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FUNCTIONAL ANALYSIS

On Convex Sets with Convex-Hereditary CEP

by

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Summary. CEP stands for the compact extension property. We characterize nonlocally convex complete metric linear spaces with convex-hereditary CEP.

1. Introduction. Let $E = (E, |\cdot|)$ stand for a real metric linear space (m.l.s.), whose metric d is determined by an F-norm $|\cdot|$ satisfying

$$|x+y| \le |x| + |y| \quad \text{and} \quad |tx| \le |x|$$

for $x, y \in E$ and $|t| \le 1$. If, additionally, E is *nonlocally convex*, then we cannot assume that either the F-norm $|\cdot|$ is homogeneous, or the balls $\{x \in E \mid |x| < \varepsilon\}$ are convex; furthermore, such a space must be infinite-dimensional. A m.l.s. E is referred to as an *F-space* if the norm $|\cdot|$ is complete.

Let K be a convex subset of E. Recall that K has the *compact extension property* (CEP) if every mapping from a closed subset of the Hilbert cube to K extends over the whole cube. K has the compact extension property if and only if, for every compact subset A of K, the inclusion mapping of $A \to K$ can be uniformly approximated by mappings with finite-dimensional (equivalently, finite-dimensional convex) ranges (see [Do] and [vBvM]). The latter property is called *Klee-admissibility* (cf. [K1] and [K2]). If K is compact, then CEP becomes the classical *absolute retract property* (AR). It is known that a compact K with CEP is homeomorphic to either a finite-dimensional cube, or the Hilbert cube [DT]. For that reason, the following long-standing problem is of great importance:

PROBLEM. *Does every compact K have CEP?*

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This problem is also open when the compactness assumption on K is dropped (in particular, when K is a linear subspace of E). Perhaps the pathology goes to the extreme and **every** nonlocally convex F-space contains a convex subset K without CEP. Specifically, it is very intriguing whether every infinite-dimensional linear subspace of the classical F-space ℓ_p, $0 < p < 1$, has CEP. Let us ask:

QUESTION 1.1. *Is there a nonlocally convex F-space all of whose convex subsets (or all linear subspaces) have CEP?*

The property required in Question 1.1 will be referred to as *convex-hereditary* CEP. Recall that, in [DvM], a m.l.s. all of whose convex subsets are AR's has been called a convex-hereditary AR. No nonlocally convex F-space which is a convex-hereditary AR is known either. Also it is unknown whether CEP and the absolute retract property coincide for F-spaces. However, there exist m.l.s. that have CEP but are not AR's (see [vBvM] and [DK]); those spaces are incomplete and have a complex Borel structure. Clearly, the notion of convex-hereditary CEP can be formally extended from F-spaces to convex sets in m.l.s. The aim of this note is to characterize convex sets that have convex-hereditary CEP.

Independently, Le Hoang Tri and Nguyen Hoang Thanh have publicized similar work in [TT].

2. Properties. In addition to CEP, we will also consider the following fixed-point properties. We say that a convex set K has the *compact fixed-point property* (CFPP) if every *compact* mapping $K \to K$ has a fixed point. Here, and everywhere in the paper, $f : X \to Y$ is compact if the closure of $f(X)$ is a compact subset of Y. If K is compact, CFPP becomes the classical *fixed-point property* (FPP). In [Ca3], Cauty claims that every K has CFPP; however, his previous proof of this claim [Ca2] contains a fatal error: see [DS].

NOTATION. Let $E = (E, |\cdot|)$ be a m.l.s. and W be a convex subset of E. For $A \subset E$ and $x \in E$, let $d(A, x) = \inf\{|a - x| \mid a \in A\}$. Following the notation in [KPR], for $B \subset E$, we write

$$D(A, B) = \sup\{d(A, x) \mid x \in B\}.$$

Note that, for $\varepsilon > 0$, we have $D(A, B) \le \varepsilon$ if and only if, for every $x \in B$ and every $\delta > 0$, there exists $a \in A$ such that $|a - x| \le \varepsilon + \delta$. (So, the Hausdorff metric is $H(A, B) = \max\{D(A, B), D(B, A)\}$.)

