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PROBABILITY THEORY AND STOCHASTIC PROCESSES

A Weak-Type Inequality for Orthogonal Submartingales and Subharmonic Functions

by

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Summary. Let X be a submartingale starting from 0, and Y be a semimartingale which is orthogonal and strongly differentially subordinate to X. The paper contains the proof of the sharp estimate

$$\mathbb{P}\Big(\sup_{t\geq 0}|Y_t|\geq 1\Big)\leq 3.375\ldots\|X\|_1.$$

As an application, a related weak-type inequality for smooth functions on Euclidean domains is established.

1. Introduction. A celebrated theorem of Kolmogorov [K] states that if $\tilde{f}$ is the conjugate function of an integrable real-valued function f on the unit circle $\mathbb{T}$, then

$$|\{\zeta\in\mathbb{T}:|\tilde{f}(\zeta)|\geq 1\}|\leq K\|f\|_1$$

for some absolute constant K. The optimal value of this constant was determined by Davis [D]:

$$K=\frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots}{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots}=1.347\ldots. \tag{1.1}$$

We shall study a certain probabilistic version of this result. Assume that $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a non-decreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X=(X_t)_{t\geq 0}$ and $Y=(Y_t)_{t\geq 0}$ be adapted real-valued cadlag semimartingales. The symbol $[X,Y]$ will denote the quadratic covariance

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process (the square bracket) of X and Y (see e.g. Dellacherie and Meyer [DM] for details). Following Bañuelos and Wang [BW1] and Wang [W] (see also Burkholder [B1]), we say that Y is *differentially subordinate* to X if the process $([X,X]_t-[Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. We say that the semimartingales X and Y are *orthogonal* if their square bracket $[X,Y]$ is constant with probability 1.

The following statement was established by Bañuelos and Wang in [BW2] (see also Choi [C]). Here and below, we use the notation $X^*=\sup_{t\geq 0}|X_t|$ and $\|X\|_1=\sup_{t\geq 0}\|X_t\|_1$.

THEOREM 1.1. *Assume that X, Y are orthogonal martingales such that Y is differentially subordinate to X. Then*

$$\mathbb{P}(Y^*\geq 1)\leq K\|X\|_1, \tag{1.2}$$

where K is given by (1.1). *The inequality is sharp.*

The objective of this paper is to study the extension of Theorem 1.1 to a wider setting in which the dominating process X is a submartingale and Y is a semimartingale. It turns out that the differential subordination is too weak in this new setting and must be strengthened so that the finite variation parts of the processes can be controlled. We will work under the condition of so-called *strong differential subordination*, introduced by Wang in [W] (consult Burkholder [B2] for the discrete-time case). The definition is the following: Let X be a submartingale, Y be a semimartingale with the Doob–Meyer decompositions

$$X=X_0+M+A,\qquad Y=Y_0+N+C, \tag{1.3}$$

where M, N are local martingales and A, C are finite variation processes. In general the decompositions may not be unique; however, we assume that A is predictable and this determines the first of them. We say that Y is *strongly subordinate* to X if Y is differentially subordinate to X and there is a decomposition (1.3) for Y such that the process $(A_t-|C|_t)_{t\geq 0}$ is nondecreasing as a function of t. Here $|C|_t$ denotes the total variation of C on the interval $[0,t]$. To give an example, let $B=(B^{(1)},B^{(2)})$ be a Brownian motion in $\mathbb{R}^2$ and consider the Itô processes

$$\begin{aligned} X_t&=X_0+\int_{0+}^{t}\phi_s\,dB_s^{(1)}+\int_{0+}^{t}\psi_s\,ds,\\ Y_t&=Y_0+\int_{0+}^{t}\zeta_s\,dB_s^{(2)}+\int_{0+}^{t}\xi_s\,ds, \end{aligned} \tag{1.4}$$

where $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$, $(\zeta_t)_{t\geq 0}$, $(\xi_t)_{t\geq 0}$ are predictable and satisfy the usual assumptions

$$\mathbb{P}\Big(\int_{0+}^{t} |\phi_s|^2\, ds < \infty \text{ and } \int_{0+}^{t} |\psi_s|\, ds < \infty \text{ for all } t > 0\Big) = 1,$$

$$\mathbb{P}\Big(\int_{0+}^{t} |\zeta_s|^2\, ds < \infty \text{ and } \int_{0+}^{t} |\xi_s|\, ds < \infty \text{ for all } t > 0\Big) = 1.$$

Then whenever $|Y_0| \le |X_0|$, $|\phi_s| \ge |\zeta_s|$ and $\psi_s \ge |\xi_s|$ for all s, then X, Y are orthogonal and Y is strongly differentially subordinate to X. Indeed, the orthogonality is clear and if we set $A_t = \int_{0+}^{t} \psi_s\, ds$, $C_t = \int_{0+}^{t} \xi_s\, ds$ for $t \ge 0$, then both differences

$$[X,X]_t - [Y,Y]_t = X_0^2 - Y_0^2 + \int_{0+}^{t} (|\phi_s|^2 - |\zeta_s|^2)\, ds,$$

$$A_t - |C|_t = \int_{0+}^{t} (\psi_s - |\xi_s|)\, ds$$

are nonnegative and nondecreasing as functions of t.

