A Discretized Approach to W. T. Gowers’ Game

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Summary. We give an alternative proof of W. T. Gowers’ theorem on block bases by reducing it to a discrete analogue on specific countable nets. We also give a Ramsey type result on $k$-tuples of block sequences in a normed linear space with a Schauder basis.

1. Introduction. W. T. Gowers in [11] (see also [10] and [12]) proved a fundamental Ramsey-type theorem for block bases in Banach spaces which led to important discoveries in the geometry of Banach spaces. By now there are several approaches to Gowers’ theorem (see [1, 2, 3, 4, 14, 21]; also in [7, 15, 18] there are direct proofs of Gowers’ dichotomy, and in [6, 8, 19, 22, 24] extensions and further applications).

Our aim in this note is to prove a discrete analogue of Gowers’ theorem which is free of approximations. To state our results we will need the following notation. Let $X$ be a real linear space with an infinite countable Hamel basis $(e_n)_n$ (actually the field over which the linear space $X$ is defined plays no role in the arguments; it is only for the sake of convenience that we will assume that $X$ is a real linear space). For a subset $A \subseteq X$ we denote by $\langle A \rangle$ the linear span of $A$. Let $D$ be a subset of $X$. We denote by $B_D^\infty$ the set of all block sequences $(x_n)_n$ with $x_n \in D$ for all $n$. For a block sequence $Z \in B_D^\infty$ let $B_D^\infty(Z)$ be the set of all block sequences of $B_D^\infty$ which are block subsequences of $Z$.

Assume that $B_D^\infty$ is non-empty and let $Z \in B_D^\infty$ and $G \subseteq B_D^\infty$. We define the $D$-Gowers’ game in $Z$, denoted by $G_D(Z)$, as follows. Player I starts the game by choosing $W_0 \in B_D^\infty(Z)$ and player II responds with a vector $w_0 \in \langle W_0 \rangle \cap D$. Then player I chooses $W_1 \in B_D^\infty(Z)$ and player II chooses...
a vector \(w_1 \in \langle W_1 \rangle \cap \mathcal{D}\) and so on. Player II wins the game if the sequence \((w_0, w_1, \ldots)\) belongs to \(\mathcal{G}\).

Suppose that \(\mathcal{D}\) is a subset of \(\mathfrak{X}\) with the following properties:

\((\mathcal{D}1)\) (Asymptotic property) \(\mathcal{D} \cap \{(e_i)_{i \geq n}\} \neq \emptyset\) for all \(n \in \mathbb{N}\).

\((\mathcal{D}2)\) (Finitization property) \(\mathcal{D} \cap \{(e_i)_{i < n}\}\) is finite for all \(n \in \mathbb{N}\).

Property \((\mathcal{D}1)\) simply means that the set \(B_{\mathcal{D}}^\infty\) of all block sequences is non-empty. Property \((\mathcal{D}2)\) implies that \(\mathcal{D}\) is countable. Hence, if we endow \(\mathcal{D}\) with the discrete topology, the space \(\mathcal{D}^N\) of all infinite countable sequences in \(\mathcal{D}\) equipped with the product topology is a Polish space.

We can now state our first main result.

**Theorem 1.** Let \(\mathfrak{X}\) be a real linear space with a countable Hamel basis \((e_n)_{n}\) and let \(\mathcal{D} \subseteq \mathfrak{X}\) have properties \((\mathcal{D}1)\) and \((\mathcal{D}2)\). Also let \(\mathcal{G} \subseteq B_{\mathcal{D}}^\infty\) be an analytic subset of \(\mathcal{D}^N\). Then for every \(U \in B_{\mathcal{D}}^\infty\) there exists \(Z \in B_{\mathcal{D}}^\infty(U)\) such that either \(B_{\mathcal{D}}^\infty(Z) \cap \mathcal{G} = \emptyset\), or player II has a winning strategy in \(G_{\mathcal{D}}(Z)\) for \(\mathcal{G}\).

While discrete in nature, Theorem 1 can be used to derive Gowers’ original result provided that \(\mathcal{D}\) has an additional property (see Section 4).

Our second main result concerns \(k\)-tuples of block sequences in normed linear spaces with a Schauder basis. Precisely, let \(\mathfrak{X}\) be a real normed linear space with a Schauder basis \((e_n)_{n}\). We denote by \(B_{\mathfrak{X}}^\infty\) the set of block sequences in \(\mathfrak{X}\) and by \(B_{B_{\mathfrak{X}}}^\infty\) the set of all block sequences in the unit ball \(B_{\mathfrak{X}}\) of \(\mathfrak{X}\). Two block sequences \(Z_1 = (z^1_n)_{n}\) and \(Z_2 = (z^2_n)_{n}\) in \(B_{\mathfrak{X}}^\infty\) are said to be **disjointly supported** if \(\text{supp } z^1_n \cap \text{supp } z^2_m = \emptyset\) for all \(m, n\). For a positive integer \(k \geq 2\) and \(Z \in B_{\mathfrak{X}}^\infty\), the set of all \(k\)-tuples of pairwise disjointly supported block subsequences of \(Z\) in \(B_{\mathfrak{X}}\) will be denoted by \((B_{B_{\mathfrak{X}}}^\infty(Z))_k\).

Also, for a family \(\mathfrak{F} \subseteq (B_{\mathfrak{X}}^\infty)^k\) of \(k\)-tuples of block sequences in \(\mathfrak{X}\), the **upward closure** of \(\mathfrak{F}\) is defined to be the set

\[ \mathfrak{F}^\uparrow = \{(U_i)_{i=0}^{k-1} \in (B_{\mathfrak{X}}^\infty)^k : \exists (V_i)_{i=0}^{k-1} \in \mathfrak{F} \text{ such that } \forall i \text{ } V_i \text{ is a block subsequence of } U_i\}. \]

If \(\Delta = (\delta_n)_{n}\) is a sequence of positive reals, then the **\(\Delta\)-expansion** of \(\mathfrak{F}\) is defined to be the set

\[ \mathfrak{F}_\Delta = \{(U_i)_{i=0}^{k-1} \in (B_{\mathfrak{X}}^\infty)^k : \exists (V_i)_{i=0}^{k-1} \in \mathfrak{F} \text{ such that } \forall i \text{ } \text{dist}(U_i, V_i) \leq \delta_i\}. \]

We prove the following.

**Theorem 2.** Let \(\mathfrak{X}\) be a real normed linear space with a Schauder basis, \(k \geq 2\), and \(\mathfrak{F}\) an analytic subset of \((B_{B_{\mathfrak{X}}}^\infty)^k\). Then for every sequence \(\Delta = (\delta_n)_{n}\) of positive real numbers there is \(Y \in B_{\mathfrak{X}}^\infty\) such that either \((B_{B_{\mathfrak{X}}}^\infty(Y))^k \cap \mathfrak{F} = \emptyset\), or \((B_{B_{\mathfrak{X}}}^\infty(Y))^k \subseteq (\mathfrak{F}_\Delta)^\uparrow\).
In the above theorem the topology of $B^{\infty}_{\mathcal{B}_{\mathcal{X}}} \times B^{\infty}_{\mathcal{B}_{\mathcal{X}}}$ is the one induced by the product of the norm topology. Theorem 2 applied for $k = 2$ and the family $\mathfrak{F} = \{(U_1, U_2) \in (B^{\infty}_{\mathcal{B}_{\mathcal{X}}})^2 : U_1, U_2 \text{ are } C\text{-equivalent}\}$, where $C \geq 1$ is a constant, yields Gowers’ second dichotomy (see Lemma 7.3 in [11]).

2. Notation. Let $\mathcal{X}$ be a real linear space with an infinite countable Hamel basis $(e_n)_n$. For two non-zero vectors $x, y$ in $\mathcal{X}$, we write $x < y$ if $\max \supp x < \min \supp y$ (where $\supp x$ is the support of $x$, i.e. if $x = \sum_n \lambda_n e_n$ then $\supp x = \{n \in \mathbb{N} : \lambda_n \neq 0\}$). A sequence $(x_n)_n$ of vectors in $\mathcal{X}$ is called a block sequence (or block basis) if $x_n < x_{n+1}$ for all $n$.

