On Convex Sets with Convex-Hereditary CEP
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Summary. CEP stands for the compact extension property. We characterize nonlocally convex complete metric linear spaces with convex-hereditary CEP.

1. Introduction. Let $E = (E, |·|)$ stand for a real metric linear space (m.l.s.), whose metric $d$ is determined by an $F$-norm $|·|$ satisfying

$$|x + y| \leq |x| + |y| \quad \text{and} \quad |tx| \leq |x|$$

for $x, y \in E$ and $|t| \leq 1$. If, additionally, $E$ is nonlocally convex, then we cannot assume that either the $F$-norm $|·|$ is homogeneous, or the balls $\{x \in E \mid |x| < \varepsilon\}$ are convex; furthermore, such a space must be infinite-dimensional. A m.l.s. $E$ is referred to as an $F$-space if the norm $|·|$ is complete.

Let $K$ be a convex subset of $E$. Recall that $K$ has the compact extension property (CEP) if every mapping from a closed subset of the Hilbert cube to $K$ extends over the whole cube. $K$ has the compact extension property if and only if, for every compact subset $A$ of $K$, the inclusion mapping of $A \to K$ can be uniformly approximated by mappings with finite-dimensional (equivalently, finite-dimensional convex) ranges (see [Do] and [vBvM]). The latter property is called Klee-admissibility (cf. [K1] and [K2]). If $K$ is compact, then CEP becomes the classical absolute retract property (AR). It is known that a compact $K$ with CEP is homeomorphic to either a finite-dimensional cube, or the Hilbert cube [DT]. For that reason, the following long-standing problem is of great importance:

**Problem.** Does every compact $K$ have CEP?

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This problem is also open when the compactness assumption on $K$ is dropped (in particular, when $K$ is a linear subspace of $E$). Perhaps the pathology goes to the extreme and every nonlocally convex $F$-space contains a convex subset $K$ without CEP. Specifically, it is very intriguing whether every infinite-dimensional linear subspace of the classical $F$-space $\ell_p$, $0 < p < 1$, has CEP. Let us ask:

**Question 1.1.** Is there a nonlocally convex $F$-space all of whose convex subsets (or all linear subspaces) have CEP?

The property required in Question 1.1 will be referred to as **convex-hereditary CEP**. Recall that, in [DvM], a m.l.s. all of whose convex subsets are AR’s has been called a convex-hereditary AR. No nonlocally convex $F$-space which is a convex-hereditary AR is known either. Also it is unknown whether CEP and the absolute retract property coincide for $F$-spaces. However, there exist m.l.s. that have CEP but are not AR’s (see [vBvM] and [DK]); those spaces are incomplete and have a complex Borel structure. Clearly, the notion of convex-hereditary CEP can be formally extended from $F$-spaces to convex sets in m.l.s. The aim of this note is to characterize convex sets that have convex-hereditary CEP.

Independently, Le Hoang Tri and Nguyen Hoang Thanh have publicized similar work in [TT].

2. **Properties.** In addition to CEP, we will also consider the following fixed-point properties. We say that a convex set $K$ has the **compact fixed-point property** (CFPP) if every compact mapping $K \to K$ has a fixed point. Here, and everywhere in the paper, $f : X \to Y$ is compact if the closure of $f(X)$ is a compact subset of $Y$. If $K$ is compact, CFPP becomes the classical **fixed-point property** (FPP). In [Ca3], Cauty claims that every $K$ has CFPP; however, his previous proof of this claim [Ca2] contains a fatal error: see [DS].

**Notation.** Let $E = (E, \| \cdot \|)$ be a m.l.s. and $W$ be a convex subset of $E$. For $A \subseteq E$ and $x \in E$, let $d(A, x) = \inf\{|a - x| \mid a \in A\}$. Following the notation in [KPR], for $B \subseteq E$, we write

$$D(A, B) = \sup\{d(A, x) \mid x \in B\}.$$ 

Note that, for $\varepsilon > 0$, we have $D(A, B) \leq \varepsilon$ if and only if, for every $x \in B$ and every $\delta > 0$, there exists $a \in A$ such that $|a - x| \leq \varepsilon + \delta$. (So, the Hausdorff metric is $H(A, B) = \max\{D(A, B), D(B, A)\}$.)