Throughout the paper, we assume that the semimartingales X, Y start from 0. Our main result is the following. Let $c = 3.375\ldots$ be given by (2.2) below.

THEOREM 1.2. *Assume that X is a submartingale such that $X_0 = 0$ and Y is a semimartingale which is orthogonal and strongly differentially subordinate to X. Then*

$$\mathbb{P}(Y^* \ge 1) \le c\|X\|_1 \tag{1.5}$$

and the constant c is the best possible, even in the setting of Itô processes (1.4) *described above.*

This should be compared to the following result of Hammack [H] concerning similar bounds, but without the orthogonality assumption.

THEOREM 1.3. *Assume that X is a submartingale such that $X_0 = 0$ and Y is a semimartingale which is strongly differentially subordinate to X. Then*

$$\mathbb{P}(Y^* \ge 1) \le 4\|X\|_1$$

and the constant 4 *is the best possible.*

The paper is organized as follows. We shall use Burkholder's method and construct first a certain special superharmonic function on $\mathbb{R}^2$: this is done in the next section. In Section 3 we exploit the properties of this function to obtain (1.5); furthermore, we construct appropriate examples showing the optimality of the constant c. The final part of the paper is devoted to an application of Theorem 1.2: we obtain a certain weak-type estimate for smooth functions on Euclidean domains, which can be regarded as a generalization of Kolmogorov's estimate.

2. A special function. The following statement is the main result of this section. Let $x^+ = \max\{x, 0\}$ stand for the positive part of the real number x and let $c = 3.375\ldots$ be given by (2.2).

THEOREM 2.1. *There is a function $U : \mathbb{R}^2 \to \mathbb{R}$ such that*

(i) *U is continuous and superharmonic,*
(ii) *for any $(x, y) \in \mathbb{R}^2$, the functions $t \mapsto U(x + t, y + t)$ and $t \mapsto U(x + t, y - t)$ are nonincreasing on $\mathbb{R}$,*
(iii) *U is concave along the horizontal lines,*
(iv) *$U(0, 0) = -c^{-1}$,*
(v) *$U(x, y) \geq U(0, 0)1_{\{|y|<1\}} - x^+$ for all $x, y \in \mathbb{R}$.*

To prove this theorem, we need to introduce some auxiliary objects. Assume that $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc of $\mathbb{C}$ and let

$$J = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq 1,\ |\operatorname{Im} z| \leq \operatorname{Re} z\},$$
$$K = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1 \text{ or } |\operatorname{Im} z| \leq 1\}.$$

Define $h : \partial K \to \mathbb{R}$ by setting $h(x, \pm 1) = 1 - |x|$ if $|x| \geq 1$ and $h(\pm 1, y) = |y| - 1$ if $|y| \geq 1$. Consider the function F given by

$$F(z) = \alpha \int_0^z \frac{\sqrt{1 - w^4}}{1 + w^4}\, dw,$$

where

$$\alpha = \sqrt{2i}\biggl(\int_0^1 \frac{\sqrt{1 - t^4}}{1 + t^4}\, dt\biggr)^{-1}.$$

Let us gather some information on F. This function is a conformal mapping of D onto the interior of K (see Tomaszewski [T]). Furthermore, F sends the arcs $\{e^{i\theta} : |\theta| < \pi/4\}$, $\{e^{i\theta} : \theta \in (\pi/4, 3\pi/4)\}$, $\{e^{i\theta} : \theta \in (3\pi/4, 5\pi/4)\}$ and $\{e^{i\theta} : \theta \in (5\pi/4, 7\pi/4)\}$ onto the sets $\partial K \cap (0, \infty)^2$, $\partial K \cap (-\infty, 0) \times (0, \infty)$, $\partial K \cap (-\infty, 0)^2$ and $\partial K \cap (0, \infty) \times (-\infty, 0)$, respectively; finally, we have $F(e^{\pm \pi i/4}) = F(e^{\pm 3\pi i/4}) = \infty$. Let G be the inverse of F and define $u : D \to \mathbb{R}$ by the Poisson integral