Capital letters (such as $U, V, Y, Z, \ldots$) refer to infinite block sequences, and overlined lower case letters (such as $\overline{u}, \overline{v}, \overline{y}, \overline{z}, \ldots$) to finite block sequences. We write $Y \preceq Z$ to denote that $Y$ is a block subsequence of $Z$, that is, $Y = (y_n)_n, Z = (z_n)_n$ are block sequences and $y_n \in \langle (z_i)_i \rangle$ for all $n$. The notations $\overline{y} \preceq Z$ and $\overline{y} \preceq \overline{z}$ are defined analogously. For $\overline{x} = (x_n)_{n=0}^k$ and $Y = (y_n)_n$ we write $\overline{x} < Y$ if $x_k < y_0$. For $\overline{x} < Y$, $\overline{x} \prec Y$ denotes the block sequence $(z_n)_n$ that starts with the elements of $\overline{x}$ and continues with those of $Y$. Also for $\overline{x} < \overline{y}$, the finite block sequence $\overline{x} \prec \overline{y}$ is similarly defined. For a block sequence $Z = (z_n)_n$ and an infinite subset $L$ of $\mathbb{N}$ we set $Z|_L = (z_n)_{n \in L}$. Also $Z|_k = (z_n)_{n=0}^{k-1}$ for $k \in \mathbb{N}$ ($Z|_0 = \emptyset$ for $k = 0$).

Let $\mathcal{D} \subseteq \mathcal{X}$. We denote by $B^{\infty}_{\mathcal{D}}$ (resp. $B^{<\infty}_{\mathcal{D}}$) the set of all infinite (resp. finite) block sequences $(x_n)_n$ with $x_n \in \mathcal{D}$ for all $n$. The set of all infinite (resp. finite) block sequences in $\mathcal{X}$ is denoted by $B^{\infty}_{\mathcal{X}}$ (resp. $B^{<\infty}_{\mathcal{X}}$). For $Z \in B^{\infty}_{\mathcal{X}}$ we set $B^{\infty}_{\mathcal{D}}(Z) = \{Y \in B^{\infty}_{\mathcal{D}} : Y \preceq Z\}$ and $B^{<\infty}_{\mathcal{D}}(Z) = \{\overline{y} \in B^{<\infty}_{\mathcal{D}} : \overline{y} \preceq Z\}$. Similarly $B^{\infty}_{\mathcal{D}}(\overline{z}) = \{\overline{y} \in B^{<\infty}_{\mathcal{D}} : \overline{y} \preceq \overline{z}\}$ for $\overline{z} \in B^{<\infty}_{\mathcal{X}}$. For a block sequence $Z \in B^{\infty}_{\mathcal{D}}$, we set $\langle Z \rangle_{\mathcal{D}} = \langle Z \rangle \cap \mathcal{D}$ where $\langle Z \rangle$ is the linear span of $Z$.

3. Discretization of Gowers’ game. Throughout this section, $\mathcal{X}$ is a real linear space with countable Hamel basis $(e_n)_n$ and $\mathcal{D}$ is a subset of $\mathcal{X}$ with properties (D1) and (D2) as stated in the Introduction. Notice that (D2) also gives that for every $U = (u_i)_i \in B^{\infty}_{\mathcal{D}}$ and $n \in \mathbb{N}$, the set $B^{<\infty}_{\mathcal{D}}((u_i)_{i<n})$ is finite.

3.1. Admissible families of $\mathcal{D}$-pairs. The aim of this subsection is to review some methods of handling diagonalizations (see also [11], [20]). A $\mathcal{D}$-pair is a pair $(\overline{x}, Y)$ where $\overline{x} \in B^{<\infty}_{\mathcal{D}}$ and $Y \in B^{\infty}_{\mathcal{D}}$. For $U \in B^{\infty}_{\mathcal{D}}$, a family $\mathcal{P} \subseteq B^{<\infty}_{\mathcal{D}}(U) \times B^{\infty}_{\mathcal{D}}(U)$ is called an admissible family of $\mathcal{D}$-pairs in $U$ if it has the following properties:

(P1) (Hereditiy) If $(\overline{x}, Y) \in \mathcal{P}$ and $Z \in B^{\infty}_{\mathcal{D}}(Y)$ then $(\overline{x}, Z) \in \mathcal{P}$. 


\((\mathcal{P}2)\) (Cofinality) For every \((\bar{x}, Y) \in \mathcal{B}^{<\infty}_2(U) \times \mathcal{B}^{\infty}_2(U)\), there is \(Z \in \mathcal{B}^{\infty}_2(Y)\) such that \((\bar{x}, Z) \in \mathcal{P}\).

For simplicity, when we write “pair” we will always mean a \(\mathcal{D}\)-pair. It will often happen that an admissible family of pairs has one more property:

\[(\mathcal{P}3)\] If \((\bar{x}, Y) \in \mathcal{P}, \bar{x} < Y\) and \(k = \min\{m : \bar{x} \in \mathcal{B}^{\infty}_2((u_i)_{i=1}^m)\}\) then \((\bar{x}, \bar{y} \setminus Y) \in \mathcal{P}\) for every \(\bar{y} \in \mathcal{B}^{\infty}_2((u_i)_{i=1}^k)\).

The next lemma follows by a standard diagonalization argument.

**Lemma 3.** Let \(U \in \mathcal{B}^{\infty}_2\) and let \(\mathcal{P}\) be an admissible family of pairs in \(U\). Then there is \(W \in \mathcal{B}^{\infty}_2(U)\) such that \((\bar{w}, Y) \in \mathcal{P}\) for all \(\bar{w} \in \mathcal{B}^{\infty}_2(W)\) and all \(Y \in \mathcal{B}^{\infty}_2(W)\) with \(\bar{w} < Y\). If in addition \(\mathcal{P}\) satisfies \((\mathcal{P}3)\) then \((\bar{w}, W) \in \mathcal{P}\) for all \(\bar{w} \in \mathcal{B}^{\infty}_2(W)\).

### 3.2. The discrete Gowers’ game.

Given \(Y \in \mathcal{B}^{\infty}_2\) and a family of infinite block sequences \(\mathcal{G} \subseteq \mathcal{B}^{\infty}_2\), we define the \(\mathcal{D}\)-Gowers’ game, \(G_\mathcal{D}(Y)\), as follows. Player I starts the game by choosing \(Z_0 \in \mathcal{B}^{\infty}_2(Y)\) and player II responds with a vector \(z_0 \in \langle Z_0 \rangle_\mathcal{D}\). Then player I chooses \(Z_1 \in \mathcal{B}^{\infty}_2(Y)\) and player II chooses a vector \(z_1 \in \langle Z_1 \rangle_\mathcal{D}\) with \(z_0 < z_1\), and so on. More generally, for a finite block sequence \(\bar{x} \in \mathcal{B}^{<\infty}_2\) and \(Y \in \mathcal{B}^{\infty}_2\) the game \(G_\mathcal{D}(\bar{x}, Y)\) is defined as above with the additional condition that player II in the first move chooses \(z_0 > \bar{x}\). Clearly \(G_\mathcal{D}(\emptyset, Y)\) is identical to \(G_\mathcal{D}(Y)\). We will say that player II wins the game \(G_\mathcal{D}(\bar{x}, Y)\) for \(\mathcal{G}\) if the block sequence \(\bar{x} \setminus (z_0, z_1, \ldots)\) belongs to \(\mathcal{G}\).

We will basically follow the classical Galvin–Prikry terminology (cf. [9, 5]) in the context of Gowers’ game. More precisely, for \(\bar{x} \in \mathcal{B}^{<\infty}_2, Y \in \mathcal{B}^{\infty}_2\) and \(\mathcal{G} \subseteq \mathcal{B}^{\infty}_2\) we say that \(Y \mathcal{G}\)-accepts \(\bar{x}\) if player II has a winning strategy in \(G_\mathcal{D}(\bar{x}, Y)\) for \(\mathcal{G}\), while \(Y \mathcal{G}\)-rejects \(\bar{x}\) if no \(Z \in \mathcal{B}^{\infty}_2(Y)\) which \(\mathcal{G}\)-accepts \(\bar{x}\). We also say that \(Y \mathcal{G}\)-decides \(\bar{x}\) if it either \(\mathcal{G}\)-accepts or \(\mathcal{G}\)-rejects it.