The properties introduced below describe intrinsically the convex set $W$; the case of $W = E$ is our main interest.
2.1. Properties related to CEP and CFPP

**Property 1.** There exist an $\varepsilon > 0$, a sequence $\{K_n\}$ of convex subsets of $W$, and a sequence $\{f_n\}$ of mappings, $f_n : K_n \to E$, satisfying

(i) $\lim D(K_n, f_n(K_n)) = 0$;
(ii) $\bigcup f_n(K_n)$ is totally bounded; and
(iii) $|f_n(x) - x| \geq \varepsilon$ for every $x \in K_n$ and $n \in \mathbb{N}$.

**Property 2.** There exist an $\varepsilon > 0$, a sequence $\{K'_n\}$ of convex subsets of $W$, and a sequence $\{A'_n\}$ of subsets of $E$ satisfying

(i') $\lim D(K'_n, A'_n) = 0$;
(ii') $\bigcup A'_n$ is totally bounded; and
(iii') for every $n \in \mathbb{N}$ and every mapping $g : A'_n \to K'_n$ there exists $x \in A'_n$ with $|g(x) - x| \geq \varepsilon$.

The following example shows that conditions (i) and (iii) do not contradict each other in the nonlocally convex case (see [TT]).

**Example 2.1.** In every $\ell_p$ space, $0 < p \leq 1/2$, there exists a sequence of mappings $\{f_n\}$ : $f_n : \Delta_n \to \ell_p$, such that

(i) $\lim D(\Delta_n, f_n(\Delta_n)) = 0$; and
(ii) $|f_n(x) - x|_p \geq 1$ for every $x \in \Delta_n$ and $n \in \mathbb{N}$.

Here, $\Delta_n$ is the standard simplex spanned by the unit vectors $e_1, \ldots, e_n$, that is, $\Delta_n = \text{conv}(\{e_1, \ldots, e_n\}) \subset \ell_p$.

**Proof.** Let $x_n = n^{-1/p}(e_1 + \cdots + e_n)$. Define $f_n(x) = x + x_n$, $x \in \Delta_n$. Since $|x_n|_p = 1$, we have $|f_n(x) - x|_p = 1$ for every $x \in \Delta_n$.

Let $x = \sum_{i=1}^n t_i e_i$, where $0 \leq t_i \leq 1$, $t_n \leq t_{n-1} \leq \cdots \leq t_1$, and $\sum_{i=1}^n t_i = 1$. Hence $t_1 \geq 1/n$. Thus $x + x_n = \sum_{i=1}^n (t_i + 1/n^{1/p}) e_i$. Let $x' = (1 - \sum_{i=2}^n (t_i + 1/n^{1/p})) e_1 + \sum_{i=2}^n (t_i + 1/n^{1/p}) e_i$. Then $x' \in \Delta_n$ since $\sum_{i=2}^n (t_i + 1/n^{1/p}) = 1 - t_1 + (n-1)/n^{1/p} \leq 1 + (n-1)/n^{1/p} - 1/n \leq 1$. Now $|f_n(x) - x'|_p = |x + x_n - x'|_p = |(1/n^{1/p} + (n-1)/n^{1/p}, 0, 0, \ldots)|_p = n^p/n$. Finally, $\lim_{n \to \infty} n^p/n = 0$.

It seems likely that a similar argument will work for every $\ell_p$ with $1/2 < p < 1$.

**Remark 1.** Example 2.1 satisfies items (i) and (iii) of Property 1 for $W = \ell_p$ (or for the convex set $W = \bigcup \Delta_n$). Yet, item (ii) is not satisfied. Namely, the sequence $\{x_n\}$ is discrete, and consequently the sequence $\{e_1 + x_n\} \subset \bigcup f_n(\Delta_n)$ has no convergent subsequence.