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} h(F(e^{it}))\, dt$$

for $r \in [0, 1)$ and $\theta \in [0, 2\pi)$ (this is well defined, see (2.4) below). To describe the optimal constant c in (1.5), let $R = 0.541\ldots$ be the unique solution to the equation

$$\int_0^R \frac{\sqrt{1+s^4}}{1-s^4}\,ds = \frac{\sqrt{2}}{2}\int_0^1 \frac{\sqrt{1-s^4}}{1+s^4}\,ds \tag{2.1}$$

and put

$$c = -\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{1-R^2}{1-2R\cos(t+\pi/4)+R^2} h(F(e^{it}))\,dt\right]^{-1}. \tag{2.2}$$

Computer simulations show that $c = 3.375\ldots$. Finally, let

$$\mathcal{U}(x,y) = u(G(x,y)) \quad \text{for } (x,y)\in K.$$

LEMMA 2.2. *The function* $\mathcal{U}$ *enjoys the following properties:*

(i) *It is continuous on* K *and harmonic in the interior of* K.
(ii) *The function* $(x,y)\mapsto \mathcal{U}(x,y)+x$ *is bounded on* J.
(iii) U *has the following symmetry: if* $(x,y)\in K$, *then*

$$U(x,y) = U(x,-y) = U(-x,y) \quad \textit{and} \quad U(x,y) = -U(y,x).$$

Proof. (i) It is obvious that $\mathcal{U}$ is harmonic in the interior of K, since it is the real part of an analytic function there. To see that $\mathcal{U}$ is continuous on K, observe that the function $t\mapsto h(F(e^{it}))$ is continuous on $[-\pi,\pi]\setminus\{\pm\pi/4,\pm 3\pi/4\}$. This implies that u is continuous on $\overline{D}\setminus\{e^{\pm\pi i/4}, e^{\pm 3\pi i/4}\}$ and the latter set is precisely $G(K)$.

(ii) First we prove the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} F(e^{it})\,dt = F(re^{i\theta}) \tag{2.3}$$

for $r\in[0,1)$, $\theta\in[0,2\pi)$. To do this, apply Fubini's theorem to obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} F(e^{it})\,dt \\
&\qquad = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\cdot\alpha\int_0^{e^{it}} \frac{\sqrt{1-z^4}}{1+z^4}\,dz\,dt \\
&\qquad = \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\cdot\alpha\frac{\sqrt{1-s^4e^{4it}}}{1+s^4e^{4it}} e^{it}\,dt\,ds.
\end{aligned}$$

For any fixed $s\in[0,1)$, the expression under the outer integral is the Poisson formula for the function

$$f_s : z\mapsto \alpha z\frac{\sqrt{1-s^4z^4}}{1+s^4z^4},$$

which is continuous on $\overline{D}$ and analytic on D. Thus, this expression equals $f_s(re^{i\theta})$ and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}F(e^{it})\,dt=\alpha\int_0^1\frac{\sqrt{1-s^4r^4e^{4it}}}{1+s^4r^4e^{4it}}re^{it}\,ds=F(re^{i\theta}).$$

To see that the above use of Fubini's theorem is permitted, note that for any fixed r and θ as above and some positive κ_1, κ_2,

$$\frac{\alpha}{2\pi}\int_0^{2\pi}\int_0^1\left|\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\,\frac{\sqrt{1-s^4e^{4it}}}{1+s^4e^{4it}}e^{it}\right|ds\,dt \tag{2.4}$$
$$\le\kappa_1\int_0^{2\pi}\int_0^1\frac{1}{|1+s^4e^{4it}|}\,ds\,dt\le\kappa_2\int_0^{2\pi}\int_0^1\frac{1}{|1+se^{4it}|}\,ds\,dt$$
$$=8\kappa_2\int_0^{\pi/4}\int_0^1\frac{1}{\sqrt{\left(\frac{s+\cos 4t}{\sin 4t}\right)^2+1}}\,\frac{ds\,dt}{\sin 4t}=2\kappa_2\int_0^{\pi}\log\left(1+\frac{1}{\cos(t/2)}\right)dt<\infty.$$

Therefore, if $(x,y)\in J$ and $G(x,y)=re^{i\theta}$, then we may write

$$\begin{aligned}\mathcal{U}(x,y)+x-1&\\ &=u(G(x,y))+\operatorname{Re}F(G(x,y))-1\\ &=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}[h(F(e^{it}))+\operatorname{Re}F(e^{it})-1]\,dt\\ &=\frac{1}{2\pi}\int_0^{3\pi/2}\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}[h(F(e^{it}))+\operatorname{Re}F(e^{it})-1]\,dt,\end{aligned}$$

because $h(x,y)+x-1=0$ for $(x,y)\in\partial K\cap\{z\in\mathbb{C}:\operatorname{Re}z>1\}$. Now if we take $(x,y)\in J$ with x sufficiently large, say, $x\ge 2$, then θ lies in a proper closed subinterval of $(-\pi/2,0)$ and thus the Poisson kernel is bounded uniformly in $r\in[0,1)$ and $t\in[0,3\pi/2]$. It suffices to note that by (2.4),

