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A Remark on a Paper of Crachiola and Makar-Limanov

by

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Summary. A. Crachiola and L. Makar-Limanov [J. Algebra 284 (2005)] showed the following: if X is an affine curve which is not isomorphic to the affine line $\mathbb{A}^1_k$, then $\mathrm{ML}(X \times Y) = k[X] \otimes \mathrm{ML}(Y)$ for every affine variety Y, where k is an algebraically closed field. In this note we give a simple geometric proof of a more general fact that this property holds for every affine variety X whose set of regular points is not k-uniruled.

1. Introduction. Let k be an algebraically closed field. *The Makar-Limanov invariant* of an affine variety X, denoted $\mathrm{ML}(X)$, is defined to be the ring of all regular functions on X that are constant on orbits of every algebraic k^+-action on X. Equivalently, it consists of the regular functions on X that are invariant for all exponential maps on the coordinate ring $k[X]$; in characteristic 0 these functions are exactly in the kernels of all locally nilpotent derivations on $k[X]$.

In this note we focus on the following property due to A. Crachiola and L. Makar-Limanov [2]:

PROPOSITION 1. *If X is an affine curve which is not isomorphic to the affine line $\mathbb{A}^1$, then*

$$\mathrm{ML}(X \times Y) = k[X] \otimes \mathrm{ML}(Y)$$

for every affine variety Y.

The proof given in [2] is based on algebraic methods. We give a simple geometric proof of a more general fact, which partially answers a question in [2] on a higher-dimensional analogue of Proposition 1. For this purpose we make use of the notion of k-uniruledness introduced by Jelonek [6]. An

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algebraic variety X is said to be *k-uniruled* if for a generic point $x \in X$ there exists a non-constant regular map $f : \mathbb{A}^1 \to X$ such that $x \in f(\mathbb{A}^1)$. If k is uncountable, then X is k-uniruled if and only if there exists a variety Y of dimension $\dim X - 1$ and a dominant regular map $Y \times \mathbb{A}^1 \to X$ (see [6, Prop. 5.1] or [7, Th. 3.1]). We show the following:

THEOREM 2. *If X is an affine variety whose set of regular points* $\mathrm{Reg}(X)$ *is not k-uniruled, then*

$$\mathrm{ML}(X \times Y) = k[X] \otimes \mathrm{ML}(Y)$$

for every affine variety Y.

Note that this implies Proposition 1, since for an affine curve X we have $X \cong \mathbb{A}^1$ iff $\mathrm{Reg}(X)$ is k-uniruled. (Namely, if $f : \mathbb{A}^1 \to \mathrm{Reg}(X)$ is a non-constant regular map, then $f(\mathbb{A}^1) = X$, since the image of $\mathbb{A}^1$ in every affine variety is closed. Thus X is smooth and has only one point at infinity. By Lüroth's theorem a smooth compactification of X is isomorphic to $\mathbb{P}^1$, so $X \cong \mathbb{P}^1 - \{\infty\} \cong \mathbb{A}^1$.)

The main application of Proposition 1 given in [2] was a new proof of the cancellation theorem for curves due to Abhyankar, Eakin and Heinzer [1]: if X, Y are affine curves such that $X \times Z \cong Y \times Z$ and $\mathrm{ML}(Z) = k$, then $X \cong Y$ (originally this was proved in [1] for $Z = \mathbb{A}^n$). Analogously, we get the following:

COROLLARY 3. *Let X, Z be affine varieties such that* $\mathrm{Reg}(X)$ *is not k-uniruled and* $\mathrm{ML}(Z) = k$. *If $f : X \times Z \to Y \times Z$ is an isomorphism, then there exists an induced isomorphism $\tilde{f} : X \to Y$ such that $\pi_Y \circ f = \tilde{f} \circ \pi_X$, where π_X, π_Y are the projections.*

For $Z = \mathbb{A}^n$ this fact was obtained in [3]. Note that it is related to the following Iitaka–Fujita theorem [4]: if X is an affine variety over $\mathbb{C}$ with the logarithmic Kodaira dimension $\bar{\kappa}(X) \geq 0$ and $f : X \times \mathbb{A}^n \to Y \times \mathbb{A}^n$ is an isomorphism, then there exists an induced isomorphism $\tilde{f} : X \to Y$ such that $\pi_Y \circ f = \tilde{f} \circ \pi_X$.

If X is a $\mathbb{C}$-uniruled affine variety, then $\bar{\kappa}(X) = -\infty$, since there exists a dominant generically finite regular map $Y \times \mathbb{A}^1 \to X$, which implies that $\bar{\kappa}(X) \leq \bar{\kappa}(Y \times \mathbb{A}^1) = \bar{\kappa}(Y) + \bar{\kappa}(\mathbb{A}^1) = -\infty$ (see [5] for properties of the Kodaira dimension).

2. Proofs.

We will need the following facts.

(2.1) *If $f : X \times Y \to Z$ is a regular map of affine varieties, then*

$$W = \{x \in X : f(x \times Y) \text{ is a point}\}$$

is closed in X.