**Question 2.2.** Is there $0 < p < 1$ such that $\ell_p$ has Property 1 (or Property 2)?
Later, we will show that if such a $p$ existed then the space $\ell_p$ could not serve as an example required in Question 1.1; see item (1) of the “Dichotomy” Statement below.

2.2. Properties related to AR and FPP for compacta

**Property 1**. There exist an $\varepsilon > 0$, a sequence $\{K_n\}$ of convex subsets of $W$, and a sequence of mappings $\{f_n\}$, $f_n : K_n \to E$, satisfying

(i) $\lim D(K_n, f_n(K_n)) = 0$;
(ii) $\bigcup K_n$ is totally bounded; and
(iii) $|f_n(x) - x| \geq \varepsilon$ for every $x \in K_n$ and $n \in \mathbb{N}$.

**Property 2**. There exist a sequence $\{K'_n\}$ of convex subsets of $W$ and a sequence $\{A'_n\}$ of subsets of $E$ satisfying

(i') $\lim D(K'_n, A'_n) = 0$;
(ii') $\bigcup K'_n$ is totally bounded; and
(iii') for every $n \in \mathbb{N}$ and every mapping $g : A'_n \to K'_n$ there exists $x \in A'_n$ with $|g(x) - x| \geq \varepsilon$.

**Remark 2.** For every convex set $W$, we have

(1) Property 1 $\Rightarrow$ Property 1 $\Rightarrow$ Property 2; and
(2) Property 1 $\Rightarrow$ Property 2 $\Rightarrow$ Property 2.

**Proof.** We show only the implication Property 1 $\Rightarrow$ Property 2. (The other implications can be proved in a similar way.) We will follow the argument of [KPR, p. 218]. Assume Property 1. Using the fact that $f(K_n)$ is totally bounded, pick a finite set $D_n \subset K_n$ with $D(K_n, f_n(K_n)) < D(D_n, f_n(K_n)) + 1/n$. Define $K'_n = \text{conv}(D_n)$. Set $A'_n = f_n(K'_n)$. Note that $\bigcup A'_n$ is totally bounded since $A'_n \subset f_n(K_n)$. Let $g : A'_n \to K'_n$ be a mapping. Then, by the Brouwer Fixed-Point Theorem, $g \circ f_n|K'_n$ has a fixed point, say $k \in K'_n$. It follows that $|g(x) - x| \geq \varepsilon$ for $x = f_n(k) \in A'_n$. So, Property 2 holds.

**Remark 3.** Properties 1, 1', 2, and 2' do not depend on the choice of an $F$-norm on $E$.

3. Characterizing convex sets via properties

**Theorem 3.1.** Let $E = (E, |\cdot|)$ stand for a m.l.s. and $W$ for a convex subset of $E$. Every convex (resp., convex compact) subset of the convex set $W$ without Property 1 (resp., without Property 1') has CFPP (resp., FPP).

**Proof.** Suppose there exist a convex subset $K$ of $W$, an $\varepsilon > 0$, and a compact mapping $f : K \to K$ such that $|f(x) - x| \geq \varepsilon$ for all $x \in K$ (and some $F$-norm $|\cdot|$). Pick a $1/n$-net $N_n$ in $f(K)$ and let $K_n = \text{conv}(N_n)$. 

Then $\bigcup f(K_n)$ is totally bounded and $\lim D(K_n, f(K_n)) = 0$. Hence, letting $f_n = f|K_n$, Property 1 follows, a contradiction.

If a compact convex set $K$ fails FPP then, taking $f$ as above and letting $K_n = K$ and $f_n = f$, Property 1’ will hold, a contradiction.

**Theorem 3.2.** Let $E$ be an $F$-space and $W$ a closed convex subset of $E$.

1. $W$ has Property 2 if and only if it contains a convex subset $K$ without CEP.
2. $W$ has Property 2’ if and only if it contains a compact convex subset $K$ which is not an AR.