Notice that if \(\bar{x} = \emptyset\) then to say that “\(Y \mathcal{G}\)-accepts the empty sequence” means that player II has a winning strategy in \(G_\mathcal{D}(Y)\) for \(\mathcal{G}\). Similarly the statement that “\(Y \mathcal{G}\)-rejects the empty sequence” is equivalent to saying that for no \(Z \in \mathcal{B}^{\infty}_2(Y)\) does player II have a winning strategy in \(G_\mathcal{D}(Z)\) for \(\mathcal{G}\). The following lemma is easily verified.

**Lemma 4.** For every \(U \in \mathcal{B}^{\infty}_2\) and every \(\mathcal{G} \subseteq \mathcal{B}^{\infty}_2\), the family

\[\mathcal{P} = \{(\bar{x}, Y) \in \mathcal{B}^{<\infty}_2(U) \times \mathcal{B}^{\infty}_2(U) : Y \mathcal{G}\text{-decides }\bar{x}\}\]

is an admissible family of pairs in \(U\) with property \((\mathcal{P}3)\).

Actually the family \(\mathcal{P}\) of the above lemma satisfies the following condition stronger than \((\mathcal{P}3)\): If \((\bar{x}, Y) \in \mathcal{P}\) and \(Z \in \mathcal{B}^{\infty}_2\) are such that \(Z|_{[n, \infty)} \leq Y\) for some \(n \in \mathbb{N}\), then \((\bar{x}, Z) \in \mathcal{P}\).
For the sake of simplicity, in the following we will omit the letter $G$ in front of “accepts”, “rejects” and “decides”. The next lemma is a consequence of Lemmas 4 and 3.

**Lemma 5.** For every $U \in \mathcal{B}_D^\infty$ there is $W \in \mathcal{B}_D^{<\infty}(U)$ such that $W$ decides all $\overline{w} \in \mathcal{B}_D^{<\infty}(W)$.

The crucial point where the above notions of “accept-reject” essentially differ from the original ones reveals itself in the next lemma. Here the notion of the winning strategy replaces successfully the traditional pigeonhole principle.

**Lemma 6.** Let $W \in \mathcal{B}_D^\infty$ decide all $\overline{w} \in \mathcal{B}_D^{<\infty}(W)$ and assume that it rejects some $\overline{w}_0 \in \mathcal{B}_D^{<\infty}(W)$. Then for every $Y \in \mathcal{B}_D^\infty(W)$ there is $Z \in \mathcal{B}_D^{<\infty}(Y)$ such that $W$ rejects $\overline{w}_0 z$ for every $z \in \langle Z \rangle_D$ with $\overline{w}_0 < z$.

**Proof.** If the conclusion is false then there is $Y \in \mathcal{B}_D^\infty(W)$ such that for every $Z \in \mathcal{B}_D^{<\infty}(Y)$ there is $z \in \langle Z \rangle_D$ with $\overline{w}_0 < z$ such that $W$ accepts $\overline{w}_0 z$. It is easy to see that this means that player II has a winning strategy in $G_D(\overline{w}_0, Y)$ for $G$, and thus $Y$ accepts $\overline{w}_0$. But this is a contradiction since $Y \in \mathcal{B}_D^\infty(W)$ and $W$ rejects $\overline{w}_0$. ■

**Lemma 7.** For every $U \in \mathcal{B}_D^\infty$ there exists $Z \in \mathcal{B}_D^{<\infty}(U)$ such that either $Z$ rejects all $\overline{z} \in \mathcal{B}_D^{<\infty}(Z)$, or player II has a winning strategy in $G_D(Z)$ for $G$.

**Proof.** By Lemma 5 there is $W \in \mathcal{B}_D^{<\infty}(U)$ such that $W$ decides all $\overline{w} \in \mathcal{B}_D^{<\infty}(W)$. If $W$ accepts the empty sequence then we readily have the second alternative of the conclusion for $Z = W$. In the opposite case consider the following family in $\mathcal{B}_D^{<\infty}(W) \times \mathcal{B}_D^\infty(W)$:

$$\mathcal{P} = \{(\overline{x}, Y) : \text{either } W \text{ accepts } \overline{x},$$

$$\text{or } \forall y \in \langle Y \rangle_D \text{ with } \overline{x} < y, \text{ } W \text{ rejects } \overline{x} y \}.\$$

Using Lemma 6 we easily verify that $\mathcal{P}$ is an admissible family in $W$ which also satisfies $(\mathcal{P}_3)$. Hence by Lemma 3 there is $Z \in \mathcal{B}_D^{<\infty}(W)$ with $(\overline{z}, Z) \in \mathcal{P}$ for every $\overline{z} \in \mathcal{B}_D^{<\infty}(Z)$. By our assumption $W$ rejects the empty sequence. Since $(\emptyset, Z) \in \mathcal{P}$ we infer that $W$, and so $Z$, rejects all $z \in \langle Z \rangle_D$. By induction on the length of finite block sequences in $\mathcal{B}_D^{<\infty}(Z)$, it is easily shown that $Z$ rejects all $\overline{z} \in \mathcal{B}_D^{<\infty}(Z)$. ■

We have finally arrived at our first stop which is an analog of the well known result of Nash-Williams ([17]). Consider the set $\mathcal{D}$ with the discrete topology and $\mathcal{D}^N$ with the product topology.

**Lemma 8.** Let $G \subseteq \mathcal{B}_D^\infty$ be open in $\mathcal{D}^N$. Then for every $U \in \mathcal{B}_D^\infty$ there exists $Z \in \mathcal{B}_D^{<\infty}(U)$ such that either $\mathcal{B}_D^{<\infty}(Z) \cap G = \emptyset$, or player II has a winning strategy in $G_D(Z)$ for $G$. 
For each a continuous function from finally, recall the following terminology from [11]. Let $N$ and let $W = (w_n)_{n \in \mathbb{N}}$ be the Baire space, i.e. the space of all infinite sequences in $\mathbb{N}$ with the topology generated by the sets $N_s = \{\sigma \in N : \exists n \text{ with } \sigma|n = s\}$, $s \in \mathbb{N}^\mathbb{N}$. A subset of a Polish space $X$ is called analytic if it is the image of a continuous function from $N$ into $X$.

For the next lemmas we fix the following:

(a) a family $(G^s)_{s \in \mathbb{N}^\mathbb{N}}$ of subsets of $\mathcal{B}_D^\infty$ such that $G^s = \bigcup_n G^s \cap U_n$ for all $s$,
(b) a bijection $\varphi : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that $\varphi(\emptyset) = 0$ and $\varphi(s \cap n) > \varphi(s)$ for all $s, n$.

For each $\bar{x}$ in $\mathcal{B}_D^{<\infty}$ we set $s_{\bar{x}}$ to be the unique element of $\mathbb{N}^\mathbb{N}$ such that $\varphi(s_{\bar{x}})$ equals the length of $\bar{x}$. For a $\mathcal{D}$-pair $(\bar{x}, Y)$ we set

$$\mathcal{B}_D^\infty(\bar{x}, Y) = \{V \in \mathcal{B}_D^\infty : \exists k \text{ such that } V|_k = \bar{x} \text{ and } V|_{k, \infty} \subseteq Y\}.$$ 

Finally, recall the following terminology from [11]. For a family $G \subseteq \mathcal{B}_D^\infty$ we say that $G$ is large for $(\bar{x}, Y)$ if $G \cap \mathcal{B}_D^\infty(\bar{x}, Z) \neq \emptyset$ for all $Z \in \mathcal{B}_D^\infty(Y)$. In the case $\bar{x} = \emptyset$ we simply say that $G$ is large for $Y$.

\textbf{Lemma 9.} For every $U \in \mathcal{B}_D^\infty$ there is $W \in \mathcal{B}_D^{<\infty}(U)$ such that for every $\bar{w} \in \mathcal{B}_D^\infty(W)$, either $G^\bar{w} \cap \mathcal{B}_D^\infty(\bar{w}, W) = \emptyset$, or $G^\bar{w}$ is large for $(\bar{w}, W)$.

