BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 58, No. 1, 2010

COMBINATORICS

A Discretized Approach to W. T. Gowers' Game

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Summary. We give an alternative proof of W. T. Gowers' theorem on block bases by reducing it to a discrete analogue on specific countable nets. We also give a Ramsey type result on k-tuples of block sequences in a normed linear space with a Schauder basis.

1. Introduction. W. T. Gowers in [11] (see also [10] and [12]) proved a fundamental Ramsey-type theorem for block bases in Banach spaces which led to important discoveries in the geometry of Banach spaces. By now there are several approaches to Gowers' theorem (see [1, 2, 3, 4, 14, 21]; also in [7, 15, 18] there are direct proofs of Gowers' dichotomy, and in [6, 8, 19, 22, 24] extensions and further applications).

Our aim in this note is to prove a discrete analogue of Gowers' theorem which is free of approximations. To state our results we will need the following notation. Let $\mathfrak{X}$ be a real linear space with an infinite countable Hamel basis $(e_n)_n$ (actually the field over which the linear space $\mathfrak{X}$ is defined plays no role in the arguments; it is only for the sake of convenience that we will assume that $\mathfrak{X}$ is a real linear space). For a subset $A \subseteq \mathfrak{X}$ we denote by $\langle A\rangle$ the linear span of A. Let $\mathfrak{D}$ be a subset of $\mathfrak{X}$. We denote by $\mathcal{B}^\infty_{\mathfrak{D}}$ the set of all block sequences $(x_n)_n$ with $x_n \in \mathfrak{D}$ for all n. For a block sequence $Z \in \mathcal{B}^\infty_{\mathfrak{D}}$ let $\mathcal{B}^\infty_{\mathfrak{D}}(Z)$ be the set of all block sequences of $\mathcal{B}^\infty_{\mathfrak{D}}$ which are block subsequences of Z.

Assume that $\mathcal{B}^\infty_{\mathfrak{D}}$ is non-empty and let $Z \in \mathcal{B}^\infty_{\mathfrak{D}}$ and $\mathcal{G} \subseteq \mathcal{B}^\infty_{\mathfrak{D}}$. We define *the $\mathfrak{D}$-Gowers' game in Z*, denoted by $G_{\mathfrak{D}}(Z)$, as follows. Player I starts the game by choosing $W_0 \in \mathcal{B}^\infty_{\mathfrak{D}}(Z)$ and player II responds with a vector $w_0 \in \langle W_0\rangle \cap \mathfrak{D}$. Then player I chooses $W_1 \in \mathcal{B}^\infty_{\mathfrak{D}}(Z)$ and player II chooses

2010 *Mathematics Subject Classification*: 05D10, 46B03.
Key words and phrases: Ramsey theory, games in Banach spaces.

a vector $w_1 \in \langle W_1 \rangle \cap \mathfrak{D}$ and so on. Player II wins the game if the sequence $(w_0, w_1, \ldots)$ belongs to $\mathcal{G}$.

Suppose that $\mathfrak{D}$ is a subset of $\mathfrak{X}$ with the following properties:

($\mathfrak{D}$1) (*Asymptotic property*) $\mathfrak{D} \cap \langle (e_i)_{i \geq n} \rangle \neq \emptyset$ for all $n \in \mathbb{N}$.

($\mathfrak{D}$2) (*Finitization property*) $\mathfrak{D} \cap \langle (e_i)_{i<n} \rangle$ is finite for all $n \in \mathbb{N}$.

Property ($\mathfrak{D}$1) simply means that the set $\mathcal{B}^\infty_{\mathfrak{D}}$ of all block sequences is nonempty. Property ($\mathfrak{D}$2) implies that $\mathfrak{D}$ is countable. Hence, if we endow $\mathfrak{D}$ with the discrete topology, the space $\mathfrak{D}^{\mathbb{N}}$ of all infinite countable sequences in $\mathfrak{D}$ equipped with the product topology is a Polish space.

We can now state our first main result.

Theorem 1. *Let $\mathfrak{X}$ be a real linear space with a countable Hamel basis $(e_n)_n$ and let $\mathfrak{D} \subseteq \mathfrak{X}$ have properties* ($\mathfrak{D}$1) *and* ($\mathfrak{D}$2)*. Also let $\mathcal{G} \subseteq \mathcal{B}^\infty_{\mathfrak{D}}$ be an analytic subset of $\mathfrak{D}^{\mathbb{N}}$. Then for every $U \in \mathcal{B}^\infty_{\mathfrak{D}}$ there exists $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(U)$ such that either $\mathcal{B}^\infty_{\mathfrak{D}}(Z) \cap \mathcal{G} = \emptyset$, or player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$.*

While discrete in nature, Theorem 1 can be used to derive Gowers' original result provided that $\mathfrak{D}$ has an additional property (see Section 4).

Our second main result concerns k-tuples of block sequences in normed linear spaces with a Schauder basis. Precisely, let $\mathfrak{X}$ be a real normed linear space with a Schauder basis $(e_n)_n$. We denote by $\mathcal{B}^\infty_{\mathfrak{X}}$ the set of block sequences in $\mathfrak{X}$ and by $\mathcal{B}^\infty_{B_{\mathfrak{X}}}$ the set of all block sequences in the unit ball $B_{\mathfrak{X}}$ of $\mathfrak{X}$. Two block sequences $Z_1 = (z^1_n)_n$ and $Z_2 = (z^2_n)_n$ in $\mathcal{B}^\infty_{\mathfrak{X}}$ are said to be *disjointly supported* if $\operatorname{supp} z^1_n \cap \operatorname{supp} z^2_m = \emptyset$ for all m, n. For a positive integer $k \geq 2$ and $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$, the set of all k-tuples of pairwise disjointly supported block subsequences of Z in $B_{\mathfrak{X}}$ will be denoted by $(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z))^k_\perp$. Also, for a family $\mathfrak{F} \subseteq (\mathcal{B}^\infty_{\mathfrak{X}})^k$ of k-tuples of block sequences in $\mathfrak{X}$, the *upward closure* of $\mathfrak{F}$ is defined to be the set

$$\mathfrak{F}^{\uparrow} = \big\{ (U_i)_{i=0}^{k-1} \in (\mathcal{B}^\infty_{\mathfrak{X}})^k : \exists (V_i)_{i=0}^{k-1} \in \mathfrak{F} \text{ such that } \forall i \; V_i \text{ is a block subsequence of } U_i \big\}.$$

If $\Delta = (\delta_n)_n$ is a sequence of positive reals, then the *Δ-expansion of* $\mathfrak{F}$ is defined to be the set

$$\mathfrak{F}_\Delta = \big\{ (U_i)_{i=0}^{k-1} \in (\mathcal{B}^\infty_{\mathfrak{X}})^k : \exists (V_i)_{i=0}^{k-1} \in \mathfrak{F} \text{ such that } \forall i \; \operatorname{dist}(U_i, V_i) \leq \Delta \big\}.$$

We prove the following.

Theorem 2. *Let $\mathfrak{X}$ be a real normed linear space with a Schauder basis, $k \geq 2$, and $\mathfrak{F}$ an analytic subset of $(\mathcal{B}^\infty_{B_{\mathfrak{X}}})^k$. Then for every sequence $\Delta = (\delta_n)_n$ of positive real numbers there is $Y \in \mathcal{B}^\infty_{\mathfrak{X}}$ such that either $(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Y))^k_\perp \cap \mathfrak{F} = \emptyset$, or $(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Y))^k \subseteq (\mathfrak{F}_\Delta)^{\uparrow}$.*

In the above theorem the topology of $\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}$ is the one induced by the product of the norm topology. Theorem 2 applied for $k = 2$ and the family

$$\mathfrak{F} = \{(U_1, U_2) \in (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}})^2 : U_1, U_2 \text{ are } C\text{-equivalent}\},$$

where $C \geq 1$ is a constant, yields Gowers' second dichotomy (see Lemma 7.3 in [11]).