Moreover, if $\{K'_n\}$ is as required in Property 2 (resp., Property 2’), then

$$K = \lim K'_{n(k)}$$

for a certain sequence $n(1) < n(2) < \cdots$; here, $\lim K'_{n(k)}$ stands for the metric limit of the sequence $\{K'_{n(k)}\}$ (see below).

**Metric limits.** For a sequence of sets $\{B_n\}$ in a metric space $(M, \rho)$, we use Kuratowski’s notation $[\text{Ku}]$ of $\text{Li} B_n$, $\text{Ls} B_n$, and $\lim B_n$ for the standard set-topology metric limits. The lower and upper metric limits are

$$\text{Li} B_n = \{ p \in M \mid p = \lim b_n, b_n \in B_n \};$$
$$\text{Ls} B_n = \{ p = \lim b_{n(k)} \mid b_{n(k)} \in B_{n(k)}, n(1) < n(2) < \cdots \},$$

respectively. In case $\text{Li} B_n = \text{Ls} B_n$, the metric limit is

$$\lim B_n = \text{Li} B_n = \text{Ls} B_n.$$

For general properties of those limits, see $[\text{Ku}]$. Let us list some:

1. The limits are closed subsets of $M$ and $\text{Li} B_n \subset \text{Ls} B_n$.
2. The limits are compact if $\bigcup B_n$ is totally bounded and $(M, \rho)$ is complete.
3. If $M$ is separable then, for some subsequence $n(1) < n(2) < \cdots$, $\lim B_{n(k)}$ exists (the generalized Bolzano–Weierstrass theorem).
4. If each $B_n$ is convex (resp., a linear space) then the limits $\text{Li} B_n$ and $\lim B_n$ are convex (resp., linear spaces).

**Proof of Theorem 3.2.** We will only prove item (1); the proof of (2) is similar.

$\Rightarrow$ Since $\bigcup A'_n$ is totally bounded, we can replace the original sets $K'_n$ by their convex subsets so that their union is separable. So, we can assume that $E$ is separable. Hence, for some increasing sequence $(n(k))$,

$$A_0 = \lim A'_{n(k)} \quad \text{and} \quad K = \lim K'_{n(k)}$$

exist. It follows that $A_0$ is a nonempty compactum, $K$ is convex, closed in $W$, and $A_0 \subset K$, and $A_0$ is a subset of $A = \text{cl}(\bigcup A'_n)$. Note that $A$ is compact.
We will show that $K$ does not have CEP. Heading for a contradiction, assume $K$ has CEP. Then the inclusion mapping $A \cap K \to K$ extends to a mapping $r : A \to K$. By the uniform continuity of $r$, for some $\delta > 0$, if $x \in A$ and $d(x, K) < \delta$ then $|r(x) - x| < 1/4$. Since $A_0 \subset K$, there exists $n_0 \in \mathbb{N}$ such that $D(K, A'_{k(n)}) < \delta$ for every $n \geq n_0$. The Klee-admissibility of $K$ (see Introduction) implies that $r$ can be approximated by mappings into finite-dimensional subcompacta of $K$. Therefore, we may assume that the range of $r$ is a finite-dimensional subcompactum $B$ of $K$. The following claim yields a desired contradiction:

**Claim.** For every $\eta > 0$, there exists $m_0 \in \mathbb{N}$ such that for every $n \geq m_0$ there exists a mapping $\varepsilon : B \to K'_{k(n)}$ such that $|\varepsilon(x) - x| < 2\eta$ for all $x \in B$.

Pick $n \in \mathbb{N}$ with $n > \max(n_0, m_0)$, where $m_0$ is chosen for $\eta = 1/8$. For $x \in A'_{k(n)}$, $|\varepsilon \circ r(x) - x| \leq |\varepsilon(r(x)) - r(x)| + |r(x) - x| < 1/4 + 1/4 < 1$. This violates item (iii') of Property 2 applied to $g = \varepsilon \circ r : A'_{k(n)} \to K'_{k(n)}$.