$$\int_0^{3\pi/2}|h(F(e^{it}))+\operatorname{Re}F(e^{it})-1|\,dt<\infty.$$

(iii) We have $F(iz)=iF(z)$ for any $z\in D$, so $G(iz)=iG(z)$ for all $z\in K$; that is, if $(x,y)\in K$, then $G(-y,x)=iG(x,y)$. Consequently, if $G(x,y)=re^{i\theta}$, then

$$\begin{aligned}\mathcal{U}(-y,x)&=u(iG(x,y))=u(re^{i(\pi/2+\theta)})\\ &=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\pi/2-\theta)+r^2}h(F(e^{it}))\,dt\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} h(F(ie^{it}))\,dt$$
$$= \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} h(iF(e^{it}))\,dt,$$

which is equal to $-\mathcal{U}(x,y)$, since $h(iz) = -h(z)$ for $z \in \partial K$. Therefore,

$$\mathcal{U}(x,y) = -\mathcal{U}(-y,x) = \mathcal{U}(x,y) \tag{2.5}$$

and it remains to show that $\mathcal{U}(x,y) = \mathcal{U}(x,-y)$ for $(x,y) \in J$. But this follows from the previous part: the function $(x,y) \mapsto \mathcal{U}(x,y) - \mathcal{U}(x,-y)$ is continuous and bounded on J, harmonic in the interior of this set and vanishes at its boundary. Thus it is identically 0. ■

Further properties of $\mathcal{U}$ are given in the following lemma.

LEMMA 2.3. *If $(x,y) \in J$, then*

$$|y| - x \le \mathcal{U}(x,y) \le \min\{1-x, 0\}, \tag{2.6}$$

$$\mathcal{U}_x(x,y) \pm \mathcal{U}_y(x,y) \le 0, \tag{2.7}$$

$$\mathcal{U}_{xx} \le 0. \tag{2.8}$$

Proof. Note that the function $(x,y) \mapsto |y| - x$ is subharmonic and agrees with $\mathcal{U}$ at the boundary of J. Combining this with Lemma 2.2(ii), we obtain the lower bound in (2.6). To show the upper bound, note that the functions $(x,y) \mapsto 1-x$, $(x,y) \mapsto 0$ are harmonic and majorize $\mathcal{U}$ at ∂J; it remains to apply Lemma 2.2(ii) to get (2.6).

We turn to (2.7). By the symmetry of $\mathcal{U}$ (see part (iii) of the previous lemma), it suffices to show that $\mathcal{U}_x(x,y) + \mathcal{U}_y(x,y) \le 0$. Using the Schwarz reflection principle, we can extend $\mathcal{U}$ to a function continuous on $J \cup (J+2i)$ and harmonic inside this set (here we have used the usual notation $J + w = \{z \in \mathbb{C} : z - w \in J\}$ for $w \in \mathbb{C}$). The extension is given by

$$\mathcal{U}(x,y) = 2 - 2x - \mathcal{U}(x, 2-y)$$

for $(x,y) \in J + 2i$. Fix $k \in (0,1)$ and define the function V on J by

$$V(x,y) = \mathcal{U}(x,y) - \mathcal{U}(x+k, y+k).$$

Using (2.6) several times, one can show that V is nonnegative at the boundary of J. Indeed: if $y = -x \in [0,1]$, then $\mathcal{U}(x,y) = 0$ and $\mathcal{U}(x+k,y+k) \le 0$; if $y = x \in [0, 1-k]$, then $\mathcal{U}(x,y) = \mathcal{U}(x+k,y+k) = 0$; if $y = x \in (1-k, 1]$, then $\mathcal{U}(x,y) = 0$ and

$$\begin{aligned}\mathcal{U}(x+k,y+k) &= 2 - 2x - 2k - \mathcal{U}(x+k, 2-y-k)\\ &\le 2 - 2x - 2k - (2 - y - k - x - k) = 0.\end{aligned}$$

Next, $\mathcal{U}(x,-1)=1-x>1-x-k\geq\mathcal{U}(x+k,-1+k)$; finally, $U(x,1)=1-x$ and $\mathcal{U}(x+k,1+k)=2-2k-2x-\mathcal{U}(x+k,1-k)\leq 1-x$. Furthermore, by Lemma 2.2, V is continuous and bounded on J and harmonic in the interior of J. Consequently, $V\geq 0$ and since $k\in(0,1)$ was arbitrary, (2.7) follows.