2. Notation. Let $\mathfrak{X}$ be a real linear space with an infinite countable Hamel basis $(e_n)_n$. For two non-zero vectors x, y in $\mathfrak{X}$, we write $x < y$ if $\max \operatorname{supp} x < \min \operatorname{supp} y$ (where $\operatorname{supp} x$ is the *support* of x, i.e. if $x = \sum_n \lambda_n e_n$ then $\operatorname{supp} x = \{n \in \mathbb{N} : \lambda_n \neq 0\}$). A sequence $(x_n)_n$ of vectors in $\mathfrak{X}$ is called a *block sequence* (or *block basis*) if $x_n < x_{n+1}$ for all n.

Capital letters (such as $U, V, Y, Z, \ldots$) refer to infinite block sequences, and overlined lower case letters (such as $\overline{u}, \overline{v}, \overline{y}, \overline{z}, \ldots$) to finite block sequences. We write $Y \preceq Z$ to denote that Y is a *block subsequence* of Z, that is, $Y = (y_n)_n$, $Z = (z_n)_n$ are block sequences and $y_n \in \langle (z_i)_i \rangle$ for all n. The notations $\overline{y} \preceq Z$ and $\overline{y} \preceq \overline{z}$ are defined analogously. For $\overline{x} = (x_n)_{n=0}^{k}$ and $Y = (y_n)_n$ we write $\overline{x} < Y$ if $x_k < y_0$. For $\overline{x} < Y$, $\overline{x}^\frown Y$ denotes the block sequence $(z_n)_n$ that starts with the elements of $\overline{x}$ and continues with those of Y. Also for $\overline{x} < \overline{y}$, the finite block sequence $\overline{x}^\frown \overline{y}$ is similarly defined. For a block sequence $Z = (z_n)_n$ and an infinite subset L of $\mathbb{N}$ we set $Z|_L = (z_n)_{n \in L}$. Also $Z|_k = (z_n)_{n=0}^{k-1}$ for $k \in \mathbb{N}$ ($Z|_0 = \emptyset$ for $k = 0$).

Let $\mathfrak{D} \subseteq \mathfrak{X}$. We denote by $\mathcal{B}^{\infty}_{\mathfrak{D}}$ (resp. $\mathcal{B}^{<\infty}_{\mathfrak{D}}$) the set of all infinite (resp. finite) block sequences $(x_n)_n$ with $x_n \in \mathfrak{D}$ for all n. The set of all infinite (resp. finite) block sequences in $\mathfrak{X}$ is denoted by $\mathcal{B}^{\infty}_{\mathfrak{X}}$ (resp. $\mathcal{B}^{<\infty}_{\mathfrak{X}}$). For $Z \in \mathcal{B}^{\infty}_{\mathfrak{X}}$ we set $\mathcal{B}^{\infty}_{\mathfrak{D}}(Z) = \{Y \in \mathcal{B}^{\infty}_{\mathfrak{D}} : Y \preceq Z\}$ and $\mathcal{B}^{<\infty}_{\mathfrak{D}}(Z) = \{\overline{y} \in \mathcal{B}^{<\infty}_{\mathfrak{D}} : \overline{y} \preceq Z\}$. Similarly $\mathcal{B}^{<\infty}_{\mathfrak{D}}(\overline{z}) = \{\overline{y} \in \mathcal{B}^{<\infty}_{\mathfrak{D}} : \overline{y} \preceq \overline{z}\}$ for $\overline{z} \in \mathcal{B}^{<\infty}_{\mathfrak{X}}$. For a block sequence $Z \in \mathcal{B}^{\infty}_{\mathfrak{D}}$, we set $\langle Z \rangle_{\mathfrak{D}} = \langle Z \rangle \cap \mathfrak{D}$ where $\langle Z \rangle$ is the linear span of Z.

3. Discretization of Gowers' game. Throughout this section, $\mathfrak{X}$ is a real linear space with countable Hamel basis $(e_n)_n$ and $\mathfrak{D}$ is a subset of $\mathfrak{X}$ with properties ($\mathfrak{D}$1) and ($\mathfrak{D}$2) as stated in the Introduction. Notice that ($\mathfrak{D}$2) also gives that for every $U = (u_i)_i \in \mathcal{B}^{\infty}_{\mathfrak{D}}$ and $n \in \mathbb{N}$, the set $\mathcal{B}^{<\infty}_{\mathfrak{D}}((u_i)_{i<n})$ is finite.

3.1. Admissible families of $\mathfrak{D}$-pairs. The aim of this subsection is to review some methods of handling diagonalizations (see also [11], [20]). A $\mathfrak{D}$-*pair* is a pair $(\overline{x}, Y)$ where $\overline{x} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}$ and $Y \in \mathcal{B}^{\infty}_{\mathfrak{D}}$. For $U \in \mathcal{B}^{\infty}_{\mathfrak{D}}$, a family $\mathcal{P} \subseteq \mathcal{B}^{<\infty}_{\mathfrak{D}}(U) \times \mathcal{B}^{\infty}_{\mathfrak{D}}(U)$ is called *an admissible family of $\mathfrak{D}$-pairs in U* if it has the following properties:

($\mathcal{P}$1) (*Heredity*) If $(\overline{x}, Y) \in \mathcal{P}$ and $Z \in \mathcal{B}^{\infty}_{\mathfrak{D}}(Y)$ then $(\overline{x}, Z) \in \mathcal{P}$.

($\mathcal{P}2$) (*Cofinality*) For every $(\overline{x}, Y) \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(U) \times \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$, there is $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ such that $(\overline{x}, Z) \in \mathcal{P}$.

For simplicity, when we write "pair" we will always mean a $\mathfrak{D}$-pair. It will often happen that an admissible family of pairs has one more property:

($\mathcal{P}3$) If $(\overline{x}, Y) \in \mathcal{P}$, $\overline{x} < Y$ and $k = \min\{m : \overline{x} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}((u_i)_{i=1}^m)\}$ then $(\overline{x}, \overline{y}^\frown Y) \in \mathcal{P}$ for every $\overline{y} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}((u_i)_{i=1}^k)$.

The next lemma follows by a standard diagonalization argument.

Lemma 3. *Let $U \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ and let $\mathcal{P}$ be an admissible family of pairs in U. Then there is $W \in \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$ such that $(\overline{w}, Y) \in \mathcal{P}$ for all $\overline{w} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(W)$ and all $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$ with $\overline{w} < Y$. If in addition $\mathcal{P}$ satisfies ($\mathcal{P}3$) then $(\overline{w}, W) \in \mathcal{P}$ for all $\overline{w} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(W)$.*

3.2. The discrete Gowers' game. Given $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ and a family of infinite block sequences $\mathcal{G} \subseteq \mathcal{B}_{\mathfrak{D}}^{\infty}$, we define the $\mathfrak{D}$-*Gowers' game*, $G_{\mathfrak{D}}(Y)$, as follows. Player I starts the game by choosing $Z_0 \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ and player II responds with a vector $z_0 \in \langle Z_0 \rangle_{\mathfrak{D}}$. Then player I chooses $Z_1 \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ and player II chooses a vector $z_1 \in \langle Z_1 \rangle_{\mathfrak{D}}$ with $z_0 < z_1$, and so on. More generally, for a finite block sequence $\overline{x} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}$ and $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ the game $G_{\mathfrak{D}}(\overline{x}, Y)$ is defined as above with the additional condition that player II in the first move chooses $z_0 > \overline{x}$. Clearly $G_{\mathfrak{D}}(\emptyset, Y)$ is identical to $G_{\mathfrak{D}}(Y)$. We will say that player II *wins the game* $G_{\mathfrak{D}}(\overline{x}, Y)$ *for* $\mathcal{G}$ if the block sequence $\overline{x}^\frown(z_0, z_1, \ldots)$ belongs to $\mathcal{G}$.