Proof. If $Z \subset \mathbb{A}^n$ and $f = (f_1, \ldots, f_n)$, then

$$W = \bigcap_{i=1}^{n} \bigcap_{y,z \in Y} \{x \in X : f_i(x,z) = f_i(x,y)\}. \ \blacksquare$$

(2.2) $\mathrm{ML}(X \times Y) \subset \mathrm{ML}(X) \otimes \mathrm{ML}(Y)$ *for arbitrary affine varieties* X, Y.

Proof. This fact appeared in [2], but the proof seems to be slightly incomplete, therefore we give an alternative argument. This is obvious if X and Y each have no non-trivial k^+-actions. Suppose to the contrary that $\mathrm{ML}(Y) \neq k[Y]$ and there exists a regular function $f \in \mathrm{ML}(X \times Y) \setminus (\mathrm{ML}(X) \otimes \mathrm{ML}(Y))$. Write $f = \sum_{i=1}^{n} f_i \otimes g_i$ with minimal n, where $f_i \in k[X]$, $g_i \in k[Y]$. By symmetry we may assume that $g_1 \notin \mathrm{ML}(Y)$. Let τ be a k^+-action on Y such that g_1 is not constant on the orbit of τ passing through a point $y \in Y$. Let $h_i(t) = g_i(\tau(y,t)) \in k[t]$, $i = 1, \ldots, n$. Then $h_1 \in k[t] \setminus k$ and for each $x \in X$ the polynomial $f(x, \tau(y,t)) = \sum_{i=1}^{n} f_i(x) h_i(t) \in k[t]$ is constant (since f is invariant for the k^+-action $((x,y),t) \mapsto (x, \tau(y,t))$). This implies that $\sum_{i=1}^{n} a_i f_i = 0$, where $a_i \in k$ is the coefficient of $t^{\deg h_1}$ in h_i. Hence $f_1 = -\sum_{i=2}^{n} \frac{a_i}{a_1} f_i$, so

$$f = \sum_{i=1}^{n} f_i \otimes g_i = \sum_{i=2}^{n} f_i \otimes \left(g_i - \frac{a_i}{a_1} g_1\right),$$

which contradicts the minimality of n. ■

LEMMA 4. *Let X be as in Theorem 2. Then every k^+-action on $X \times Y$ is induced by the trivial action on X and an action on Y.*

Proof. Let σ be a k^+-action on $X \times Y$ and Z be the set of points $z \in X \times Y$ such that the orbit of σ passing through z is contracted to a point by the projection $\pi : X \times Y \to X$. We have to show that $Z = X \times Y$. Suppose that $Z \neq X \times Y$. Applying (2.1) to the map $\pi \circ \sigma : X \times Y \times \mathbb{A}^1 \to X$, we see that Z is closed in $X \times Y$. Clearly, Z and the singular locus $\mathrm{Sing}(X \times Y)$ are invariant for σ, hence so is the open set $U = (X \times Y) \setminus (Z \cup \mathrm{Sing}(X \times Y))$. Then $\pi(U)$ has nonempty interior and is the union of k-uniruled curves that are images of σ's orbits in U. Since $\mathrm{Sing}(X \times Y) = (\mathrm{Sing}(X) \times Y) \cup (X \times \mathrm{Sing}(Y))$, we have $\pi(U) \subset \mathrm{Reg}(X)$, which contradicts the fact that $\mathrm{Reg}(X)$ is not k-uniruled. ■

This lemma implies that $k[X] \otimes \mathrm{ML}(Y) \subset \mathrm{ML}(X \times Y)$ in Theorem 2; the opposite inclusion is a consequence of (2.2). ■

In the proof of Corollary 3 we use the following:

(2.3) $k[X]$ *is algebraically closed in* $k[X \times Y]$ *for affine varieties* X, Y.

Proof. Suppose that there exists $f \in k[X \times Y] \setminus k[X]$ satisfying an equation $a_n f^n + \cdots + a_0 = 0$ with $a_i \in k[X]$, $a_n \neq 0$. By (2.1) the set of points

$x \in X$ such that f is constant on $x \times Y$ is a proper closed subset of X. It follows that the map

$$F : X \times Y \to X \times \mathbb{A}^1, \quad F(x, y) = (x, f(x, y)),$$

is dominant. Then for the monomorphism $F^* : k[X][t] \to k[X \times Y]$ we have

$$F^*(a_n t^n + \cdots + a_0) = a_n f^n + \cdots + a_0 = 0,$$

contradiction. ∎

Proof of Corollary 3. Let $\varphi : k[X \times Z] \to k[Y \times Z]$ be an isomorphism. We have to show that $\varphi(k[X]) = k[Y]$. By Theorem 2, $\mathrm{ML}(X \times Z) = k[X] \otimes \mathrm{ML}(Z) = k[X]$. From (2.2) it follows that $\varphi(k[X]) = \varphi(\mathrm{ML}(X \times Z)) = \mathrm{ML}(Y \times Z) \subset \mathrm{ML}(Y) \otimes \mathrm{ML}(Z) \subset k[Y]$. Obviously, X and Y have equal dimensions, so the extension $\varphi(k[X]) \subset k[Y]$ is algebraic. By (2.3), $\varphi(k[X])$ is algebraically closed in $k[Y \times Z]$, which implies that $\varphi(k[X]) = k[Y]$. ∎

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