\textit{Proof.} Let $P$ be the set of all pairs $(\bar{x}, Y)$ in $\mathcal{B}_D^{<\infty}(U) \times \mathcal{B}_D^\infty(Y)$ such that either $G^\bar{x} \cap \mathcal{B}_D^\infty(\bar{x}, Y) = \emptyset$, or $G^\bar{x}$ is large for $(\bar{x}, Y)$. It is easy to see that $P$ is admissible and satisfies $(P3)$. Hence the conclusion follows by Lemma 3. ■

Let $W \in \mathcal{B}_D^\infty$ be a block sequence in $\mathcal{D}$ satisfying the conclusion of Lemma 9. For $\bar{w} \in \mathcal{B}_D^{<\infty}(W)$, let $F(\bar{w})$ be the family of all $V = (v_i)_i \in \mathcal{B}_D^{<\infty}(W)$ with $\bar{w} < V$ and with the following properties. There exist $m, l \in \mathbb{N}$ with $l \geq 1$ such that

(i) $s_{\bar{w}} m = s_{\bar{x}}$, where $\bar{x} = \bar{w} \cap (v_i)_{i=0}^{l-1}$,
(ii) the family $G^\bar{w} m$ is large for $(\bar{w} \cap (v_i)_{i=0}^{l-1}, W)$.

Notice that $F(\bar{w})$ is open in $\mathcal{D}^\mathbb{N}$.

\textbf{Lemma 10.} Let $\bar{w} \in \mathcal{B}_D^{<\infty}(W)$ and assume that $G^\bar{w}$ is large for $(\bar{w}, W)$. Then $F(\bar{w})$ is large for $W$. 

Proof. Let \( Z \in B_D^\infty (W) \). Since \( G^{s \pi} \) is large for \((\vec{w}, W)\) there is \( V = (v_i)_i \) such that \( \vec{w} < V \) and \( \vec{w} \cap V \subseteq G^{s \pi} \cap B_D^\infty (\vec{w}, Z) = \bigcup_m G^{s \pi_m} \cap B_D^\infty (\vec{w}, Z) \) and so \( \vec{w} \cap V \subseteq G^{s \pi_m} \cap B_D^\infty (\vec{w}, Z) \) for some \( m \in \mathbb{N} \). Notice that for \( l = \varphi (s^{-m}) - \varphi (s) \) we have \( s_{\vec{w} m} = s_{\vec{w} x} \), where \( x = \vec{w} \cap (v_i)_{i=0}^{l-1} \), and \( \vec{w} \cap V \subseteq G^{s \pi_m} \cap B_D^\infty (\vec{w} \cap (v_i)_{i=0}^{l-1}, Z) \). Therefore \( G^{s \pi_m} \cap B_D^\infty (\vec{w} \cap (v_i)_{i=0}^{l-1}, W) \neq \emptyset \), which (by the properties of \( W \)) means that \( G^{s \pi_m} \) is large for \((\vec{w} \cap (v_i)_{i=0}^{l-1}, W)\). Hence \( V \in F(\vec{w}) \cap B_D^\infty (Z) \).

Lemma 11. There is \( Z \in B_D^\infty (W) \) such that for every \( z \in B_D^{< \infty} (Z) \), either \( G^{s \pi} \cap B_D^\infty (\vec{w}, Y) = \emptyset \) or player II has a winning strategy in the game \( G_2 (Z) \) for the family \( F(z) \).

Proof. Let \( P \) be the family of pairs \((\vec{w}, Y) \in B_D^{< \infty} (W) \times B_D^\infty (W) \) such that either \( G^{s \pi} \cap B_D^\infty (\vec{w}, Y) = \emptyset \), or player II has a winning strategy in \( G_2 (Y) \) for \( F(\vec{w}) \).

By Lemma 3 it suffices to show that \( P \) is an admissible family of pairs in \( W \) which in addition satisfies \((P3)\). It is easy to see that only the cofinality property needs some explanation. Let \((\vec{w}, Y) \in B_D^{< \infty} (W) \times B_D^\infty (W) \). Since \( \vec{w} \in B_D^{< \infty} (W) \), either \( G^{s \pi} \cap B_D^\infty (\vec{w}, W) = \emptyset \), or \( G^{s \pi} \) is large for \((\vec{w}, W)\). In the first case, \( G^{s \pi} \cap B_D^\infty (\vec{w}, Y) = \emptyset \) and so \((\vec{w}, Y) \in P \). In the second case, Lemma 10 implies that \( F(\vec{w}) \) is large for \( W \). Hence by Lemma 8 there is \( V \in B_D^\infty (Y) \) such that player II has a winning strategy in \( G_2 (V) \) for \( F(\vec{w}) \), and so \((\vec{w}, V) \in P \).

We are now ready for the proof of the main result.

Proof of Theorem 1. Assume that there is no \( Z \in B_D^\infty (U) \) such that \( B_D^\infty (Z) \cap G = \emptyset \), that is, \( G \) is large for \( U \). Let \( f : N \to D^N \) be a continuous map with \( f[N] = G \), and for \( s \in N^{< N} \), let \( G^s = f[N_s] \). Then \( G^0 = G \) and \( G^s = \bigcup_n G^{s^{-n}} \). Following the process of the above lemmas let \( W \in B_D^{< \infty} (U) \) be as in Lemma 9 and \( Z \in B_D^\infty (W) \) as in Lemma 11. We claim that player II has a winning strategy in the game \( G_2 (Z) \) for \( G \).

Indeed, by our assumption \( G = G^0 \) is large in \( B_D^\infty (Z) = B_D^\infty (\emptyset, Z) \) and so player II has a winning strategy in \( G_2 (Z) \) for \( F(\emptyset) \). This means that player II is able to produce, after a finite number of moves, a finite block sequence \( \vec{y}_0 \in B_D^{< \infty} (Z) \) such that there is \( m_0 \in \mathbb{N} \) with \( s_{\vec{y}_0} = (m_0) \) and \( G^{(m_0)} \) large for \((\vec{y}_0, W)\). By Lemma 11 player II has a winning strategy in \( G_2 (Z) \) for \( F(\vec{y}_0) \), that is, player II can extend \( \vec{y}_0 \) to a finite block sequence \( \vec{y}_0 \vec{y}_1 \in B_D^{< \infty} (Z) \) such that there is \( m_1 \in \mathbb{N} \) such that \( s_{\vec{y}_0 \vec{y}_1} = (m_0, m_1) \) and \( G^{(m_0, m_1)} \) is large for \((\vec{y}_0 \vec{y}_1, W)\).

Continuing in this way we conclude that player II has a strategy in the game \( G_2 (Z) \) to construct a block sequence \( Y = \vec{y}_0 \vec{y}_1 \ldots \) such that for some
σ = (m_i) ∈ N and for every k ∈ N, G^{σ|k} is large for ((\bar{y}_0 \ldots \bar{y}_{k-1}), W).

To show that this is actually a winning strategy for G, we have to prove that Y ∈ G. Fix k ∈ N. Since G^{σ|k} is large for ((\bar{y}_0 \ldots \bar{y}_{k-1}), W), there exists Y_k ∈ B_∞^\infty(W) such that (\bar{y}_0 \ldots \bar{y}_{k-1}) Y_k ∈ G^{σ|k}. Since (G^{σ|k})_n is decreasing, Y = \lim_n (\bar{y}_0 \ldots \bar{y}_{n-1}) Y_n ∈ G^{σ|k} for all k ∈ N, and thus Y ∈ \bigcap_k G^{σ|k}. By the continuity of f, \bigcap_k G^{σ|k} = \{f(σ)\} and therefore Y = f(σ) ∈ G. ■

4. Passing from the discrete to Gowers’ game. In this section we will see how using Theorem 1 one can derive W. T. Gowers’ Ramsey theorem (see Theorem 16). Henceforth, X will be a normed linear space with a Schauder basis (e_n)_n.