To prove the concavity property (2.8), we proceed similarly. Fix $k\in(0,1)$ and consider the function $V:J\to\mathbb{R}$ given by

$$V(x,y)=2\mathcal{U}(x,y)-\mathcal{U}(x+k,y)-\mathcal{U}(x-k,y).$$

It suffices to prove that $V\geq 0$ on the boundary of J. Clearly, this is true on $[1+k,\infty)\times\{-1,1\}$ ($V\equiv 0$ there). For $x\in(1,1+k)$, using the symmetry of $\mathcal{U}$ and the lower bound from (2.6), we get

$$\mathcal{U}(x-k,\pm 1)=-\mathcal{U}(1,x-k)\leq 1-(x-k),$$

so $V(x,\pm 1)\geq 0$. Similarly, by the symmetry of $\mathcal{U}$ we get, for $x\in[0,1]$,

$$V(x,\pm x)=-\mathcal{U}(x+k,x)-\mathcal{U}(x-k,x)=-\mathcal{U}(x+k,x)+\mathcal{U}(x,x-k)\geq 0,$$

in virtue of (2.7). This completes the proof. ∎

We turn to the main result of this section.

Proof of Theorem 2.1. Let $U:\mathbb{R}^2\to\mathbb{R}$ be given by

$$U(x,y)=\begin{cases}\mathcal{U}(x+1,y) & \text{if } (x+1,y)\in J,\\ 0 & \text{if } |y|\leq 1 \text{ and } (x+1,y)\notin J,\\ -x^+ & \text{if } |y|>1.\end{cases}$$

Let us verify the stated properties of U.

(i) The continuity is straightforward. In addition, U is harmonic in the interior of $J-1$ and is majorized on $\mathbb{R}^2$ by the superharmonic function $(x,y)\mapsto -x^+$ (see (2.6)). Hence the mean-value inequality is satisfied and U is superharmonic.

(ii) Clearly, we have the monotonicity outside the set $J-1$; thus the property follows from continuity of U and (2.7).

(iii) The concavity is evident for $|y|\geq 1$. When $|y|<1$, use (2.8) and the estimate $U(x,y)\leq 0$ on $J-1$ (see the upper bound in (2.6)).

(iv) This follows immediately from the equality $F(Re^{-i\pi/4})=1$, or $G(1,0)=Re^{-i\pi/4}$ (recall that R is given by (2.1)).

(v) Both sides of the estimate are equal when $|y|\geq 1$, so let us assume that $y\in(-1,1)$. Then the majorization takes the form

$$U(x,y)\geq U(0,0)-x^+.$$

This is clear when $(x,y)\notin J-1$: the left hand side equals 0 and the right is $U(0,0)=\mathcal{U}(1,0)\leq 0$ (see (2.6)). Next, suppose that $(x,y)\in J-1$. Since U is harmonic on this set, (2.8) implies that for any $x>-1$ the function $U(x,\cdot)$ is convex on $\{y:(x,y)\in J-1\}$ and hence $U(x,y)\geq U(x,0)$ by the symmetry of U. Thus, all we need is

$$U(x,0) \geq U(0,0) - x^+ \quad \text{for } x > -1.$$

This follows at once from the fact that both sides are equal for $x = 0$ and that $U_x(x,0) \in [-1,0]$ for $x > -1$ (to see the latter, use the concavity of the function $U(\cdot,0)$ and the lower bound in (2.6)). ∎

3. Proof of Theorem 1.2. For any semimartingale X there exists a unique continuous local martingale part X^c of X satisfying

$$[X,X]_t = |X_0|^2 + [X^c,X^c]_t + \sum_{0<s\leq t} |\Delta X_s|^2$$

for all $t \geq 0$; here $\Delta X_s = X_s - X_{s-}$ stands for the jump of X at time s. Furthermore, we have $[X^c,X^c] = [X,X]^c$, the pathwise continuous part of $[X,X]$. We have the following easy fact (see Lemma 1 in [W] and Corollary 1 in [BW2]).

Lemma 3.1. *If X and Y are semimartingales and Y is differentially subordinate to X, then Y^c is differentially subordinate to X^c, the inequality $|\Delta Y_t| \leq |\Delta X_t|$ holds for all $t > 0$ and $|Y_0| \leq |X_0|$. Furthermore, if X and Y are orthogonal, then Y has continuous paths and X^c, Y are orthogonal.*

Now we are ready to present the proof of Theorem 1.2.