We will basically follow the classical Galvin–Prikry terminology (cf. [9], [5]) in the context of Gowers' game. More precisely, for $\overline{x} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}$, $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ and $\mathcal{G} \subset \mathcal{B}_{\mathfrak{D}}^{\infty}$ we say that Y $\mathcal{G}$-*accepts* $\overline{x}$ if player II has a winning strategy in $G_{\mathfrak{D}}(\overline{x}, Y)$ for $\mathcal{G}$, while Y $\mathcal{G}$-*rejects* $\overline{x}$ if no $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ which $\mathcal{G}$-accepts $\overline{x}$. We also say that Y $\mathcal{G}$-*decides* $\overline{x}$ if it either $\mathcal{G}$-accepts or $\mathcal{G}$-rejects it.

Notice that if $\overline{x} = \emptyset$ then to say that "Y $\mathcal{G}$-accepts the empty sequence" means that player II has a winning strategy in $G_{\mathfrak{D}}(Y)$ for $\mathcal{G}$. Similarly the statement that "Y $\mathcal{G}$-rejects the empty sequence" is equivalent to saying that for no $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ does player II have a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$. The following lemma is easily verified.

Lemma 4. *For every $U \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ and every $\mathcal{G} \subseteq \mathcal{B}_{\mathfrak{D}}^{\infty}$, the family*

$$\mathcal{P} = \{(\overline{x}, Y) \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(U) \times \mathcal{B}_{\mathfrak{D}}^{\infty}(U) : Y \ \mathcal{G}\text{-decides } \overline{x}\}$$

is an admissible family of pairs in U with property ($\mathcal{P}3$).

Actually the family $\mathcal{P}$ of the above lemma satisfies the following condition stronger than ($\mathcal{P}3$): If $(\overline{x}, Y) \in \mathcal{P}$ and $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ are such that $Z|_{[n,\infty)} \preceq Y$ for some $n \in \mathbb{N}$, then $(\overline{x}, Z) \in \mathcal{P}$.

For the sake of simplicity, in the following we will omit the letter $\mathcal{G}$ in front of "accepts", "rejects" and "decides". The next lemma is a consequence of Lemmas 4 and 3.

LEMMA 5. *For every $U \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ there is $W \in \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$ such that W decides all $\overline{w} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(W)$.*

The crucial point where the above notions of "accept-reject" essentially differ from the original ones reveals itself in the next lemma. Here the notion of the winning strategy replaces successfully the traditional pigeonhole principle.

LEMMA 6. *Let $W \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ decide all $\overline{w} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(W)$ and assume that it rejects some $\overline{w}_0 \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$. Then for every $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$ there is $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ such that W rejects $\overline{w}_0{}^\frown z$ for every $z \in \langle Z\rangle_{\mathfrak{D}}$ with $\overline{w}_0 < z$.*

Proof. If the conclusion is false then there is $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$ such that for every $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(Y)$ there is $z \in \langle Z\rangle_{\mathfrak{D}}$ with $\overline{w}_0 < z$ such that W accepts $\overline{w}_0{}^\frown z$. It is easy to see that this means that player II has a winning strategy in $G_{\mathfrak{D}}(\overline{w}_0, Y)$ for $\mathcal{G}$, and thus Y accepts $\overline{w}_0$. But this is a contradiction since $Y \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$ and W rejects $\overline{w}_0$. ∎

LEMMA 7. *For every $U \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ there exists $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$ such that either Z rejects all $\overline{z} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(Z)$, or player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$.*

Proof. By Lemma 5 there is $W \in \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$ such that W decides all $\overline{w} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(W)$. If W accepts the empty sequence then we readily have the second alternative of the conclusion for $Z = W$. In the opposite case consider the following family in $\mathcal{B}_{\mathfrak{D}}^{<\infty}(W) \times \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$:

$$\mathcal{P} = \{(\overline{x}, Y) : \text{either } W \text{ accepts } \overline{x}, \quad \text{or } \forall y \in \langle Y\rangle_{\mathfrak{D}} \text{ with } \overline{x} < y,\ W \text{ rejects } \overline{x}^\frown y\}.$$

Using Lemma 6 we easily verify that $\mathcal{P}$ is an admissible family in W which also satisfies ($\mathcal{P}3$). Hence by Lemma 3 there is $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(W)$ with $(\overline{z}, Z) \in \mathcal{P}$ for every $\overline{z} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(Z)$. By our assumption W rejects the empty sequence. Since $(\emptyset, Z) \in \mathcal{P}$ we infer that W, and so Z, rejects all $z \in \langle Z\rangle_{\mathfrak{D}}$. By induction on the length of finite block sequences in $\mathcal{B}_{\mathfrak{D}}^{<\infty}(Z)$, it is easily shown that Z rejects all $\overline{z} \in \mathcal{B}_{\mathfrak{D}}^{<\infty}(Z)$. ∎

We have finally arrived at our first stop which is an analog of the well known result of Nash-Williams ([17]). Consider the set $\mathfrak{D}$ with the discrete topology and $\mathfrak{D}^{\mathbb{N}}$ with the product topology.

LEMMA 8. *Let $\mathcal{G} \subseteq \mathcal{B}_{\mathfrak{D}}^{\infty}$ be open in $\mathfrak{D}^{\mathbb{N}}$. Then for every $U \in \mathcal{B}_{\mathfrak{D}}^{\infty}$ there exists $Z \in \mathcal{B}_{\mathfrak{D}}^{\infty}(U)$ such that either $\mathcal{B}_{\mathfrak{D}}^{\infty}(Z) \cap \mathcal{G} = \emptyset$, or player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$.*

Proof. By Lemma 7 we can find $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(U)$ such that either Z rejects all $\overline{z} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(Z)$, or player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$. Hence it suffices to show that the first alternative gives $\mathcal{B}^\infty_{\mathfrak{D}}(Z) \cap \mathcal{G} = \emptyset$. Indeed, let $W = (w_n)_n \in \mathcal{B}^\infty_{\mathfrak{D}}(Z)$. Then for all k, Z rejects $W|_k = (w_n)_{n<k}$. Therefore there is some $Z_k \in \mathcal{B}^\infty_{\mathfrak{D}}(Z)$ with $W|_k < Z_k$ such that $W|_k^\frown Z_k \notin \mathcal{G}$. Since the sequence $(W|_k^\frown Z_k)_k$ converges in $\mathfrak{D}^{\mathbb{N}}$ to W and the complement of $\mathcal{G}$ is closed, we conclude that $W \notin \mathcal{G}$. ∎

We now pass to the case of an analytic family $\mathcal{G}$. First let us state some basic definitions (cf. [13]). Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences in $\mathbb{N}$ and let $\mathcal{N}$ be the Baire space, i.e. the space of all infinite sequences in $\mathbb{N}$ with the topology generated by the sets $\mathcal{N}_s = \{\sigma \in \mathcal{N} : \exists n \text{ with } \sigma|n = s\}$, $s \in \mathbb{N}^{<\mathbb{N}}$. A subset of a Polish space X is called *analytic* if it is the image of a continuous function from $\mathcal{N}$ into X.

For the next lemmas we fix the following:

(a) a family $(\mathcal{G}^s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of $\mathcal{B}^\infty_{\mathfrak{D}}$ such that $\mathcal{G}^s = \bigcup_n \mathcal{G}^{s^\frown n}$ for all s,
(b) a bijection $\varphi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ such that $\varphi(\emptyset) = 0$ and $\varphi(s^\frown n) > \varphi(s)$ for all s, n.

For each $\overline{x}$ in $\mathcal{B}^{<\infty}_{\mathfrak{D}}$ we set $s_{\overline{x}}$ to be the unique element of $\mathbb{N}^{<\mathbb{N}}$ such that $\varphi(s_{\overline{x}})$ equals the length of $\overline{x}$. For a $\mathfrak{D}$-pair $(\overline{x}, Y)$ we set

$$\mathcal{B}^\infty_{\mathfrak{D}}(\overline{x}, Y) = \{V \in \mathcal{B}^\infty_{\mathfrak{D}} : \exists k \text{ such that } V|_k = \overline{x} \text{ and } V|_{[k,\infty)} \preceq Y\}.$$

Finally, recall the following terminology from [11]. For a family $\mathcal{G} \subseteq \mathcal{B}^\infty_{\mathfrak{D}}$ we say that $\mathcal{G}$ is *large for* $(\overline{x}, Y)$ if $\mathcal{G} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{x}, Z) \neq \emptyset$ for all $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(Y)$. In the case $\overline{x} = \emptyset$ we simply say that $\mathcal{G}$ is large for Y.