The properties introduced below describe intrinsically the convex set W; the case of $W = E$ is our main interest.

2.1. Properties related to CEP and CFPP

Property 1. *There exist an $\varepsilon > 0$, a sequence $\{K_n\}$ of convex subsets of W, and a sequence $\{f_n\}$ of mappings, $f_n : K_n \to E$, satisfying*

(i) $\lim D(K_n, f_n(K_n)) = 0$;
(ii) $\bigcup f_n(K_n)$ *is totally bounded; and*
(iii) $|f_n(x) - x| \geq \varepsilon$ *for every* $x \in K_n$ *and* $n \in \mathbb{N}$.

Property 2. *There exist an $\varepsilon > 0$, a sequence $\{K'_n\}$ of convex subsets of W, and a sequence $\{A'_n\}$ of subsets of E satisfying*

(i′) $\lim D(K'_n, A'_n) = 0$;
(ii′) $\bigcup A'_n$ *is totally bounded; and*
(iii′) *for every* $n \in \mathbb{N}$ *and every mapping* $g : A'_n \to K'_n$ *there exists* $x \in A'_n$ *with* $|g(x) - x| \geq \varepsilon$.

The following example shows that conditions (i) and (iii) do not contradict each other in the nonlocally convex case (see [TT]).

Example 2.1. *In every ℓ_p space, $0 < p \leq 1/2$, there exists a sequence of mappings $\{f_n\}$, $f_n : \Delta_n \to \ell_p$, such that*

(i) $\lim D(\Delta_n, f_n(\Delta_n)) = 0$; *and*
(ii) $|f_n(x) - x|_p \geq 1$ *for every* $x \in \Delta_n$ *and* $n \in \mathbb{N}$.

Here, Δ_n is the standard simplex spanned by the unit vectors $e_1, \ldots, e_n$, that is, $\Delta_n = \mathrm{conv}(\{e_1, \ldots, e_n\}) \subset \ell_p$.

Proof. Let $x_n = n^{-1/p}(e_1 + \cdots + e_n)$. Define $f_n(x) = x + x_n$, $x \in \Delta_n$. Since $|x_n|_p = 1$, we have $|f_n(x) - x|_p = 1$ for every $x \in \Delta_n$.

Let $x = \sum_{i=1}^n t_i e_i$, where $0 \leq t_i \leq 1$, $t_n \leq t_{n-1} \leq \cdots \leq t_1$, and $\sum_{i=1}^n t_i = 1$. Hence $t_1 \geq 1/n$. Thus $x + x_n = \sum_{i=1}^n (t_i + 1/n^{1/p}) e_i$. Let $x' = \big(1 - \sum_{i=2}^n (t_i + 1/n^{1/p})\big) e_1 + \sum_{i=2}^n (t_i + 1/n^{1/p}) e_i$. Then $x' \in \Delta_n$ since $\sum_{i=2}^n (t_i + 1/n^{1/p}) = 1 - t_1 + (n-1)/n^{1/p} \leq 1 + (n-1)/n^{1/p} - 1/n \leq 1$. Now $|f_n(x) - x'|_p = |x + x_n - x'|_p = |(1/n^{1/p} + (n-1)/n^{1/p}, 0, 0, 0, \ldots)|_p = n^p/n$. Finally, $\lim_{n\to\infty} n^p/n = 0$. ∎

It seems likely that a similar argument will work for every ℓ_p with $1/2 < p < 1$.

Remark 1. Example 2.1 satisfies items (i) and (iii) of Property 1 for $W = \ell_p$ (or for the convex set $W = \bigcup \Delta_n$). Yet, item (ii) is not satisfied. Namely, the sequence $\{x_n\}$ is discrete, and consequently the sequence $\{e_1 + x_n\} \subset \bigcup f_n(\Delta_n)$ has no convergent subsequence.