Proof of (1.5). Consider a C^∞ radial function $g : \mathbb{R}^2 \to [0,\infty)$, supported on the ball of center $(0,0)$ and radius 1, satisfying $\int_{\mathbb{R}^2} g = 1$. For any $\delta > 0$, define $U^\delta : \mathbb{R}^2 \to \mathbb{R}$ by the convolution

$$U^\delta(x,y) = \int_{\mathbb{R}^2} U(x+\delta r, y+\delta s) g(r,s)\, dr\, ds.$$

Clearly, U^δ has the properties (i)–(iii) listed in Theorem 2.1. Furthermore, since U is superharmonic and g is radial, we have the following version of (iv):

$$U^\delta(0,0) \leq U(0,0) = -c^{-1}. \tag{3.1}$$

Concerning (v), we see that

$$\begin{aligned} U^\delta(x,y) &\geq \int_{\mathbb{R}^2} [-c^{-1} 1_{\{|y+\delta s|<1\}} + (x+\delta r)^+] g(r,s)\, dr\, ds \\ &\geq -c^{-1} 1_{\{|y|<1+\delta\}} - (x^+ + \delta). \end{aligned} \tag{3.2}$$

Let X, Y be semimartingales as in the statement and let M, N, A, C be the processes coming from the Doob–Meyer decompositions (1.3) of X and Y. Of course we may assume that X is bounded in L^1, since otherwise the claim is trivial. Fix $\varepsilon > 0$, $\ell > 0$ and introduce the stopping time

$$\tau = \tau(\ell) = \inf\{t \geq 0 : |Y_t| \geq 1 + \varepsilon \text{ or } |X_t| \geq \ell\}.$$

An application of Itô's formula gives

$$U^\delta(X_{\tau\wedge t}, Y_{\tau\wedge t}) = I_0 + I_1 + I_2 + I_3/2 + I_4, \tag{3.3}$$

where

$$\begin{aligned}
I_0 &= U^\delta(X_0, Y_0),\\
I_1 &= \int_{0+}^{\tau\wedge t} U^\delta_x(X_{s-}, Y_s)\, dM_s + \int_{0+}^{\tau\wedge t} U^\delta_y(X_{s-}, Y_s)\, dN_s,\\
I_2 &= \int_{0+}^{\tau\wedge t} U^\delta_x(X_{s-}, Y_s)\, dA_s + \int_{0+}^{\tau\wedge t} U^\delta_y(X_{s-}, Y_s)\, dC_s,\\
I_3 &= \int_{0+}^{\tau\wedge t} U^\delta_{xx}(X_{s-}, Y_s)\, d[X^c, X^c]_s\\
&\quad + 2\int_{0+}^{\tau\wedge t} U^\delta_{xy}(X_{s-}, Y_s)\, d[X^c, Y]_s + \int_{0+}^{\tau\wedge t} U^\delta_{yy}(X_{s-}, Y_s)\, d[Y, Y]_s,\\
I_4 &= \sum_{0<s\le\tau\wedge t} [U^\delta(X_s, Y_s) - U^\delta(X_{s-}, Y_s) - U^\delta_x(X_{s-}, Y_s)\Delta X_s].
\end{aligned}$$

Let us examine the terms I_0–I_4. First, $I_0 = U^\delta(0,0) \le -c^{-1}$: see (3.1). Next, $\mathbb{E}I_1 = 0$ due to the properties of stochastic integrals. Furthermore, using strong differential subordination and the inequality $U^\delta_x + |U^\delta_y| \le 0$ (see Theorem 2.1(ii)),

$$\begin{aligned}
I_2 &\le \int_{0+}^{\tau\wedge t} U^\delta_x(X_{s-}, Y_s)\, dA_s + \int_{0+}^{\tau\wedge t} |U^\delta_y(X_{s-}, Y_s)|\, d|C|_s\\
&\le \int_{0+}^{\tau\wedge t} [U^\delta_x(X_{s-}, Y_s) + |U^\delta_y(X_{s-}, Y_s)|]\, dA_s \le 0.
\end{aligned}$$

Exploiting the orthogonality of X and Y, we see that the middle term in I_3 vanishes. Combining this with the inequality $U^\delta_{xx} \le 0$ and the differential subordination of Y to X^c, we obtain

$$I_3 \le \int_{0+}^{\tau\wedge t} U^\delta_{xx}(X_s, Y_s)\, d[Y, Y]_s + \int_{0+}^{\tau\wedge t} U^\delta_{yy}(X_s, Y_s)\, d[Y, Y]_s \le 0,$$

since U^δ is superharmonic. Finally, $I_4 \le 0$, because U^δ is concave along horizontal lines. Plugging all these into (3.3) gives $\mathbb{E}U^\delta(X_{\tau\wedge t}, Y_{\tau\wedge t}) \le -c^{-1}$ and hence, by (3.2),

$$c^{-1}\mathbb{P}(|Y_{\tau\wedge t}| \ge 1+\delta) \le \mathbb{E}(X^+_{\tau\wedge t} + \delta) \le \mathbb{E}|X_t| + \delta, \tag{3.4}$$

where the last passage follows from Doob's optional sampling theorem and the trivial bound $x^+ \le |x|$. Letting $\delta \to 0$, we get

$$\mathbb{P}(|Y_{\tau\wedge t}| > 1) \le c\mathbb{E}|X_t| \le c\|X\|_1.$$

Recall that τ depends on ℓ; if we take $\sigma = \inf\{t \ge 0 : |Y_t| \ge 1 + \varepsilon\}$, then letting $\ell \to \infty$ above gives $\mathbb{P}(|Y_{\sigma\wedge t}| > 1) \le c\|X\|_1$ and hence

$$\mathbb{P}(Y^* \ge 1 + 2\varepsilon) \le \lim_{t\to\infty} \mathbb{P}(|Y_{\sigma\wedge t}| > 1) \le c\|X\|_1.$$