Lemma 9. *For every $U \in \mathcal{B}^\infty_{\mathfrak{D}}$ there is $W \in \mathcal{B}^\infty_{\mathfrak{D}}(U)$ such that for every $\overline{w} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W)$, either $\mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, W) = \emptyset$, or $\mathcal{G}^{s_{\overline{w}}}$ is large for $(\overline{w}, W)$.*

Proof. Let $\mathcal{P}$ be the set of all pairs $(\overline{x}, Y)$ in $\mathcal{B}^{<\infty}_{\mathfrak{D}}(U) \times \mathcal{B}^\infty_{\mathfrak{D}}(Y)$ such that either $\mathcal{G}^{s_{\overline{x}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{x}, Y) = \emptyset$, or $\mathcal{G}^{s_{\overline{x}}}$ is large for $(\overline{x}, Y)$. It is easy to see that $\mathcal{P}$ is admissible and satisfies ($\mathcal{P}$3). Hence the conclusion follows by Lemma 3. ∎

Let $W \in \mathcal{B}^\infty_{\mathfrak{D}}$ be a block sequence in $\mathfrak{D}$ satisfying the conclusion of Lemma 9. For $\overline{w} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W)$, let $\mathcal{F}(\overline{w})$ be the family of all $V = (v_i)_i \in \mathcal{B}^\infty_{\mathfrak{D}}(W)$ with $\overline{w} < V$ and with the following properties. There exist $m, l \in \mathbb{N}$ with $l \geq 1$ such that

(i) $s_{\overline{w}}^\frown m = s_{\overline{x}}$, where $\overline{x} = \overline{w}^\frown (v_i)_{i=0}^{l-1}$,
(ii) the family $\mathcal{G}^{s_{\overline{w}}^\frown m}$ is large for $(\overline{w}^\frown (v_i)_{i=0}^{l-1}, W)$.

Notice that $\mathcal{F}(\overline{w})$ is open in $\mathfrak{D}^{\mathbb{N}}$.

Lemma 10. *Let $\overline{w} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W)$ and assume that $\mathcal{G}^{s_{\overline{w}}}$ is large for $(\overline{w}, W)$. Then $\mathcal{F}(\overline{w})$ is large for W.*

Proof. Let $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(W)$. Since $\mathcal{G}^{s_{\overline{w}}}$ is large for $(\overline{w}, W)$ there is $V = (v_i)_i$ such that $\overline{w} < V$ and $\overline{w}^\frown V \in \mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, Z) = \bigcup_m \mathcal{G}^{s_{\overline{w}}^\frown m} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, Z)$ and so $\overline{w}^\frown V \in \mathcal{G}^{s_{\overline{w}}^\frown m} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, Z)$ for some $m \in \mathbb{N}$. Notice that for $l = \varphi(s^\frown m) - \varphi(s)$ we have $s_{\overline{w}}^\frown m = s_{\overline{x}}$, where $\overline{x} = \overline{w}^\frown (v_i)_{i=0}^{l-1}$, and $\overline{w}^\frown V \in \mathcal{G}^{s_{\overline{w}}^\frown m} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}^\frown (v_i)_{i=0}^{l-1}, Z)$. Therefore $\mathcal{G}^{s_{\overline{w}}^\frown m} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}^\frown (v_i)_{i=0}^{l-1}, W) \neq \emptyset$, which (by the properties of W) means that $\mathcal{G}^{s_{\overline{w}}^\frown m}$ is large for $(\overline{w}^\frown (v_i)_{i=0}^{l-1}, W)$. Hence $V \in \mathcal{F}(\overline{w}) \cap \mathcal{B}^\infty_{\mathfrak{D}}(Z)$. ∎

LEMMA 11. *There is $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(W)$ such that for every $\overline{z} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(Z)$, either $\mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{z}, Z) = \emptyset$ or player II has a winning strategy in the game $G_{\mathfrak{D}}(Z)$ for the family $\mathcal{F}(\overline{z})$.*

Proof. Let $\mathcal{P}$ be the family of pairs $(\overline{w}, Y) \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W) \times \mathcal{B}^\infty_{\mathfrak{D}}(W)$ such that either $\mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, Y) = \emptyset$, or player II has a winning strategy in $G_{\mathfrak{D}}(Y)$ for $\mathcal{F}(\overline{w})$.

By Lemma 3 it suffices to show that $\mathcal{P}$ is an admissible family of pairs in W which in addition satisfies ($\mathcal{P}3$). It is easy to see that only the cofinality property needs some explanation. Let $(\overline{w}, Y) \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W) \times \mathcal{B}^\infty_{\mathfrak{D}}(W)$. Since $\overline{w} \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(W)$, either $\mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, W) = \emptyset$, or $\mathcal{G}^{s_{\overline{w}}}$ is large for $(\overline{w}, W)$. In the first case, $\mathcal{G}^{s_{\overline{w}}} \cap \mathcal{B}^\infty_{\mathfrak{D}}(\overline{w}, Y) = \emptyset$ and so $(\overline{w}, Y) \in \mathcal{P}$. In the second case, Lemma 10 implies that $\mathcal{F}(\overline{w})$ is large for W. Hence by Lemma 8, there is $V \in \mathcal{B}^\infty_{\mathfrak{D}}(Y)$ such that player II has a winning strategy in $G_{\mathfrak{D}}(V)$ for $\mathcal{F}(\overline{w})$, and so $(\overline{w}, V) \in \mathcal{P}$. ∎

We are now ready for the proof of the main result.

Proof of Theorem 1. Assume that there is no $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(U)$ such that $\mathcal{B}^\infty_{\mathfrak{D}}(Z) \cap \mathcal{G} = \emptyset$, that is, $\mathcal{G}$ is large for U. Let $f : \mathcal{N} \to \mathfrak{D}^{\mathbb{N}}$ be a continuous map with $f[\mathcal{N}] = \mathcal{G}$, and for $s \in \mathbb{N}^{<\mathbb{N}}$, let $\mathcal{G}^s = f[\mathcal{N}_s]$. Then $\mathcal{G}^\emptyset = \mathcal{G}$ and $\mathcal{G}^s = \bigcup_n \mathcal{G}^{s^\frown n}$. Following the process of the above lemmas let $W \in \mathcal{B}^\infty_{\mathfrak{D}}(U)$ be as in Lemma 9 and $Z \in \mathcal{B}^\infty_{\mathfrak{D}}(W)$ as in Lemma 11. We claim that player II has a winning strategy in the game $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$.

Indeed, by our assumption $\mathcal{G} = \mathcal{G}^\emptyset$ is large in $\mathcal{B}^\infty_{\mathfrak{D}}(Z) = \mathcal{B}^\infty_{\mathfrak{D}}(\emptyset, Z)$ and so player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{F}(\emptyset)$. This means that player II is able to produce, after a finite number of moves, a finite block sequence $\overline{y}_0 \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(Z)$ such that there is $m_0 \in \mathbb{N}$ with $s_{\overline{y}_0} = (m_0)$ and $\mathcal{G}^{(m_0)}$ large for $(\overline{y}_0, W)$. By Lemma 11, player II has a winning strategy in $G_{\mathfrak{D}}(Z)$ for $\mathcal{F}(\overline{y}_0)$, that is, player II can extend $\overline{y}_0$ to a finite block sequence $\overline{y}_0^\frown \overline{y}_1 \in \mathcal{B}^{<\infty}_{\mathfrak{D}}(Z)$ such that there is $m_1 \in \mathbb{N}$ such that $s_{\overline{y}_0^\frown \overline{y}_1} = (m_0, m_1)$ and $\mathcal{G}^{(m_0, m_1)}$ is large for $(\overline{y}_0^\frown \overline{y}_1, W)$.