First let us recall some relevant definitions. Let B_\infty^\infty (resp. B_\infty^\infty) be the set of all block sequences in X (resp. in the unit ball B_X). Let U = (u_n)_n, V = (v_n)_n ∈ B_\infty^\infty and Δ = (δ_n)_n a sequence of positive real numbers. We say that U, V are Δ-near and we write dist(U, V) ≤ Δ if \|u_n - v_n\| ≤ δ_n for all n ∈ N. For a family F ⊆ B_\infty^\infty and a sequence Δ = (δ_n)_n of positive real numbers the Δ-expansion of F is the set

\[ F_Δ = \{U ∈ B_\infty^\infty : ∃V ∈ F such that \text{dist}(U, V) ≤ Δ\}. \]

For Y ∈ B_\infty^\infty and a family F ⊆ B_\infty^\infty, the Gowers’ game G_X(Y) is defined as the D-Gowers game by replacing D and G ⊆ B_\infty^\infty with B_X and F ⊆ B_\infty^\infty respectively.

For the next two lemmas we fix the following:

(i) a subset D of \{(e_n)_n\} with the asymptotic property (D1),
(ii) a family F ⊆ B_\infty^\infty of block sequences in B_X,
(iii) a sequence Δ = (δ_n)_n of positive real numbers.

**Lemma 12.** Let G = F_Δ ∩ B_\infty^\infty and suppose that B_\infty^\infty (\tilde{Z}) ∩ G = ∅ for some \tilde{Z} ∈ B_\infty^\infty. Assume that there exists Z ∈ B_\infty^\infty such that

\[ B_\infty^\infty (Z) ⊆ (B_\infty^\infty (\tilde{Z}))_Δ \]

(that is, for every block subsequence U = (u_n)_n of Z with \|u_n\| ≤ 1 there is a block subsequence \tilde{U} = (\tilde{u}_n)_n of \tilde{Z} with \tilde{u}_n ∈ D such that dist(U, \tilde{U}) ≤ Δ).

Then B_\infty^\infty (Z) ∩ F = ∅.

**Proof.** Let U ∈ B_\infty^\infty (Z). By our assumptions there is \tilde{U} ∈ B_\infty^\infty (\tilde{Z}) such that dist(U, \tilde{U}) ≤ Δ and \tilde{U} ∉ G. Then U ∉ F, otherwise \tilde{U} ∈ F_Δ ∩ B_\infty^\infty (\tilde{Z}) which is a contradiction. ■

**Lemma 13.** Let δ_0 ≤ 1 and \sum_{j=n+1}^\infty δ_j ≤ δ_n for all n. Let G = F_Δ/10C ∩ B_\infty^\infty, where C is the basis constant of (e_n)_n, and suppose that for some
\[ \tilde{Z} \in B_\mathcal{D}^\infty \text{ player II has a winning strategy in the discrete game } G_\mathcal{D}(\tilde{Z}) \text{ for } \mathcal{G}. \]

Assume that there exists \( Z \in B_\mathcal{X}^\infty \) such that

\[ B_\mathcal{B}_\mathcal{X}^\infty (Z) \subseteq (B_\mathcal{D}(\tilde{Z}))_{\Delta/10C}. \]

Then player II has a winning strategy in Gowers' game \( G_\mathcal{X}(Z) \) for \( \mathcal{F}_\Delta \).

Proof. We will define a winning strategy for player II in Gowers' game \( G_\mathcal{X}(Z) \) for \( \mathcal{F}_\Delta \) provided that he has one in the discrete game \( G_\mathcal{D}(Z) \) for \( \mathcal{G} \).

Suppose that we have just completed the \( n \)th move of \( G_\mathcal{X}(Z) \) (resp. \( G_\mathcal{D}(\tilde{Z}) \)) and \( x_0 < \cdots < x_{n-1} \) (resp. \( \tilde{x}_0 < \cdots < \tilde{x}_{n-1} \)) have been chosen by player II in \( G_\mathcal{X}(Z) \) (resp. in \( G_\mathcal{D}(\tilde{Z}) \)).

Suppose that in \( G_\mathcal{X}(Z) \) player I chooses a block sequence \( Z_n = (z^n_k)_k \in B_\mathcal{X}(Z) \). By normalizing we may suppose that \( \|z^n_k\| = 1 \) for every \( k \), and so by our assumptions on \( \tilde{Z} \) and \( Z \) there exists \( Z_n = (\tilde{z}^n_k)_k \in B_\mathcal{D}(\tilde{Z}) \) such that \( \text{dist}(Z_n, \tilde{Z}_n) \leq \Delta/10C \). Then for all \( k \), \( \|z^n_k - \tilde{z}^n_k\| \leq \delta_k/10C \) and so \( \|\tilde{z}^n_k\| \geq 1 - \delta_k/10C \). Let \( k_0 \geq n \) be such that \( x_{n-1} < z^n_{k_0} \) and let player I play \( \tilde{Z}_n|_{k_0, \infty} = (\tilde{z}^n_k)_{k \geq k_0} \) in the \( n \)th move of the discrete game \( G_\mathcal{D}(\tilde{Z}) \). Then player II extends \( (\tilde{x}_0, \ldots, \tilde{x}_{n-1}) \) according to his strategy in \( G_\mathcal{D}(\tilde{Z}) \) for \( \mathcal{G} \), by picking \( \tilde{x}_n \in (\tilde{z}^n_k)_{k \geq k_0} \). Then \( \tilde{x}_n = \sum_{k \in I_n} \lambda^n_k \tilde{z}_k^n \), where \( I_n \) is a finite segment in \( \mathbb{N} \) with \( \min I_n \geq k_0 \) and \( \lambda_k^n \in \mathbb{R} \). Going back to Gowers’ game \( G_\mathcal{X}(Z) \), let player II play \( x_n = \sum_{k \in I_n} \lambda^n_k z^n_k \). Then \( x_n > x_{n-1} \) and so player II forms in this way a block sequence in \( B_\mathcal{X}(Z) \).

It remains to show that \( (x_n)_n \in \mathcal{F}_\Delta \). Since \( (\tilde{x}_n)_n \in \mathcal{G} \subseteq \mathcal{F}_{\Delta/10C} \subseteq (B_\mathcal{B}_\mathcal{X}^\infty)_{\Delta/10C} \), we see that \( \|\tilde{x}_n\| \leq 1 + \delta_n/10C \) for all \( n \). Hence

\[ \|x_n - \tilde{x}_n\| \leq \sum_{k \in I_n} |\lambda^n_k| \|z^n_k - \tilde{z}^n_k\| \leq 4C \sum_{k \in I_n} \frac{\delta_k}{10C} \leq \frac{4}{5}\delta_{\min I_n} \leq \frac{4}{5}\delta_n. \]

Since \( (\tilde{x}_n)_n \in \mathcal{F}_{\Delta/10C} \), the last inequality gives \( (x_n)_n \in \mathcal{F}_{\Delta/5 + \Delta/10C} \subseteq \mathcal{F}_\Delta \). □

The above lemmas lead us to define the next property for a subset \( \mathcal{D} \) of \( \mathcal{X} \) and a given sequence \( \Delta = (\delta_n)_n \) of positive real numbers.

\[ (\mathcal{D}3) \text{ (}\Delta\text{-}block covering property) \text{ For every } \tilde{Z} \in B_\mathcal{D}^\infty \text{ there exists } Z \in B_\mathcal{X}^\infty \text{ such that } B_\mathcal{B}_\mathcal{X}^\infty (Z) \subseteq (B_\mathcal{D}(\tilde{Z}))_{\Delta}. \]

In the next proposition we give an example of a subset \( \mathcal{D} \) of \( \mathcal{X} \) with properties \( (\mathcal{D}1) - (\mathcal{D}3) \). Actually we show that a property much stronger than (\( \mathcal{D}3 \)) can be satisfied. In particular, for every \( \tilde{Z} \in B_\mathcal{D}^\infty \) with \( \tilde{Z} = (\tilde{z}_n)_n \), if we set \( Z = (z_n)_n \) with \( z_n = \tilde{z}_{2n} + \tilde{z}_{2n+1} \) then \( B_\mathcal{B}_\mathcal{X}^\infty (Z) \subseteq (B_\mathcal{D}(\tilde{Z}))_{\Delta}. \)
Finally, we set \( V. Kanellopoulos and K. Tyros \)

For every finite non-empty segment \( I = [n_1, n_2] \) of \( \mathbb{N} \), \( n_1 \leq n_2 \), define \( \mathcal{D}(I) = \mathcal{D}([n_1, n_2]) \) to be the set of all \( x = \sum_{i=n_1}^{n_2} \lambda_i e_i \) with the following properties:

(i) For all \( n_1 \leq i \leq n_2 \), \( \lambda_i \in \Lambda(i, l) \), where \( l = n_2 - n_1 + 1 \) is the length of \( I \).