It remains to justify the Claim. Let $p = \dim(B)$. Choose $m_0$ so that $D(K'_{k(n)}, B) \leq \eta/2(p+1)$ for $n \geq m_0$. Now, for every $b \in B$, pick $k(b) \in K'_{k(n)}$ such that $|b - k(b)| < \eta/(p+1)$. Inscribe an open cover $\mathcal{U} = \{U_1, \ldots, U_l\}$ of order $p+1$ in the cover of $B$ by the balls centered at $b \in B$ of radius $\eta/(p+1)$. Hence, each $U_j$ is contained in a ball centered at $b_j \in B$ of radius $\eta/(p+1)$. Let $\{\lambda_j\}_{j=1}^l$ be a partition subordinated to $\mathcal{U}$ and set $\varepsilon(x) = \sum_{j=1}^l \lambda_j(x) k(b_j)$.

By our assumption, there exists a convex set $K \subset W$ and a compactum $A \subset K$ such that the identity on $A$ cannot be approximated by mappings into finite-dimensional convex subsets of $K$. Let $N_n$ be a finite $1/n$-net in $A$. Set $K'_n = \text{conv}(N_n)$ and $A'_n = A$. Then $\lim D(K'_n, A'_n) = 0$. If there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that, for every $n \geq n_0$ and every mapping $g : A \to K'_n$, we have $|g(x) - x| \geq \delta$ for some $x = x(n, g) \in A$, then Property 2 holds. Otherwise, for some sequence $k(1) < k(2) < \cdots$, there are mappings $g_{k(n)} : A \to K'_{k(n)}$ with $\lim \sup \{|g_{k(n)}(x) - x| \mid x \in A\} = 0$. This contradicts the initial statement.

**“Dichotomy” Statement.** Let $E$ be an $F$-space.

1. If Property 2 holds, then $E$ contains a convex subset without CEP.
2. If Property 2 fails, then every convex subset of $E$ has CFPP.
3. If Property 2′ holds, then $E$ contains a non-AR compact convex subset.
4. If Property 2′ fails, then every compact convex subset of $E$ has FPP.

Obviously, in the above statement, $E$ can be replaced by a closed convex subset $W$ of $E$. 
Corollary 3.3. Property 2′ (also Property 1′) fails in each F-space E with a separating sequence of continuous linear functionals (e.g., in each $\ell_p$, $0 < p < 1$).

Proof. Every convex compactum in such an E is an absolute retract (a so-called Keller cube, see [BP, p. 98] or [BD]).

Example 2.1 shows that in $\ell_p$, $0 < p < 1/2$, there are sequences that satisfy conditions (i) and (iii) of Property 1′ (resp., conditions (i′) and (iii′) of Property 2′). However, it is not possible that these sequences meet condition (ii) because all convex compacta in $\ell_p$ are absolute retracts.

4. Remarks. Possibly all nonlocally convex F-spaces have Property 2 because no example of an F-space all of whose convex subsets are absolute retracts is known.

Remark 4. The implication Property 2′ $\Rightarrow$ Property 2 in Remark 2 cannot be reversed. According to [DK] there exists an F-space $\tilde{C}$ without CEP and whose convex compacta are absolute retracts. (The space $\tilde{C}$ is a refined version of the famous space $C$ constructed by Cauty [Ca1].)

It is likely that no other implications in Remark 2 can be reversed either.

Question 4.1. Does C (or $\tilde{C}$) have Property 1?

Obviously, $\tilde{C}$ fails Property 1′ because it fails Property 2′.

Remark 5. If, in Properties 1, 2, 1′, or 2′, every $K_n$ is a linear space then $K$ obtained in Theorem 3.2 is a linear space.

Perhaps condition (ii′) in Property 2′ can be weakened so that its modified variant would imply the existence of a convex set without CEP. Can this be achieved by replacing, in (ii′), total boundedness by local total boundedness?

References