Continuing in this way we conclude that player II has a strategy in the game $G_{\mathfrak{D}}(Z)$ to construct a block sequence $Y = \overline{y}_0^\frown \overline{y}_1^\frown \ldots$ such that for some

$\sigma = (m_i)_i \in \mathcal{N}$ and for every $k \in \mathbb{N}$, $\mathcal{G}^{\sigma|k}$ is large for $((\overline{y}_0^\frown \ldots {}^\frown \overline{y}_{k-1}), W)$. To show that this is actually a winning strategy for $\mathcal{G}$, we have to prove that $Y \in \mathcal{G}$. Fix $k \in \mathbb{N}$. Since $\mathcal{G}^{\sigma|k}$ is large for $((\overline{y}_0^\frown \ldots {}^\frown \overline{y}_{k-1}), W)$, there exists $Y_k \in \mathcal{B}^\infty_{\mathfrak{D}}(W)$ such that $(\overline{y}_0^\frown \ldots {}^\frown \overline{y}_{k-1})^\frown Y_k \in \mathcal{G}^{\sigma|k}$. Since $(\mathcal{G}^{\sigma|n})_n$ is decreasing, $Y = \lim_n (\overline{y}_0^\frown \ldots {}^\frown \overline{y}_{n-1})^\frown Y_n \in \overline{\mathcal{G}^{\sigma|k}}$ for all $k \in \mathbb{N}$, and thus $Y \in \bigcap_k \overline{\mathcal{G}^{\sigma|k}}$. By the continuity of f, $\bigcap_k \overline{\mathcal{G}^{\sigma|k}} = \{f(\sigma)\}$ and therefore $Y = f(\sigma) \in \mathcal{G}$. ∎

4. Passing from the discrete to Gowers' game. In this section we will see how using Theorem 1 one can derive W. T. Gowers' Ramsey theorem (see Theorem 16). Henceforth, $\mathfrak{X}$ will be a normed linear space with a Schauder basis $(e_n)_n$.

First let us recall some relevant definitions. Let $\mathcal{B}^\infty_{\mathfrak{X}}$ (resp. $\mathcal{B}^\infty_{B_{\mathfrak{X}}}$) be the set of all block sequences in $\mathfrak{X}$ (resp. in the unit ball $B_{\mathfrak{X}}$). Let $U = (u_n)_n, V = (v_n)_n \in \mathcal{B}^\infty_{\mathfrak{X}}$ and $\Delta = (\delta_n)_n$ a sequence of positive real numbers. We say that U, V are *Δ-near* and we write $\mathrm{dist}(U,V) \le \Delta$ if $\|u_n - v_n\| \le \delta_n$ for all $n \in \mathbb{N}$. For a family $\mathcal{F} \subseteq \mathcal{B}^\infty_{\mathfrak{X}}$ and a sequence $\Delta = (\delta_n)_n$ of positive real numbers the *Δ-expansion* of $\mathcal{F}$ is the set

$$\mathcal{F}_\Delta = \{U \in \mathcal{B}^\infty_{\mathfrak{X}} : \exists V \in \mathcal{F} \text{ such that } \mathrm{dist}(U,V) \le \Delta\}.$$

For $Y \in \mathcal{B}^\infty_{B_{\mathfrak{X}}}$ and a family $\mathcal{F} \subseteq \mathcal{B}^\infty_{B_{\mathfrak{X}}}$ the Gowers' game $G_{\mathfrak{X}}(Y)$ is defined as the $\mathfrak{D}$-Gowers game by replacing $\mathfrak{D}$ and $\mathcal{G} \subseteq \mathcal{B}^\infty_{\mathfrak{D}}$ with $B_{\mathfrak{X}}$ and $\mathcal{F} \subseteq \mathcal{B}^\infty_{B_{\mathfrak{X}}}$ respectively.

For the next two lemmas we fix the following:

(i) a subset $\mathfrak{D}$ of $\langle (e_n)_n \rangle$ with the asymptotic property ($\mathfrak{D}$1),
(ii) a family $\mathcal{F} \subseteq \mathcal{B}^\infty_{B_{\mathfrak{X}}}$ of block sequences in $B_{\mathfrak{X}}$,
(iii) a sequence $\Delta = (\delta_n)_n$ of positive real numbers.

Lemma 12. *Let $\mathcal{G} = \mathcal{F}_\Delta \cap \mathcal{B}^\infty_{\mathfrak{D}}$ and suppose that $\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}) \cap \mathcal{G} = \emptyset$ for some $\widetilde{Z} \in \mathcal{B}^\infty_{\mathfrak{D}}$. Assume that there exists $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$ such that*

$$\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \subseteq (\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_\Delta$$

(that is, for every block subsequence $U = (u_n)_n$ of Z with $\|u_n\| \le 1$ there is a block subsequence $\widetilde{U} = (\widetilde{u}_n)_n$ of $\widetilde{Z}$ with $\widetilde{u}_n \in \mathfrak{D}$ such that $\mathrm{dist}(U, \widetilde{U}) \le \Delta$*). Then $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \cap \mathcal{F} = \emptyset$.*

Proof. Let $U \in \mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z)$. By our assumptions there is $\widetilde{U} \in \mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z})$ such that $\mathrm{dist}(U, \widetilde{U}) \le \Delta$ and $\widetilde{U} \notin \mathcal{G}$. Then $U \notin \mathcal{F}$, otherwise $\widetilde{U} \in \mathcal{F}_\Delta \cap \mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z})$ which is a contradiction. ∎

Lemma 13. *Let $\delta_0 \le 1$ and $\sum_{j=n+1}^\infty \delta_j \le \delta_n$ for all n. Let $\mathcal{G} = \mathcal{F}_{\Delta/10C} \cap \mathcal{B}^\infty_{\mathfrak{D}}$, where C is the basis constant of $(e_n)_n$, and suppose that for some*

$\widetilde{Z} \in \mathcal{B}^\infty_{\mathfrak{D}}$ player II has a winning strategy in the discrete game $G_{\mathfrak{D}}(\widetilde{Z})$ for $\mathcal{G}$. Assume that there exists $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$ such that

$$\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \subseteq (\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_{\Delta/10C}.$$

Then player II has a winning strategy in Gowers' game $G_{\mathfrak{X}}(Z)$ for $\mathcal{F}_\Delta$.

Proof. We will define a winning strategy for player II in Gowers' game $G_{\mathfrak{X}}(Z)$ for $\mathcal{F}_\Delta$ provided that he has one in the discrete game $G_{\mathfrak{D}}(Z)$ for $\mathcal{G}$. Suppose that we have just completed the nth move of $G_{\mathfrak{X}}(Z)$ (resp. $G_{\mathfrak{D}}(\widetilde{Z})$) and $x_0 < \cdots < x_{n-1}$ (resp. $\widetilde{x}_0 < \cdots < \widetilde{x}_{n-1}$) have been chosen by player II in $G_{\mathfrak{X}}(Z)$ (resp. in $G_{\mathfrak{D}}(\widetilde{Z})$).

