$ such that $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge \beta$ implies $\|(x+y)/2\| \le 1-\varphi(\beta)$.

EXAMPLE 3. Let $(Y, \|\cdot\|)$ be a uniformly convex space. By considering $(h-x)/r$, $(h+x)/r$ in place of x, y respectively we can see that $\|x-h\| \le r$, $\|x+h\| \le r$ and $2\|x\|/r \ge \beta$ implies $\|h\| \le r(1-\varphi(\beta))$. Since φ is an increasing surjection, it is continuous. For $z = 0$ we take

$$\varrho(\delta, r) = \|h\| = \begin{cases} r(1-\varphi(\delta/r)), & \delta \in [0, 2r], \\ 0, & \delta > 2r. \end{cases}$$

Consequently, each nonempty convex set $X \subset Y$ is a discus space with the same ϱ.

In the two lemmas to follow we present some properties of discus spaces.

LEMMA 4. *If (X,d) is a complete discus space then* (4) *can be replaced by*

(5) *for each $x, y \in X$ and $r > 0$ there exists a $z \in X$ such that*
$$B(x,r) \cap B(y,r) \subset B(z, \varrho(d(x,y), r)).$$

Proof. Set $\alpha = \varrho(d(x,y), r)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ decreasing to α be such that there are $x_n \in X$ with $B(x,r) \cap B(y,r) \subset B(x_n, \alpha_n)$. Assume $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, i.e. there is a $\beta > 0$ such that $d(x_n, x_k) \ge \beta$ for infinitely many $k < n$. Set $2\gamma = \alpha - \varrho(\beta, \alpha) = \varrho(0,\alpha) - \varrho(\beta,\alpha) > 0$ (see (1)). We have

$$\begin{aligned} B(x,r) \cap B(y,r) &\subset B(x_n, \alpha_n) \cap B(x_k, \alpha_k) \subset B(x_n, \alpha_k) \cap B(x_k, \alpha_k) \\ &\subset B(z_{n,k}, \varrho(d(x_n, x_k), \alpha_k) + \gamma) \end{aligned}$$

for some $z_{n,k} \in X$ (see (4)). On the other hand, $\varrho(d(x_n,x_k), \alpha_k) \le \varrho(\beta, \alpha_k)$ (see (2)) and $\varrho(\beta, \alpha_k) \le \varrho(\beta, \alpha) + \gamma$ for sufficiently large k (see (3)). Now we obtain

$$\begin{aligned} B(x,r) \cap B(y,r) &\subset B(z_{n,k}, \varrho(\beta,\alpha) + \gamma) = B(z_{n,k}, \alpha - 2\gamma + \gamma) \\ &= B(z_{n,k}, \alpha - \gamma) \subset B(z_{n,k}, \alpha), \end{aligned}$$

i.e. (5) is satisfied. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence convergent to a z then $B(x_n, \alpha_n) \subset B(z, \alpha+\beta)$ for arbitrary $\beta > 0$ and all sufficiently large n, which means $B(x,r) \cap B(y,r) \subset B(z, \alpha+\beta)$ for all $\beta > 0$ and $B(x,r) \cap B(y,r) \subset \overline{B}(z,\alpha)$. Since $B(x,r) \cap B(y,r)$ is open, we obtain (5).

DEFINITION 5. Let (X,d) be a metric space and A a nonempty subset of X. An $x \in X$ is a *central point* for A if

(6) $$\begin{aligned} r(A) &:= \inf\{t \in (0,\infty] : \text{there exists a } z \in X \text{ with } A \subset B(z,t)\} \\ &= \inf\{t \in (0,\infty] : A \subset B(x,t)\}. \end{aligned}$$

The *centre* $c(A)$ for A is the set of all central points for A, and $r(A)$ is the *radius* of A.

LEMMA 6. *Let (X,d) be a complete discus space and let $A \subset X$ be nonempty and bounded. Then $c(A)$ is a singleton.*

Proof. Let $(r_n)_{n\in\mathbb{N}}$ decrease to $r = r(A)$ while $A \subset B(x_n, r_n)$. Suppose $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geq \beta > 0$ for infinitely many $k < n$. We have

$$\begin{aligned} A &\subset B(x_n, r_n) \cap B(x_k, r_k) \subset B(x_n, r_k) \cap B(x_k, r_k) \\ &\subset B(z_{n,k}, \varrho(d(x_n, x_k), r_k)) \subset B(z_{n,k}, \varrho(\beta, r_k)) \end{aligned}$$

(see (5), (2)) and consequently $A \subset B(z_{n,k}, r(A) - \gamma)$ (see the previous proof), a contradiction. Let $(x_n)_{n\in\mathbb{N}}$ converge to an x. Then for any $\beta > 0$ we have $B(x_n, r_n) \subset B(x, r + \beta)$ for all sufficiently large n, which means $A \subset B(x, r+\beta)$ for all $\beta > 0$ and consequently $x \in c(A)$. Suppose $x, y \in c(A)$ and $d(x, y) \geq \beta > 0$. Then by (5) we obtain

$$A \subset \overline{B}(x,r) \cap \overline{B}(y,r) \subset \overline{B}(z, \varrho(\beta, r)) \subset \overline{B}(z, r-\gamma)$$

for a $\gamma > 0$, a contradiction. Thus $c(A)$ consists of a single point.

Now we are going to present a lemma which concerns mappings.