(ii) The coefficients \( \lambda_{n_1} \) and \( \lambda_{n_2} \) are both non-zero.

(iii) \( \|x\| \leq 1 \).

Finally, we set

\[
\mathcal{D} = \bigcup_{n_1 \leq n_2} \mathcal{D}([n_1, n_2]).
\]

It is easy to see that \( \mathcal{D} \) satisfies (D1)-(D2). In particular, \( (e_n)_n \in \mathcal{B}_\mathcal{D}^\infty \). It remains to show that \( \mathcal{D} \) has the \( \Delta \)-block covering property. Actually, we will prove that \( \mathcal{D} \) has a stronger property; to do this we first state the following.

**Claim.** Let \( \tilde{Z} \in \mathcal{B}_\mathcal{D}^\infty \) and let \( w \in \langle \tilde{Z} \rangle \) be such that \( \text{card}(\text{supp}_{\tilde{Z}}(w)) \geq 2 \) and \( \|w\| \leq 1 \). Then there is \( \tilde{w} \in \langle \tilde{Z} \rangle_\mathcal{D} \) such that

1. \( \text{supp}_{\tilde{Z}}(\tilde{w}) = \text{supp}_{\tilde{Z}}(w) \).
2. \( \|w - \tilde{w}\| \leq 2^{-k_{m_1}+1}, \) where \( m_1 = \min \text{supp}_{\tilde{Z}}(w) \).

**Proof of the claim.** Let \( \tilde{Z} = (\tilde{z}_j)_j \) and let \( (I_j)_j, I_j = [n_1(j), n_2(j)] \), \( n_1(j) \leq n_2(j) \), be the sequence of successive finite non-empty segments of \( \mathbb{N} \) such that \( \tilde{z}_j \in \mathcal{D}(I_j) \). Let \( m_1 < m_2 \) in \( \mathbb{N} \), let \( (\mu_j)_{j=m_1}^{m_2} \) be scalars such that \( \mu_{m_1}, \mu_{m_2} \) are both non-zero and let \( w = \sum_{j \in [m_1, m_2]} \mu_j \tilde{z}_j \) in \( \mathcal{B}_\mathcal{X} \).

Set

\[
w' = (1 - 2^{-k_{m_1}})w = \sum_{j \in [m_1, m_2]} (1 - 2^{-k_{m_1}}) \mu_j \tilde{z}_j \quad \text{and} \quad \tilde{w} = \sum_{j \in [m_1, m_2]} \tilde{\mu}_j \tilde{z}_j,
\]

where \( \tilde{\mu}_j = s_j \cdot 2^{-(k_{m_1}+1)} \) and

\[
s_j = \begin{cases} 
[(1 - 2^{-k_{m_1}}) \mu_j 2^{k_{m_1}+1}] & \text{if } \mu_j \geq 0, \\
[(1 - 2^{-k_{m_1}}) \mu_j 2^{k_{m_1}+1}] & \text{if } \mu_j < 0,
\end{cases}
\]

i.e. \( \tilde{\mu}_j \) are of the form \( s_j \cdot 2^{-(k_{m_1}+1)} \) with \( |\tilde{\mu}_j| \geq |\mu_j (1 - 2^{-k_{m_1}})| \) and \( |\tilde{\mu}_j - (1 - 2^{-k_{m_1}}) \mu_j | < 2^{-(k_{m_1}+1)} \).

It is easy to see that \( \tilde{\mu}_j = 0 \) if and only if \( \mu_j = 0 \) and so \( \text{supp}_{\tilde{Z}}(\tilde{w}) = \text{supp}_{\tilde{Z}}(w) \). Moreover, for all \( j \), \( |(1 - 2^{-k_{m_1}}) \mu_j - \tilde{\mu}_j| \leq 2^{-(k_{m_1}+1)} \) and so
(1) \[ \| w' - \tilde{w} \| \leq \sum_{j \in [m_1, m_2]} \| (1 - 2^{-k_1}) \mu_j - \tilde{\mu}_j \| \tilde{\zeta}_j \| \leq \sum_{j \in [m_1, m_2]} 2^{-(k_1(j)+1)} \leq 2^{-k_1(m_1)}, \]
and therefore \( \| w' - \tilde{w} \| \leq 2^{-k_1 m_1}, \) since \( m_1 \leq n_1(m_1). \) As \( \| w - w' \| \leq 2^{-k_1 m_1}, \) we obtain \( \| w - \tilde{w} \| \leq 2^{-k_1 m_1 + 1}. \)

It remains to show that \( \tilde{w} \in \mathcal{D}. \) Since for all \( j \in [m_1, m_2] \) we have \( \tilde{\zeta}_j \in \mathcal{D}(I_j), \) it follows that \( \tilde{\zeta}_j = \sum_{i \in I_j} t_i^j 2^{-l_j(k_i+1)} e_i, \) where \( l_j = n_2(j) - n_1(j) + 1 \) is the length of \( I_j \) and \( t_{n_1(j)}^j, t_{n_2(j)}^j \) are both non-zero. Therefore setting \( I = [n_1(m_1), n_2(m_2)], \) we have

(2) \[ \tilde{w} = \sum_{j \in [m_1, m_2]} \tilde{\mu}_j \tilde{\zeta}_j = \sum_{j \in [m_1, m_2]} \tilde{\mu}_j \left( \sum_{i \in I_j} t_i^j 2^{-l_j(k_i+1)} e_i \right) = \sum_{i \in I} \lambda_i e_i \]
where \( \lambda_i = t_i^j 2^{-l_j(k_i+1)} \tilde{\mu}_j \) for all \( i \in I_j \) and \( j \in [m_1, m_2], \) and \( \lambda_i = 0 \) for all \( i \in I \setminus \bigcup_{j \in [m_1, m_2]} I_j. \)

We first show that condition (i) of the definition of \( \mathcal{D} \) is satisfied, that is, \( \lambda_i \in \Lambda(i, l) \) for all \( i \in I, \) where \( l = n_2(m_2) - n_1(m_1) + 1 \) is the length of \( I. \) Since \( 0 \in \Lambda(i, l), \) it suffices to check this for each \( i \in \bigcup_{j \in [m_1, m_2]} I_j. \) So fix \( j \in [m_1, m_2] \) and \( i \in I_j. \) Then

(3) \[ \lambda_i = t_i^j 2^{-l_j(k_i+1)} \tilde{\mu}_j = t_i^j 2^{-l_j(k_i+1)} s_j 2^{-(k_1(j)+1)} = \tau_i^j 2^{-(k_1(j)+1)} \]
where \( \tau_i^j = t_i^j s_j 2^{(l-l_j)(k_i+1)-(k_1(j)+1)}. \) Since \( m_1 < m_2 \) we have \( l > l_j. \) Also \( n_1(j) \leq i \) and hence \( (l - l_j)(k_i+1) - (k_1(j)+1) \geq 0. \) Therefore \( \tau_i^j \in \mathbb{Z}, \) which gives that \( \lambda_i \in \Lambda(i, l). \)

Moreover, since \( \tilde{\mu}_{m_1}, \tilde{\mu}_{m_2}, \tilde{\mu}_{n_1(m_1)}, \tilde{\mu}_{n_2(m_2)} \) are all non-zero we deduce that \( \lambda_{n_1(m_1)} \) and \( \lambda_{n_2(m_2)} \) are also non-zero and so condition (ii) of the definition of \( \mathcal{D} \) is satisfied. Finally, by \( \| \tilde{w} \| \leq \| w' \| + 2^{-k_1(m_1)} \leq 1 \) and so condition (iii) is fulfilled. We conclude that \( \tilde{w} \in \mathcal{D}, \) and the proof of the claim is complete.