Suppose that in $G_{\mathfrak{X}}(Z)$ player I chooses a block sequence $Z_n = (z^n_k)_k \in \mathcal{B}^\infty_{\mathfrak{X}}(Z)$. By normalizing we may suppose that $\|z^n_k\| = 1$ for every k, and so by our assumptions on $\widetilde{Z}$ and Z there exists $\widetilde{Z}_n = (\widetilde{z}^n_k)_k \in \mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z})$ such that $\mathrm{dist}(Z_n, \widetilde{Z}_n) \le \Delta/10C$. Then for all k, $\|z^n_k - \widetilde{z}^n_k\| \le \delta_k/10C$ and so $\|\widetilde{z}^n_k\| \ge 1 - \delta_k/10C$. Let $k_0 \ge n$ be such that $x_{n-1} < z^n_{k_0}$ and let player I play $\widetilde{Z}_n|_{[k_0,\infty]} = (\widetilde{z}^n_k)_{k\ge k_0}$ in the nth move of the discrete game $G_{\mathfrak{D}}(\widetilde{Z})$. Then player II extends $(\widetilde{x}_0, \ldots, \widetilde{x}_{n-1})$ according to his strategy in $G_{\mathfrak{D}}(\widetilde{Z})$ for $\mathcal{G}$, by picking $\widetilde{x}_n \in \langle (\widetilde{z}^n_k)_{k\ge k_0}\rangle_{\mathfrak{D}}$. Then $\widetilde{x}_n = \sum_{k\in I_n} \lambda^n_k \widetilde{z}^n_k$, where I_n is a finite segment in $\mathbb{N}$ with $\min I_n \ge k_0$ and $\lambda^n_k \in \mathbb{R}$. Going back to Gowers' game $G_{\mathfrak{X}}(Z)$, let player II play $x_n = \sum_{k\in I_n} \lambda^n_k z^n_k$. Then $x_n > x_{n-1}$ and so player II forms in this way a block sequence in $\mathcal{B}_{\mathfrak{X}}(Z)$.

It remains to show that $(x_n)_n \in \mathcal{F}_\Delta$. Since $(\widetilde{x}_n)_n \in \mathcal{G} \subseteq \mathcal{F}_{\Delta/10C} \subseteq (\mathcal{B}^\infty_{B_{\mathfrak{X}}})_{\Delta/10C}$, we see that $\|\widetilde{x}_n\| \le 1 + \delta_n/10C$ for all n. Hence

$$|\lambda^n_k| \le 2C\,\frac{\|\widetilde{x}_n\|}{\|\widetilde{z}^n_k\|} \le 2C\,\frac{1+\delta_n/10C}{1-\delta_k/10C} \le 2C\,\frac{1+\delta_0/10C}{1-\delta_0/10C} \le 4C$$

for all $k \in I_n$. Therefore,

$$\|x_n - \widetilde{x}_n\| \le \sum_{k\in I_n} |\lambda^n_k|\,\|z^n_k - \widetilde{z}^n_k\| \le 4C \sum_{k\in I_n} \frac{\delta_k}{10C} \le \frac{4}{5}\delta_{\min I_n} \le \frac{4}{5}\delta_n.$$

Since $(\widetilde{x}_n)_n \in \mathcal{F}_{\Delta/10C}$, the last inequality gives $(x_n)_{n\in\mathbb{N}} \in \mathcal{F}_{4\Delta/5+\Delta/10C} \subseteq \mathcal{F}_\Delta$. ■

The above lemmas lead us to define the next property for a subset $\mathfrak{D}$ of $\mathfrak{X}$ and a given sequence $\Delta = (\delta_n)_n$ of positive real numbers.

($\mathfrak{D}$3) (*Δ-block covering property*) For every $\widetilde{Z} \in \mathcal{B}^\infty_{\mathfrak{D}}$ there exists $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$ such that $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \subseteq (\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_\Delta$.

In the next proposition we give an example of a subset $\mathfrak{D}$ of $\mathfrak{X}$ with properties ($\mathfrak{D}$1)–($\mathfrak{D}$3). Actually we show that a property much stronger than ($\mathfrak{D}$3) can be satisfied. In particular, for every $\widetilde{Z} \in \mathcal{B}^\infty_{\mathfrak{D}}$ with $\widetilde{Z} = (\widetilde{z}_n)_n$, if we set $Z = (z_n)_n$ with $z_n = \widetilde{z}_{2n} + \widetilde{z}_{2n+1}$ then $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \subseteq (\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_\Delta$.

PROPOSITION 14. *For every sequence $\Delta = (\delta_n)_n$ of positive real numbers there is $\mathfrak{D} \subseteq B_{\mathfrak{X}} \cap \langle (e_n)_n \rangle$ satisfying ($\mathfrak{D}$1)–($\mathfrak{D}$3) and such that $(e_n)_n \in \mathcal{B}^{\infty}_{\mathfrak{D}}$.*

Proof. Let $(k_n)_n$ be a strictly increasing sequence of positive integers such that $2^{-k_n+1} \leq \delta_n$ for every n. For $i, l \in \mathbb{N}$, $l \geq 1$, let

$$\Lambda(i,l) = \{t \cdot 2^{-l\cdot(k_i+1)} : t \in \mathbb{Z}\}$$

For every finite non-empty segment $I = [n_1, n_2]$ of $\mathbb{N}$, $n_1 \leq n_2$, define $\mathfrak{D}(I) = \mathfrak{D}([n_1, n_2])$ to be the set of all $x = \sum_{i=n_1}^{n_2} \lambda_i e_i$ with the following properties:

(i) For all $n_1 \leq i \leq n_2$, $\lambda_i \in \Lambda(i,l)$, where $l = n_2 - n_1 + 1$ is the length of I.
(ii) The coefficients λ_{n_1} and λ_{n_2} are both non-zero.
(iii) $\|x\| \leq 1$.

Finally, we set

$$\mathfrak{D} = \bigcup_{n_1 \leq n_2} \mathfrak{D}([n_1, n_2]).$$

It is easy to see that $\mathfrak{D}$ satisfies ($\mathfrak{D}$1)–($\mathfrak{D}$2). In particular, $(e_n)_n \in \mathcal{B}^{\infty}_{\mathfrak{D}}$. It remains to show that $\mathfrak{D}$ has the Δ-block covering property. Actually, we will prove that $\mathfrak{D}$ has a stronger property; to do this we first state the following.

CLAIM. *Let $\widetilde{Z} \in \mathcal{B}^{\infty}_{\mathfrak{D}}$ and let $w \in \langle \widetilde{Z} \rangle$ be such that* $\mathrm{card}(\mathrm{supp}_{\widetilde{Z}}(w)) \geq 2$ *and $\|w\| \leq 1$. Then there is $\widetilde{w} \in \langle \widetilde{Z} \rangle_{\mathfrak{D}}$ such that*

(1) $\mathrm{supp}_{\widetilde{Z}}(\widetilde{w}) = \mathrm{supp}_{\widetilde{Z}}(w)$.
(2) $\|w - \widetilde{w}\| \leq 2^{-k_{m_1}+1}$, *where* $m_1 = \min \mathrm{supp}_{\widetilde{Z}}(w)$.

Proof of the claim. Let $\widetilde{Z} = (\widetilde{z}_j)_j$ and let $(I_j)_j$, $I_j = [n_1(j), n_2(j)]$, $n_1(j) \leq n_2(j)$, be the sequence of successive finite non-empty segments of $\mathbb{N}$ such that $\widetilde{z}_j \in \mathfrak{D}(I_j)$. Let $m_1 < m_2$ in $\mathbb{N}$, let $(\mu_j)_{j=m_1}^{m_2}$ be scalars such that μ_{m_1}, μ_{m_2} are both non-zero and let $w = \sum_{j\in[m_1,m_2]} \mu_j \widetilde{z}_j$ in $B_{\mathfrak{X}}$.

Set

$$w' = (1 - 2^{-k_{m_1}})w = \sum_{j\in[m_1,m_2]} (1 - 2^{-k_{m_1}})\mu_j \widetilde{z}_j \quad \text{and} \quad \widetilde{w} = \sum_{j\in[m_1,m_2]} \widetilde{\mu}_j \widetilde{z}_j,$$

where $\widetilde{\mu}_j = s_j \cdot 2^{-(k_{n_1(j)}+1)}$ and

$$s_j = \begin{cases} \lceil (1 - 2^{-k_{m_1}})\mu_j 2^{k_{n_1(j)}+1} \rceil & \text{if } \mu_j \geq 0, \\ \lfloor (1 - 2^{-k_{m_1}})\mu_j 2^{k_{n_1(j)}+1} \rfloor & \text{if } \mu_j < 0, \end{cases}$$

i.e. $\widetilde{\mu}_j$ are of the form $s_j \cdot 2^{-(k_{n_1(j)+1})}$ with $|\widetilde{\mu}_j| \geq |\mu_j(1 - 2^{-k_{m_1}})|$ and $|\widetilde{\mu}_j - (1 - 2^{-k_{m_1}})\mu_j| < 2^{-(k_{n_1(j)}+1)}$.