Let 2^X be the family of all subsets of X and let $F : X \to 2^X$ be a multivalued mapping (we assume that $F(x) \neq \emptyset$, $x \in X$).

DEFINITION 7. Let (X,d) be a metric space, $\emptyset \neq Y \subset X$ and $F : Y \to 2^Y$ a mapping. An $x \in X$ is a *central point* for F if

$$\begin{aligned} (7) \quad r(F) := \inf\{t \in (0,\infty] &: \text{there exists } n_0 \text{ such that for each } n > n_0 \\ &\qquad \text{there is a } z \in X \text{with } F^n(Y) \subset B(z,t)\} \\ = \inf\{t \in (0,\infty] &: \text{there exists } n_0 \text{ such that} \\ &\qquad F^n(Y) \subset B(x,t) \text{ for each } n > n_0\}. \end{aligned}$$

The *centre* $c(F)$ for F is the set of all central points for F, and $r(F)$ is the *radius* of F.

LEMMA 8. *Let (X,d) be a complete discus space. If $\emptyset \neq Y \subset X$ is bounded and $F : Y \to 2^Y$ is a mapping then $c(F)$ is a singleton.*

Proof. Set $r = r(F)$. We have $F^{n+1}(Y) \subset F^n(Y)$ and therefore there exists a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ convergent to r and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $F^n(Y) \subset B(x_n, r_n)$ for all $n \in \mathbb{N}$. Suppose $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, i.e. $d(x_n, x_k) \geq \beta > 0$ for infinitely many $k < n$. We have

$$F^n(Y) \subset F^n(Y) \cap F^k(Y) \subset B(x_n, r_n) \cap B(x_k, r_k) \subset B(z_{n,k}, \varrho(\beta, r_k))$$

and consequently $F^n(Y) \subset B(z_{n,k}, r - \gamma)$ for a $\gamma > 0$ (see the previous proof), a contradiction. Now let $(x_n)_{n\in\mathbb{N}}$ converge to x. We obtain $F^n(Y) \subset B(x, r+\beta)$ for any $\beta > 0$ and sufficiently large n. Consequently, $x \in c(F)$. The uniqueness of $x \in c(F)$ can be obtained as in the proof of Lemma 6 for $c(A)$.

Definition 9. Let (X,d) be a metric space, $\emptyset \neq Y \subset X$ a bounded set and $F : X \to 2^X$ a mapping. Then F is *Y-stable* if $F(Y) \subset Y$, $c(F_{|Y}) \neq \emptyset$ and

(8) for some $x \in c(F_{|Y})$ and each $t > r(F_{|Y})$ there exist $n \in \mathbb{N}$ and $y \in F(x)$ such that $F^n(Y) \subset B(y,t)$.

If F is X-stable then we say F is *stable*.

Theorem 10. *Let (X,d) be a complete discus space, $\emptyset \neq Y \subset X$ a bounded set and $F : X \to 2^X$ a Y-stable mapping (which implies that $c(F_{|Y})$ is a singleton). If $F(c(F_{|Y}))$ is closed then F has a fixed point.*

Proof. Let $\{x\} = c(F_{|Y})$ and $x_n \in F(x)$ be such that $F^n(Y) \subset B(x_n, r_n)$ with $(r_n)_{n\in\mathbb{N}}$ decreasing to $r(F_{|Y})$. The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to the unique point x of $c(F_{|Y})$ (see the proof of Lemma 8). All the points x_n belong to $F(x)$ which is closed and therefore $x \in F(x)$.

Theorem 11. *Let (X,d) be a complete discus space and let $f : X \to X$ be a mapping with $Y = \bigcup\{f^n(y) : n \in \mathbb{N}\}$ bounded for some $y \in X$. If*

(9) *for $\{x\} = c(f_{|Y})$ we have $d(f(x), f(y)) \le d(x,y)$ for all $y \in Y$,*

then f has a fixed point.

Proof. We have $f(Y) \subset Y$ and Y is bounded. If $f^{n-1}(Y) \subset B(x,t)$ then $f^n(Y) \subset f(B(x,t))$. For $d(x,y) < t$ we obtain $d(f(x), f(y)) \le d(x,y) < t$ (see (9)), which means $f(y) \in B(f(x),t)$ and consequently $f(B(x,t)) \subset B(f(x),t)$, i.e. (8) is satisfied and we apply Theorem 10.

Let us recall that for (X,d) being a metric space a mapping $f : X \to X$ is *non-expansive* if $d(f(x), f(y)) \le d(x,y)$ for all $x, y \in X$. Clearly each non-expansive mapping f with $c(f)$ nonempty satisfies (9).

In view of Example 3, Theorem 11 extends the classical theorem of Browder, Göhde and Kirk [1, (7.9) (b), p. 34] for non-expansive mappings (what is more, we do not assume $f(X)$ to be bounded). See also the paper of Goebel and Kirk [2].

References

[1] J. Dugundji and A. Granas, *Fixed Point Theory*, Vol. I, PWN, Warszawa, 1982.

[2] K. Goebel and W. A. Kirk, *Classical theory of nonexpansive mappings*, in: Handbook of Metric Fixed Point Theory, W. A. Kirk and B. Sims (eds.), Kluwer, Dordrecht, 2001, 49–91.

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30-059 Kraków, Poland
E-mail: pasicki@wms.mat.agh.edu.pl

Received March 6, 2006;
received in final form May 24, 2006 (7517)