We continue with the proof of the proposition. Let \( \tilde{Z} = (\tilde{z}_j)_j \) in \( \mathcal{B}_\alpha^\infty \) and let \( Z = (z_j)_j \) where \( z_j = \tilde{z}_{2j} + \tilde{z}_{2j+1} \) for all \( j. \) Pick \( W = (w_i)_i \) in \( \mathcal{B}_\beta^\infty(\tilde{Z}). \) Then for each \( i \) there exist \( m_i^1 < m_i^2 \) and scalars \( (\mu_j)_j \) such that \( w_i = \sum_{j \in [m_i^1, m_i^2]} \mu_j \tilde{z}_j \in \mathcal{B}_\alpha \) and \( \mu_{m_i^1}, \mu_{m_i^2} \) are both non-zero. By the claim, for each \( i \) there exist scalars \( (\tilde{\mu}_j)_j \) such that \( \tilde{w}_i = \sum_{j \in [m_i^1, m_i^2]} \tilde{\mu}_j \tilde{z}_j \in \mathcal{D} \) and \( \| w_i - \tilde{w}_i \| \leq 2^{-k_{m_i^2}+1} \leq 2^{-k_i+1} \leq \delta_i. \) We set \( \tilde{W} = (\tilde{w}_i)_i; \) then \( \tilde{W} \in \mathcal{B}_\alpha^\infty(\tilde{Z}) \) and \( \text{dist}(\tilde{W}, W) \leq \Delta. \) Hence \( \mathcal{B}_\beta^\infty(\tilde{Z}) \subseteq (\mathcal{B}_\alpha^\infty(\tilde{Z}))_\Delta \) and the proof is complete. 

It is easy to see that

\[ \rho(x, y) = \| x - y \| + \frac{1}{\| x \|} - \frac{1}{\| y \|}, \quad x, y \in \mathcal{X} \setminus \{ 0 \}, \]
is an equivalent metric on \((X \setminus \{0\}, \| \cdot \|)\) and that the product topology on \((X \setminus \{0\}, \rho)^N\) makes \(B^\infty_X\) a Polish space.

**Lemma 15.** Let \(F\) be an analytic subset of \(B^\infty_X\) and \(\Delta = (\delta_n)_n\) be a sequence of positive real numbers. Then

(i) \(F_\Delta\) is analytic in \(B^\infty_X\).
(ii) For every countable \(D \subseteq X\), \(F_\Delta \cap B^\infty_D\) is analytic in \(D^N\) (where \(D\) is endowed with the discrete topology).

**Proof.** (i) It is easy to see that \(Q = \{(U, V) : \text{dist}(U, V) \leq \Delta\}\) is closed in \(B^\infty_X \times B^\infty_X\). Let proj\(_1\) (resp. proj\(_2\)) be the projection of \(B^\infty_X \times B^\infty_X\) onto the first (resp. second) coordinate. Then \(F_\Delta = \text{proj}_1[Q \cap (B_X \times F)] = \text{proj}_1[Q \cap \text{proj}_2^{-1}(F)]\).

(ii) Let \(I : D^N \rightarrow X^N\) be the identity map. Then \(I\) is clearly continuous and \(F_\Delta \cap B^\infty_D = I^{-1}(F_\Delta)\). \(\blacksquare\)

**Theorem 16** (W. T. Gowers). Let \(X\) be a normed linear space with a basis and let \(F \subseteq B^\infty_{B_X}\) be an analytic family of block sequences in the unit ball \(B_X\) of \(X\). Then for every \(\Delta > 0\) there exists a block sequence \(Z \in B^\infty_X\) such that either \(B^\infty_{B_X}(Z) \cap F = \emptyset\), or player II has a winning strategy in Gowers’ game \(G_X(Z)\) for \(F_\Delta\).

**Proof.** Let \((e_n)_n\) be a normalized basis for \(X\) with constant \(C\). Let \(\Delta' = (\delta'_n)_n\) be a sequence of positive real numbers such that \(\delta'_0 \leq 1\), \(\delta'_n \leq \delta_n\), and \(\sum_{i>n} \delta'_i \leq \delta_n\). By Proposition 14, there is \(D \subseteq X\) with \((e_n)_n \in B^\infty_{B_D}\) satisfying (D1)-(D3) for \(\Delta'/10C\). Let also \(G = F_{\Delta'/10C} \cap B^\infty_D\). By Lemma 15, \(G\) is analytic in \(D^N\), and applying Theorem 1, we obtain a block sequence \(Z \in B^\infty_D\) such that either \(B^\infty_{B_X}(Z) \cap G = \emptyset\), or player II has a winning strategy in \(G_D(Z)\) for \(\Delta'\). Choose \(Z \in B^\infty_X\) such that \(B^\infty_X(Z) \subseteq (B^\infty_{\tilde{Z}}(\tilde{Z}))_{\Delta'/10C}\). From Lemmas 12 and 13 either \(B^\infty_{B_X}(Z) \cap F = \emptyset\), or player II has a winning strategy in Gowers’ game \(G_X(Z)\) for \(F_{\Delta'}\), and so (as \(\Delta' \leq \Delta\)) for \(F_\Delta\) as well. \(\blacksquare\)

**5. A Ramsey consequence on \(k\)-tuples of block bases.** The main goal of this section is to prove Theorem 2. First we need to do some preliminary work and introduce some notation. Fix a positive integer \(k \geq 2\). For each \(0 \leq i \leq k - 1\) and every infinite subset \(L = \{l_0 < l_1 < \cdots\}\) of \(\mathbb{N}\) we set \(L_i(\text{mod } k) = \{kn+i : n \in \mathbb{N}\}\) and we define

\[
([L]^\infty)_0 = \prod_{i=0}^{k-1} ([L_i(\text{mod } k)]^\infty) = \{(L_i)_{i=0}^{k-1} \in ([L]^\infty) : \forall i L_i \subseteq L_i(\text{mod } k)\}.
\]

Notice that \(([L]^\infty)_0\) is not hereditary, that is, generally \(([L']^\infty)_0 \not\subseteq ([L]^\infty)_0\) for \(L' \subseteq L\). Let also

\[
([L]^\infty)_{\perp} = \{(L_i)_{i=0}^{k-1} \in ([L]^\infty) : \forall i \neq j L_i \cap L_j = \emptyset\}.
\]
We have the following elementary lemma which relates the above types of products.

**Lemma 17.** Let \( N = \{(2n+1)k : n \in \mathbb{N}\} \). Then
\[
([N]^{\infty})_k \subseteq \bigcup_{L \in [N]^\infty} ([L]^{\infty})_k.
\]

**Proof.** Let \((M_i)_{i=0}^{k-1} \in ([N]^{\infty})_k\). Let \( M = \bigcup_{i=0}^{k-1} M_i \) and for each \( m \in M \) define the interval \( I_m = [m-i_m, m-i_m+k-1] \) of \( \mathbb{N} \) where \( i_m \) is the unique natural number \( i \) such that \( m \in M_i \). Notice that the length of each \( I_m \) is \( k \) while the length of an interval with unequal endpoints in \( N \) is at least \( 2k+1 \). Hence \( I_{m_1} \cap I_{m_2} = \emptyset \) for \( m_1 \neq m_2 \), and \( I_m \cap N = \{m\} \) for all \( m \in M \).

Let \( L = \bigcup_{m \in M} I_m \). We claim that \((M_i)_{i=0}^{k-1} \in ([N]^{\infty})_k\). Indeed, let \( L = (l_n)_n \) be the increasing enumeration of \( L \). For each \( 0 \leq i \leq k-1 \) and \( m \in M \) let \( I_m(i) = m-i_m+i \) be the \( i \)th element of \( I_m \). Since \((I_m)_{m \in M}\) is a sequence of pairwise disjoint intervals of \( \mathbb{N} \) of length \( k \), we easily see that \( L_i_{(\text{mod}k)} = \bigcup_{m \in M} I_m(i) \). Fix \( 0 \leq i \leq k-1 \). Then \( m \in M_i \) if and only if \( i_m = i \) if and only if \( I_m(i) = m \). Hence \( M_i = \bigcup_{m \in M_i} \{I_m(i)\} \subseteq \bigcup_{m \in M} \{I_m(i)\} = L_i_{(\text{mod}k)} \).