It is easy to see that $\widetilde{\mu}_j = 0$ if and only if $\mu_j = 0$ and so $\mathrm{supp}_{\widetilde{Z}}(\widetilde{w}) = \mathrm{supp}_{\widetilde{Z}}(w)$. Moreover, for all j, $|(1 - 2^{-k_{m_1}})\mu_j - \widetilde{\mu}_j| \leq 2^{-(k_{n_1(j)}+1)}$ and so

$$\text{(1)}\qquad \|w' - \widetilde{w}\| \le \sum_{j\in[m_1,m_2]} \big|(1-2^{-k_{m_1}})\mu_j - \widetilde{\mu}_j\big|\,\|\widetilde{z}_j\| \le \sum_{j\in[m_1,m_2]} 2^{-(k_{n_1(j)}+1)} \le 2^{-k_{n_1(m_1)}},$$

and therefore $\|w' - \widetilde{w}\| \le 2^{-k_{m_1}}$, since $m_1 \le n_1(m_1)$. As $\|w - w'\| \le 2^{-k_{m_1}}$, we obtain $\|w - \widetilde{w}\| \le 2^{-k_{m_1}+1}$.

It remains to show that $\widetilde{w} \in \mathfrak{D}$. Since for all $j \in [m_1, m_2]$ we have $\widetilde{z}_j \in \mathfrak{D}(I_j)$, it follows that $\widetilde{z}_j = \sum_{i\in I_j} t_i^j 2^{-l_j(k_i+1)} e_i$, where $l_j = n_2(j) - n_1(j) + 1$ is the length of I_j and $t^j_{n_1(j)}, t^j_{n_2(j)}$ are both non-zero. Therefore setting $I = [n_1(m_1), n_2(m_2)]$, we have

$$\text{(2)}\qquad \widetilde{w} = \sum_{j\in[m_1,m_2]} \widetilde{\mu}_j \widetilde{z}_j = \sum_{j\in[m_1,m_2]} \widetilde{\mu}_j\Big(\sum_{i\in I_j} t_i^j 2^{-l_j(k_i+1)} e_i\Big) = \sum_{i\in I} \lambda_i e_i$$

where $\lambda_i = t_i^j 2^{-l_j(k_i+1)}\widetilde{\mu}_j$ for all $i \in I_j$ and $j \in [m_1, m_2]$, and $\lambda_i = 0$ for all $i \in I \setminus \bigcup_{j\in[m_1,m_2]} I_j$.

We first show that condition (i) of the definition of $\mathfrak{D}$ is satisfied, that is, $\lambda_i \in \Lambda(i,l)$ for all $i \in I$, where $l = n_2(m_2) - n_1(m_1) + 1$ is the length of I. Since $0 \in \Lambda(i,l)$, it suffices to check this for each $i \in \bigcup_{j\in[m_1,m_2]} I_j$. So fix $j \in [m_1, m_2]$ and $i \in I_j$. Then

$$\text{(3)}\qquad \lambda_i = t_i^j 2^{-l_j(k_i+1)}\widetilde{\mu}_j = t_i^j 2^{-l_j(k_i+1)} s_j 2^{-(k_{n_1(j)}+1)} = \tau_i^j 2^{-l(k_i+1)}$$

where $\tau_i^j = t_i^j s_j 2^{(l-l_j)(k_i+1)-(k_{n_1(j)}+1)}$. Since $m_1 < m_2$ we have $l > l_j$. Also $n_1(j) \le i$ and hence $(l - l_j)(k_i + 1) - (k_{n_1(j)} + 1) \ge 0$. Therefore $\tau_i^j \in \mathbb{Z}$, which gives that $\lambda_i \in \Lambda(i,l)$.

Moreover, since $\widetilde{\mu}_{m_1}, \widetilde{\mu}_{m_2}, t^{m_1}_{n_1(m_1)}, t^{m_2}_{n_2(m_2)}$ are all non-zero we deduce that $\lambda_{n_1(m_1)}$ and $\lambda_{n_2(m_2)}$ are also non-zero and so condition (ii) of the definition of $\mathfrak{D}$ is satisfied. Finally, by (1), $\|\widetilde{w}\| \le \|w'\| + 2^{-k_{n_1(m_1)}} \le 1$ and so condition (iii) is fulfilled. We conclude that $\widetilde{w} \in \mathfrak{D}$, and the proof of the claim is complete.

We continue with the proof of the proposition. Let $\widetilde{Z} = (\widetilde{z}_j)_j$ in $\mathcal{B}^\infty_{\mathfrak{D}}$ and let $Z = (z_j)_j$ where $z_j = \widetilde{z}_{2j} + \widetilde{z}_{2j+1}$ for all j. Pick $W = (w_i)_i$ in $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z)$. Then for each i there exist $m_1^i < m_2^i$ and scalars $(\mu_j)_j$ such that $w_i = \sum_{j\in[m_1^i,m_2^i]} \mu_j \widetilde{z}_j \in B_{\mathfrak{X}}$ and $\mu_{m_1^i}, \mu_{m_2^i}$ are both non-zero. By the claim, for each i there exist scalars $(\widetilde{\mu}_j)_j$ such that $\widetilde{w}_i = \sum_{j\in[m_1^i,m_2^i]} \widetilde{\mu}_j \widetilde{z}_j \in \mathfrak{D}$ and $\|w_i - \widetilde{w}_i\| \le 2^{-k_{m_1^i}+1} \le 2^{-k_i+1} \le \delta_i$. We set $\widetilde{W} = (\widetilde{w}_i)_i$; then $\widetilde{W} \in \mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z})$ and $\operatorname{dist}(\widetilde{W}, W) \le \Delta$. Hence $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z) \subseteq (\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_\Delta$ and the proof is complete. ■

It is easy to see that

$$\rho(x,y) = \|x-y\| + \big|1/\|x\| - 1/\|y\|\big|, \qquad x, y \in \mathfrak{X} \setminus \{0\},$$

is an equivalent metric on $(\mathfrak{X}\setminus\{0\},\|\cdot\|)$ and that the product topology on $(\mathfrak{X}\setminus\{0\},\rho)^{\mathbb{N}}$ makes $\mathcal{B}^\infty_{\mathfrak{X}}$ a Polish space.

LEMMA 15. *Let $\mathcal{F}$ be an analytic subset of $\mathcal{B}^\infty_{\mathfrak{X}}$ and $\Delta=(\delta_n)_n$ be a sequence of positive real numbers. Then*

(i) *$\mathcal{F}_\Delta$ is analytic in $\mathcal{B}^\infty_{\mathfrak{X}}$.*
(ii) *For every countable $\mathfrak{D}\subseteq\mathfrak{X}$, $\mathcal{F}_\Delta\cap\mathcal{B}^\infty_{\mathfrak{D}}$ is analytic in $\mathfrak{D}^{\mathbb{N}}$ (where $\mathfrak{D}$ is endowed with the discrete topology).*

Proof. (i) It is easy to see that $\mathcal{Q}=\{(U,V):\mathrm{dist}(U,V)\le\Delta\}$ is closed in $\mathcal{B}^\infty_{\mathfrak{X}}\times\mathcal{B}^\infty_{\mathfrak{X}}$. Let proj_1 (resp. proj_2) be the projection of $\mathcal{B}^\infty_{\mathfrak{X}}\times\mathcal{B}^\infty_{\mathfrak{X}}$ onto the first (resp. second) coordinate. Then $\mathcal{F}_\Delta=\mathrm{proj}_1[\mathcal{Q}\cap(\mathcal{B}_{\mathfrak{X}}\times\mathcal{F})]=\mathrm{proj}_1[\mathcal{Q}\cap\mathrm{proj}_2^{-1}(\mathcal{F})]$.