Question 2.2. *Is there $0 < p < 1$ such that ℓ_p has Property 1 (or Property 2)?*

Later, we will show that if such a p existed then the space ℓ_p could not serve as an example required in Question 1.1; see item (1) of the "Dichotomy" Statement below.

2.2. Properties related to AR and FPP for compacta

PROPERTY $1'$. *There exist an $\varepsilon > 0$, a sequence $\{K_n\}$ of convex subsets of W, and a sequence of mappings $\{f_n\}$, $f_n : K_n \to E$, satisfying*

(i) $\lim D(K_n, f_n(K_n)) = 0$;
(ii) $\bigcup K_n$ *is totally bounded; and*
(iii) $|f_n(x) - x| \geq \varepsilon$ *for every* $x \in K_n$ *and* $n \in \mathbb{N}$.

PROPERTY $2'$. *There exist a sequence $\{K'_n\}$ of convex subsets of W and a sequence $\{A'_n\}$ of subsets of E satisfying*

(i$'$) $\lim D(K'_n, A'_n) = 0$;
(ii$'$) $\bigcup K'_n$ *is totally bounded; and*
(iii$'$) *for every* $n \in \mathbb{N}$ *and every mapping* $g : A'_n \to K'_n$ *there exists* $x \in A'_n$ *with* $|g(x) - x| \geq \varepsilon$.

REMARK 2. For every convex set W, we have

(1) Property $1'$ $\Rightarrow$ Property 1 $\Rightarrow$ Property 2; and
(2) Property $1'$ $\Rightarrow$ Property $2'$ $\Rightarrow$ Property 2.

Proof. We show only the implication Property 1 $\Rightarrow$ Property 2. (The other implications can be proved in a similar way.) We will follow the argument of [KPR, p. 218]. Assume Property 1. Using the fact that $f(K_n)$ is totally bounded, pick a finite set $D_n \subset K_n$ with $D(K_n, f_n(K_n)) < D(D_n, f_n(K_n)) + 1/n$. Define $K'_n = \mathrm{conv}(D_n)$. Set $A'_n = f_n(K'_n)$. Note that $\bigcup A'_n$ is totally bounded since $A'_n \subset f_n(K_n)$. Let $g : A'_n \to K'_n$ be a mapping. Then, by the Brouwer Fixed-Point Theorem, $g \circ f_n | K'_n$ has a fixed point, say $k \in K'_n$. It follows that $|g(x) - x| \geq \varepsilon$ for $x = f_n(k) \in A'_n$. So, Property 2 holds. ∎

REMARK 3. Properties 1, $1'$, 2, and $2'$ do not depend on the choice of an F-norm on E.

3. Characterizing convex sets via properties

THEOREM 3.1. *Let $E = (E, |\cdot|)$ stand for a m.l.s. and W for a convex subset of E. Every convex* (*resp., convex compact*) *subset of the convex set W without Property* 1 (*resp., without Property* $1'$) *has CFPP* (*resp., FPP*).

Proof. Suppose there exist a convex subset K of W, an $\varepsilon > 0$, and a compact mapping $f : K \to K$ such that $|f(x) - x| \geq \varepsilon$ for all $x \in K$ (and some F-norm $|\cdot|$). Pick a $1/n$-net N_n in $f(K)$ and let $K_n = \mathrm{conv}(N_n)$.

Then $\bigcup f(K_n)$ is totally bounded and $\lim D(K_n, f(K_n)) = 0$. Hence, letting $f_n = f|K_n$, Property 1 follows, a contradiction.

If a compact convex set K fails FPP then, taking f as above and letting $K_n = K$ and $f_n = f$, Property 1′ will hold, a contradiction. ■

THEOREM 3.2. *Let E be an F-space and W a closed convex subset of E.*

1. *W has Property 2 if and only if it contains a convex subset K without CEP.*
2. *W has Property 2′ if and only if it contains a compact convex subset K which is not an AR.*

Moreover, if $\{K'_n\}$ is as required in Property 2 (resp., Property 2′), then

$$K = \operatorname{Lim} K'_{n(k)}$$

for a certain sequence $n(1) < n(2) < \cdots$; here, $\operatorname{Lim} K'_{n(k)}$ stands for the metric limit of the sequence $\{K'_{n(k)}\}$ (see below).