The above notation is easily extended to block sequences in the unit ball \( B_X \) of a Banach space \( X \) as follows. For every \( Z \in B_{B_X}^{\infty} \) let
\[
(B_{B_X}^{\infty}(Z))_k = \{(Z_i)_{i=0}^{k-1} \in (B_{B_X}^{\infty})^k : \forall i \ Z_i \preceq Z|_{N_i_{(\text{mod}k)}}\},
\]
and generally for \( L \in [N]^{\infty} \), set
\[
(B_{B_X}^{\infty}(Z|_L))_k = \{(Z_i)_{i=0}^{k-1} \in (B_{B_X}^{\infty})^k : \forall i \ Z_i \preceq Z|_{L_i_{(\text{mod}k)}}\}.
\]
The next lemma is an immediate consequence of Lemma 17.

**Lemma 18.** Let \( Z \in B_{B_X}^{\infty} \) and \( N = \{(2n+1)k : n \in \mathbb{N}\} \). Then
\[
(B_{B_X}^{\infty}(Z|_N))_k \subseteq \bigcup_{L \in [N]^{\infty}} (B_{B_X}^{\infty}(Z|_L))_k.
\]

For a family \( \mathcal{F} \subseteq (B_{B_X}^{\infty})^k \) let
\[
\mathcal{F}^\mathcal{G} = \{Z \in B_{S_X}^{\infty} : \mathcal{G} \cap (B_{B_X}^{\infty}(Z))_k \neq \emptyset\},
\]
where \( S_X \) is the unit sphere of \( X \).

**Lemma 19.** If \( \mathcal{G} \) is analytic in \( (B_{B_X}^{\infty})^k \), then \( \mathcal{F}^\mathcal{G} \subseteq B_{S_X}^{\infty} \) is analytic in \( B_{B_X}^{\infty} \).

**Proof.** Let \( \mathcal{K} = \{(Z, (V_i)_{i=0}^{k-1} \in B_{S_X}^{\infty} \times (B_{B_X}^{\infty})^k : (V_i)_{i=0}^{k-1} \in (B_{B_X}^{\infty}(Z))_k\}. \)
Then \( \mathcal{K} \) is a closed subset of \( B_{B_X}^{\infty} \times (B_{B_X}^{\infty})^k \) and \( \mathcal{F}^\mathcal{G} = \text{proj}_1[(B_{B_X}^{\infty} \times \mathcal{G}) \cap \mathcal{K}] \).

**Proof of Theorem 3** Let \((e_n)_n\) be a normalized basis of \( X \) with basis constant \( C \). Choose \( \Delta' = (\delta'_n)_n \) such that \( 0 < \delta'_n \leq (4C)^{-1}\delta_n \) and \( \sum_{j=n+1}^{\infty} \delta'_j \leq \delta_n \). By Lemma 19 \( \mathcal{F}^\mathcal{G} \) is an analytic subset of \( B_{B_X}^{\infty} \), and by Theorem 16 there is a block subsequence \( Z = (z_n)_n \) such that either
Hence it suffices to see that \( n \in N \), where \( N = \{(2n + 1)k : n \in N \} \). We claim that \( Y \) satisfies the conclusion of the theorem.

Indeed, if \( B_{B_X}^\infty (Z) \cap F^\delta = \emptyset \) then \( F \cap (B_{B_X}^\infty (Z'))_0 = \emptyset \) for all \( Z' \in B_{B_X}^\infty (Z) \). In particular, \( F \cap (B_{B_X}^\infty (Z|L))_0 = \emptyset \) for all \( L \in [N]^\infty \), which by Lemma \ref{lem:18} gives that \( F \cap (B_{B_X}^\infty (Y))_1 = \emptyset \).

So assume that player II has a winning strategy in Gowers’ game \( G_X(Z) \) for \( (F^\delta)_{\Delta'} \). Since \( Y = Z|N \) the same holds for the game \( G_X(Y) \). Fix \( (U_i)_{i=0}^{k-1} \in (B_{B_X}^\infty (Y))^k \). We have to show that there exists \( (V_i)_{i=0}^{k-1} \in (B_{B_X}^\infty (Z))^k \) such that \( V_i \preceq U_i \) and \( (V_i)_{i=0}^{k-1} \in F_{\Delta} \). Consider a run of the game such that in the \( n \)-th move player I plays \( U_i \), where \( n = i \pmod{k} \). Then player II succeeds in constructing a block sequence \( V = (v_n)_n \) in \( (F^\delta)_{\Delta'} \) such that \( v_n \in U_i \) for all \( n = i \pmod{k} \). Choose \( W \in F^\delta \) with \( \text{dist}(V, W) \leq \Delta' \) and for each \( i, W_i \preceq W \mid N_i \pmod{k} \) such that \( (W_i)_{i=0}^{k-1} \in (B_{B_X}^\infty (W))^k \cap F \). Let \( W = (w_n)_n \) and \( W_i = (w^n_i)_n \). Then for each \( i = 1, \ldots, k \) there is a block sequence \( (F_i^n)_n \) of finite subsets of \( N_i \pmod{k} \) and a sequence \( (\lambda_j)_j \) of scalars such that \( w^n_i = \sum_{j \in F^n_i} \lambda_j w_j \) for all \( i \) and \( n \). We set \( v^n_i = \sum_{j \in F^n_i} \lambda_j v_j \) and \( V_i = (v^n_i)_n \). Then \( V_i \preceq V \mid N_i \pmod{k} \preceq U_i \) for all \( i \). It remains to show that \( (V_i)_{i=0}^{k-1} \in F_{\Delta} \). For this it suffices to see that \( \text{dist}(V_i, W_i) \leq \Delta \) for all \( i \). Indeed, fix \( 0 \leq i \leq k - 1 \) and \( n \in N \). Since \( \|w^n_i\| \leq 1 \) and \( \|w_j\| = 1 \), we get \( |\lambda_j| \leq 2C \) and therefore

\[
\|v^n_i - w^n_i\| \leq \sum_{j \in F^n_i} |\lambda_j| \|v_j - w_j\| \leq 2C \sum_{j \in F^n_i} \delta_j \leq 4C\delta_n \leq \delta_n.
\]

Hence \( (U_i)_{i=0}^{k-1} \in (F_{\Delta})^+ \). 

6. Comments. 1. C. Rosendal \cite{Rosendal21} proves a Ramsey dichotomy between winning strategies in Gowers’ game and winning strategies in the infinite asymptotic game. By appropriately modifying his argument, one can check that the proof in \cite{Rosendal21} works in the more general setting of a linear space \( X \) of countable dimension over the field of reals provided that both games are restricted to a countable subset \( D \) of \( X \) with property (D1) stated in the introduction. This modification can be used to derive an alternative proof of Theorem \textbf{1}.

2. Theorem 2 is actually an extension of the following fact concerning pairs of infinite subsets of \( N \). Given an analytic family \( F \subset [N]^\infty \times [N]^\infty \) there is an infinite subset \( L \) of \( N \) such that either all disjoint pairs of infinite subsets of \( L \) belong to the complement of \( F \), or for every \( (L_1, L_2) \in [L]^\infty \times [L]^\infty \), there is \( (L'_1, L'_2) \in \) such that \( L'_i \subseteq L_i \) for all \( i = 1, 2 \). To see this, consider the map \( \Phi : M \rightarrow (M_0, M_1) \) where if \( M = \{m_i\}_i \) is the increasing enumeration of \( L \) then \( M_0 = \{m_i\}_i \) even and \( M_1 = \{m_i\}_i \) odd. Then apply Silver’s theorem (see
for the family $\Phi^{-1}(\mathcal{F}^\uparrow)$ where $\mathcal{F}^\uparrow = \{(L, M) : \exists (L', M') \in \mathcal{F}$ with $L' \subseteq L$ and $M' \subseteq M\}$. It is easy to see that keeping the “half” of the monochromatic set, the result follows. Also, applying K. Milliken’s theorem [16], one can derive an analogue of the above result for pairs of block sequences of finite subsets of $\mathbb{N}$.

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References


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