It suffices to apply this bound to a pair $((1+2\varepsilon)X, (1+2\varepsilon)Y)$ (for which the orthogonality and the strong differential subordination holds) and let $\varepsilon \to 0$. ∎

Remark 3.2. In fact (see (3.4)) we have proved a stronger bound

$$\mathbb{P}(Y^* \ge 1) \le c\|X^+\|_1. \tag{3.5}$$

Sharpness. Let B be a two-dimensional Brownian motion starting from $(0,0)$, put $\tau = \inf\{t : B_t \in \partial J\}$ and consider the (random) interval $I = [\tau, \tau - B_\tau^{(1)}]$ if $B_\tau^{(1)} < 0$, and $I = \emptyset$ otherwise. Let X, Y be Itô processes defined by $X_0 = Y_0 = 0$ and, for $t > 0$,

$$dX_t = 1_{\{\tau\ge t\}}\, dB_t^{(1)} + 1_I(t)\, dt,$$
$$dY_t = 1_{\{\tau\ge t\}}\, dB_t^{(2)} + \operatorname{sgn} B_\tau^{(2)} 1_I(t)\, dt.$$

To gain some intuition about the pair (X,Y), note that it behaves like B until it reaches the boundary of J at time τ. Then if $X_\tau \ge 0$, the pair stops; if $X_\tau < 0$ and $Y_\tau > 0$, then the pair moves along the line segment $\{(x-1, x) : x \in [0,1]\}$ until Y reaches 1; finally, if $X_\tau < 0$ and $Y_\tau < 0$, the pair moves along the line segment $\{(x-1, -x) : x \in [0,1]\}$ until Y reaches -1. Observe that Y is strongly differentially subordinate to X, and (X,Y) is constant on the interval $[\tau+1, \infty)$.

It is easy to show that τ is exponentially integrable and hence an application of Itô's formula yields $\mathbb{E}U(X_\tau, Y_\tau) = U(0,0) = -c^{-1}$. However, we have $|Y_{\tau+1}| \ge 1$ and

$$U(X_\tau, Y_\tau) = U(X_{\tau+1}, Y_{\tau+1}) = -X_{\tau+1}^+ = -X_{\tau+1}$$

with probability 1, so

$$\mathbb{P}(Y^* \ge 1) = 1 \quad \text{and} \quad \mathbb{E}|X_{\tau+1}| = c^{-1}.$$

Unfortunately, this does not finish the proof yet: the quantity $\mathbb{E}|X_{\tau+1}|$ is strictly smaller than $\|X\|_1$, because X takes negative values. To overcome this difficulty, we shall split the probability space into several small parts and use an appropriate copy of the above pair (X,Y) on each part. To be more precise, let ε be an arbitrary positive number. We have $\mathbb{E}|X_t| \to \mathbb{E}|X_{\tau+1}|$ as $t \to \infty$, so there is a deterministic positive number $T > 0$ such that

$$\|X_t\|_1 \le c^{-1} + \varepsilon \quad \text{for } t \ge T. \tag{3.6}$$

For $j = 0, 1, 2, \ldots$, let (X^j, Y^j) be a pair given by the above construction, but with $(B_t)_{t\ge 0}$ replaced by the time-shifted Brownian motion

$$B_t^j = \begin{cases} 0 & \text{if } t \le Tj, \\ B_t - B_{Tj} & \text{if } t > Tj. \end{cases}$$

Then $(X_t^j, Y_t^j) = (0,0)$ for $t \le Tj$ and

$$((X^j_{Tj+t}, Y^j_{Tj+t}))_{t\ge 0} \text{ has the same distribution as } (X,Y). \tag{3.7}$$

Moreover, X^j, Y^j are Itô processes with respect to the original Brownian motion B, and Y^j is strongly differentially subordinate to X^j.