Metric limits. For a sequence of sets $\{B_n\}$ in a metric space (M, ρ), we use Kuratowski's notation [Ku] of $\operatorname{Li} B_n$, $\operatorname{Ls} B_n$, and $\operatorname{Lim} B_n$ for the standard set-topology metric limits. The *lower* and *upper metric limits* are

$$\operatorname{Li} B_n = \{p \in M \mid p = \lim b_n,\ b_n \in B_n\},$$
$$\operatorname{Ls} B_n = \{p = \lim b_{n(k)} \mid b_{n(k)} \in B_{n(k)},\ n(1) < n(2) < \cdots\},$$

respectively. In case $\operatorname{Li} B_n = \operatorname{Ls} B_n$, the *metric limit* is

$$\operatorname{Lim} B_n = \operatorname{Li} B_n = \operatorname{Ls} B_n.$$

For general properties of those limits, see [Ku]. Let us list some:

(1) The limits are closed subsets of M and $\operatorname{Li} B_n \subset \operatorname{Ls} B_n$.
(2) The limits are compact if $\bigcup B_n$ is totally bounded and (M, ρ) is complete.
(3) If M is separable then, for some subsequence $n(1) < n(2) < \cdots$, $\operatorname{Lim} B_{n(k)}$ exists (the generalized Bolzano–Weierstrass theorem).
(4) If each B_n is convex (resp., a linear space) then the limits $\operatorname{Li} B_n$ and $\operatorname{Lim} B_n$ are convex (resp., linear spaces).

Proof of Theorem 3.2. We will only prove item (1); the proof of (2) is similar.

$\Rightarrow$ Since $\bigcup A'_n$ is totally bounded, we can replace the original sets K'_n by their convex subsets so that their union is separable. So, we can assume that E is separable. Hence, for some increasing sequence $(n(k))$,

$$A_0 = \operatorname{Lim} A'_{n(k)} \quad \text{and} \quad K = \operatorname{Lim} K'_{n(k)}$$

exist. It follows that A_0 is a nonempty compactum, K is convex, closed in W, and $A_0 \subset K$, and A_0 is a subset of $A = \operatorname{cl}(\bigcup A'_n)$. Note that A is compact.

We will show that K does not have CEP. Heading for a contradiction, assume K has CEP. Then the inclusion mapping $A \cap K \to K$ extends to a mapping $r : A \to K$. By the uniform continuity of r, for some $\delta > 0$, if $x \in A$ and $d(x, K) < \delta$ then $|r(x) - x| < 1/4$. Since $A_0 \subset K$, there exists $n_0 \in \mathbb{N}$ such that $D(K, A'_{k(n)}) < \delta$ for every $n \geq n_0$. The Klee-admissibility of K (see Introduction) implies that r can be approximated by mappings into finite-dimensional subcompacta of K. Therefore, we may assume that the range of r is a finite-dimensional subcompactum B of K. The following claim yields a desired contradiction:

Claim. *For every $\eta > 0$, there exists $m_0 \in \mathbb{N}$ such that for every $n \geq m_0$ there exists a mapping $\varepsilon : B \to K'_{k(n)}$ such that $|\varepsilon(x) - x| < 2\eta$ for all $x \in B$.*

Pick $n \in \mathbb{N}$ with $n > \max(n_0, m_0)$, where m_0 is chosen for $\eta = 1/8$. For $x \in A'_{k(n)}$, $|\varepsilon \circ r(x) - x| \leq |\varepsilon(r(x)) - r(x)| + |r(x) - x| < 1/4 + 1/4 < 1$. This violates item (iii$'$) of Property 2 applied to $g = \varepsilon \circ r : A'_{k(n)} \to K'_{k(n)}$.