Fix a positive integer k and consider a random variable η which is independent of B and has the distribution $\mathbb{P}(\eta = j) = 1/k$ for $j = 0, 1, \ldots, k-1$. This random variable splits Ω into k parts $\{\eta = 0\}, \{\eta = 1\}, \ldots, \{\eta = k-1\}$. We define

$$(X_t, Y_t) = (X_t^j, Y_t^j) \quad \text{on } \{\eta = j\}$$

for $t \ge 0$ and $j = 0, 1, \ldots, k-1$. Then both X and Y are Itô processes with respect to B, and Y is strongly differentially subordinate to X. Observe that

$$\mathbb{P}(Y^* \ge 1) = \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{P}(Y^{j*} \ge 1) = 1$$

and for any $t \ge 0$,

$$\|X_t\|_1 = \frac{1}{k}\sum_{j=0}^{k-1} \|X_t^j\|_1.$$

Now, if $t \le Tj$, then $X_t^j = 0$, and so $\|X_t^j\|_1 = 0 \le c^{-1} + \varepsilon$. If $t \in (Tj, Tj+T)$, then $\|X_t^j\|_1 \le \|X\|_1 < \infty$ in virtue of (3.7). Finally, if $t \ge Tj + T$, then, by (3.6), $\|X_t^j\|_1 = \|X_{t-Tj}\|_1 \le c^{-1} + \varepsilon$. Consequently, we obtain

$$\sup_{t\ge 0} \|X_t\|_1 \le \frac{k-1}{k}(c^{-1} + \varepsilon) + \frac{\|X\|_1}{k} \le c^{-1} + \varepsilon,$$

provided k is sufficiently large. This proves the optimality of c. ∎

4. An inequality for smooth functions. In this section we shall prove an estimate which can be regarded as a version of Kolmogorov's weak type inequality for subharmonic functions. Assume that $n \ge 1$ is a given integer and let D be an open connected subset of $\mathbb{R}^n$. Let ξ be a fixed point lying in D. Consider two real valued C^2 functions u, v on D, satisfying $u(\xi) = v(\xi) = 0$ and the following further properties: for all $x \in D$,

$$\nabla u(x) \cdot \nabla v(x) = 0, \tag{4.1}$$

where the dot $\cdot$ denotes the scalar product in $\mathbb{R}^n$, and

$$|\nabla v(x)| \le |\nabla u(x)|, \qquad |\Delta v(x)| \le |\Delta u(x)|. \tag{4.2}$$

The condition (4.1) plays the role of orthogonality, while (4.2) corresponds to strong differential subordination, as we shall see in a moment. For any bounded subdomain D_0 of D satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$, let $\mu_{D_0}^{\xi}$ denote the harmonic measure on ∂D_0 with respect to the point ξ. Theorem 1.2 yields the following result (for related statements, see Choi [C] and Janakiraman [J]).

Theorem 4.1. *Assume that* u, v *satisfy* (4.1), (4.2) *and* $u(\xi) = v(\xi) = 0$. *If* u *is subharmonic, then for any* D_0 *as above,*

$$\mu_{D_0}^{\xi}(\{x \in \partial D_0 : |v(x)| \ge 1\}) \le c \int_{\partial D_0} |u(x)|\, \mu_{D_0}^{\xi}(dx). \tag{4.3}$$

Unfortunately, we do not know if the constant c is optimal in (4.3).

Proof of (4.3). Take an n-dimensional Brownian B motion starting from ξ and consider the stopping time $\tau = \inf\{t : B_t \notin D_0\}$. Then $X = (u(B_{\tau\wedge t}))_{t\ge 0}$ is a submartingale starting from 0 and $Y = (v(B_{\tau\wedge t}))_{t\ge 0}$ is a semimartingale starting from 0. An application of Itô's formula yields the Doob–Meyer decompositions

$$X_t = M_t + A_t = \int_{0+}^{\tau\wedge t} \nabla u(B_s)\, dB_s + \frac{1}{2}\int_{0+}^{\tau\wedge t} \Delta u(B_s)\, ds,$$

$$Y_t = N_t + C_t = \int_{0+}^{\tau\wedge t} \nabla v(B_s)\, dB_s + \frac{1}{2}\int_{0+}^{\tau\wedge t} \Delta v(B_s)\, ds.$$

Because

$$[X, Y]_t = \int_{0+}^{\tau\wedge t} \nabla u(B_s) \cdot \nabla v(B_s)\, ds,$$

the assumption (4.1) implies that X and Y are orthogonal. Moreover,

$$[X, X]_t - [Y, Y]_t = \int_{0+}^{\tau\wedge t} (|\nabla u(B_s)|^2 - |\nabla v(B_s)|^2)\, ds$$

and

$$A_t - |C|_t = \frac{1}{2}\int_{0+}^{\tau\wedge t} \Delta u(B_s) - |\Delta v(B_s)|\, ds,$$

so by (4.2), Y is strongly differentially subordinate to X. Hence, by (3.5),

$$\mu_{D_0}^{\xi}(\{x : |v(x)| \ge 1\}) \le \mathbb{P}(Y^* \ge 1) \le c\|X^+\|_1 \le c\|u^+\|_1 \le c\|u\|_1,$$

because the distribution of B_τ is precisely $\mu_{D_0}^{\xi}$. The proof is complete. ∎

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