It remains to justify the Claim. Let $p = \dim(B)$. Choose m_0 so that $D(K'_{k(n)}, B) \leq \eta/2(p+1)$ for $n \geq m_0$. Now, for every $b \in B$, pick $k(b) \in K'_{k(n)}$ such that $|b - k(b)| < \eta/(p+1)$. Inscribe an open cover $\mathcal{U} = \{U_1, \ldots, U_l\}$ of order $p+1$ in the cover of B by the balls centered at $b \in B$ of radius $\eta/(p+1)$. Hence, each U_j is contained in a ball centered at $b_j \in B$ of radius $\eta/(p+1)$. Let $\{\lambda_j\}_{j=1}^{l}$ be a partition subordinated to $\mathcal{U}$ and set $\varepsilon(x) = \sum_{j=1}^{l} \lambda_j(x) k(b_j)$.

$\Leftarrow$ By our assumption, there exists a convex set $K \subset W$ and a compactum $A \subset K$ such that the identity on A cannot be approximated by mappings into finite-dimensional convex subsets of K. Let N_n be a finite $1/n$-net in A. Set $K'_n = \mathrm{conv}(N_n)$ and $A'_n = A$. Then $\lim D(K'_n, A'_n) = 0$. If there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that, for every $n \geq n_0$ and every mapping $g : A \to K'_n$, we have $|g(x) - x| \geq \delta$ for some $x = x(n, g) \in A$, then Property 2 holds. Otherwise, for some sequence $k(1) < k(2) < \cdots$, there are mappings $g_{k(n)} : A \to K'_{k(n)}$ with $\lim[\sup\{|g_{k(n)}(x) - x| \mid x \in A\}] = 0$. This contradicts the initial statement. ■

"Dichotomy" Statement. *Let E be an F-space.*

1. *If Property 2 holds, then E contains a convex subset without CEP.*
2. *If Property 2 fails, then every convex subset of E has CFPP.*
3. *If Property 2$'$ holds, then E contains a non-AR compact convex subset.*
4. *If Property 2$'$ fails, then every compact convex subset of E has FPP.*

Obviously, in the above statement, E can be replaced by a closed convex subset W of E.

COROLLARY 3.3. *Property* $2'$ (*also Property* $1'$) *fails in each* F*-space* E *with a separating sequence of continuous linear functionals* (*e.g., in each* ℓ_p, $0 < p < 1$).

Proof. Every convex compactum in such an E is an absolute retract (a so-called Keller cube, see [BP, p. 98] or [BD]). ■

Example 2.1 shows that in ℓ_p, $0 < p < 1/2$, there are sequences that satisfy conditions (i) and (iii) of Property $1'$ (resp., conditions (i$'$) and (iii$'$) of Property $2'$). However, it is not possible that these sequences meet condition (ii) because all convex compacta in ℓ_p are absolute retracts.

4. Remarks. Possibly all nonlocally convex F-spaces have Property 2 because no example of an F-space all of whose convex subsets are absolute retracts is known.

REMARK 4. The implication Property $2' \Rightarrow$ Property 2 in Remark 2 cannot be reversed. According to [DK] there exists an F-space $\tilde{C}$ without CEP and whose convex compacta are absolute retracts. (The space $\tilde{C}$ is a refined version of the famous space C constructed by Cauty [Ca1].)

It is likely that no other implications in Remark 2 can be reversed either.

QUESTION 4.1. *Does* C (*or* $\tilde{C}$) *have Property* 1*?*

Obviously, $\tilde{C}$ fails Property $1'$ because it fails Property $2'$.

REMARK 5. If, in Properties 1, 2, $1'$, or $2'$, every K_n is a linear space then K obtained in Theorem 3.2 is a linear space.

Perhaps condition (ii$'$) in Property $2'$ can be weakened so that its modified variant would imply the existence of a convex set without CEP. Can this be achieved by replacing, in (ii$'$), total boundedness by local total boundedness?

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