(ii) Let $I:\mathfrak{D}^{\mathbb{N}}\to\mathfrak{X}^{\mathbb{N}}$ be the identity map. Then I is clearly continuous and $\mathcal{F}_\Delta\cap\mathcal{B}^\infty_{\mathfrak{D}}=I^{-1}(\mathcal{F}_\Delta)$. ∎

THEOREM 16 (W. T. Gowers). *Let $\mathfrak{X}$ be a normed linear space with a basis and let $\mathcal{F}\subseteq\mathcal{B}^\infty_{B_{\mathfrak{X}}}$ be an analytic family of block sequences in the unit ball $B_{\mathfrak{X}}$ of $\mathfrak{X}$. Then for every $\Delta>0$ there exists a block sequence $Z\in\mathcal{B}^\infty_{\mathfrak{X}}$ such that either $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z)\cap\mathcal{F}=\emptyset$, or player II has a winning strategy in Gowers' game $G_{\mathfrak{X}}(Z)$ for $\mathcal{F}_\Delta$.*

Proof. Let $(e_n)_n$ be a normalized basis for $\mathfrak{X}$ with constant C. Let $\Delta'=(\delta'_n)_n$ be a sequence of positive real numbers such that $\delta'_0\le 1$, $\delta'_n\le\delta_n$, and $\sum_{i>n}\delta'_i\le\delta'_n$. By Proposition 14, there is $\mathfrak{D}\subseteq\mathfrak{X}$ with $(e_n)_n\in\mathcal{B}^\infty_{\mathfrak{D}}$ satisfying ($\mathfrak{D}$1)–($\mathfrak{D}$3) for $\Delta'/10C$. Let also $\mathcal{G}=\mathcal{F}_{\Delta'/10C}\cap\mathcal{B}^\infty_{\mathfrak{D}}$. By Lemma 15, $\mathcal{G}$ is analytic in $\mathfrak{D}^{\mathbb{N}}$, and applying Theorem 1, we obtain a block sequence $\widetilde{Z}\in\mathcal{B}^\infty_{\mathfrak{D}}$ such that either $\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z})\cap\mathcal{G}=\emptyset$, or player II has a winning strategy in $G_{\mathfrak{D}}(\widetilde{Z})$ for $\mathcal{G}$. Choose $Z\in\mathcal{B}^\infty_{\mathfrak{X}}$ such that $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z)\subseteq(\mathcal{B}^\infty_{\mathfrak{D}}(\widetilde{Z}))_{\Delta'/10C}$. From Lemmas 12 and 13, either $\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z)\cap\mathcal{F}=\emptyset$, or player II has a winning strategy in Gowers' game $G_{\mathfrak{X}}(Z)$ for $\mathcal{F}_{\Delta'}$, and so (as $\Delta'\le\Delta$) for $\mathcal{F}_\Delta$ as well. ∎

5. A Ramsey consequence on k-tuples of block bases. The main goal of this section is to prove Theorem 2. First we need to do some preliminary work and introduce some notation. Fix a positive integer $k\ge 2$. For each $0\le i\le k-1$ and every infinite subset $L=\{l_0<l_1<\cdots\}$ of $\mathbb{N}$ we set $L_{i\,(\mathrm{mod}\,k)}=\{l_{kn+i}:n\in\mathbb{N}\}$ and we define

$$([L]^\infty)^k_\circ=\prod_{i=0}^{k-1}[L_{i\,(\mathrm{mod}\,k)}]^\infty=\{(L_i)_{i=0}^{k-1}\in([L]^\infty)^k:\forall i\ L_i\subseteq L_{i\,(\mathrm{mod}\,k)}\}.$$

Notice that $([L]^\infty)^k_\circ$ is not hereditary, that is, generally $([L']^\infty)^k_\circ\not\subseteq([L]^\infty)^k_\circ$ for $L'\subseteq L$. Let also

$$([L]^\infty)^k_\perp=\{(L_i)_{i=0}^{k-1}\in([L]^\infty)^k:\forall i\neq j\ L_i\cap L_j=\emptyset\}.$$

We have the following elementary lemma which relates the above types of products.

LEMMA 17. *Let* $N = \{(2n+1)k : n \in \mathbb{N}\}$. *Then*

$$([N]^\infty)^k_\perp \subseteq \bigcup_{L\in[\mathbb{N}]^\infty} ([L]^\infty)^k_\circ.$$

Proof. Let $(M_i)_{i=0}^{k-1} \in ([N]^\infty)^k_\perp$. Let $M = \bigcup_{i=0}^{k-1} M_i$ and for each $m \in M$ define the interval $I_m = [m - i_m, m - i_m + k - 1]$ of $\mathbb{N}$ where i_m is the unique natural number i such that $m \in M_i$. Notice that the length of each I_m is k while the length of an interval with unequal endpoints in N is at least $2k+1$. Hence $I_{m_1} \cap I_{m_2} = \emptyset$ for $m_1 \neq m_2$, and $I_m \cap N = \{m\}$ for all $m \in M$.

Let $L = \bigcup_{m\in M} I_m$. We claim that $(M_i)_{i=0}^{k-1} \in ([L]^\infty)^k_\circ$. Indeed, let $L = (l_n)_n$ be the increasing enumeration of L. For each $0 \le i \le k-1$ and $m \in M$ let $I_m(i) = m - i_m + i$ be the ith element of I_m. Since $(I_m)_{m\in M}$ is a sequence of pairwise disjoint intervals of $\mathbb{N}$ of length k, we easily see that $L_{i \,(\mathrm{mod}\, k)} = \bigcup_{m\in M} I_m(i)$. Fix $0 \le i \le k-1$. Then $m \in M_i$ if and only if $i_m = i$ if and only if $I_m(i) = m$. Hence $M_i = \bigcup_{m\in M_i}\{I_m(i)\} \subseteq \bigcup_{m\in M}\{I_m(i)\} = L_{i\,(\mathrm{mod}\, k)}$. ■

The above notation is easily extended to block sequences in the unit ball $B_{\mathfrak{X}}$ of a Banach space $\mathfrak{X}$ as follows. For every $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$ let

$$(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z))^k_\circ = \{(Z_i)_{i=0}^{k-1} \in (\mathcal{B}^\infty_{B_{\mathfrak{X}}})^k : \forall i \;\; Z_i \preceq Z|_{\mathbb{N}_{i\,(\mathrm{mod}\,k)}}\},$$

and generally for $L \in [\mathbb{N}]^\infty$, set

$$(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z|_L))^k_\circ = \{(Z_i)_{i=0}^{k-1} \in (\mathcal{B}^\infty_{B_{\mathfrak{X}}})^k : \forall i \;\; Z_i \preceq Z|_{L_{i\,(\mathrm{mod}\,k)}}\}.$$

The next lemma is an immediate consequence of Lemma 17.

LEMMA 18. *Let* $Z \in \mathcal{B}^\infty_{\mathfrak{X}}$ *and* $N = \{(2n+1)k : n \in \mathbb{N}\}$. *Then*

$$(\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z|_N))^k_\perp \subseteq \bigcup_{L\in[\mathbb{N}]^\infty} (\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z|_L))^k_\circ.$$

For a family $\mathfrak{F} \subseteq (\mathcal{B}^\infty_{B_{\mathfrak{X}}})^k$ let

$$\mathcal{F}^{\mathfrak{F}} = \{Z \in \mathcal{B}^\infty_{S_{\mathfrak{X}}} : \mathfrak{F} \cap (\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z))^k_\circ \neq \emptyset\},$$

where $S_{\mathfrak{X}}$ is the unit sphere of $\mathfrak{X}$.

LEMMA 19. *If* $\mathfrak{F}$ *is analytic in* $(\mathcal{B}^\infty_{\mathfrak{X}})^k$, *then* $\mathcal{F}^{\mathfrak{F}} \subseteq \mathcal{B}^\infty_{S_{\mathfrak{X}}}$ *is analytic in* $\mathcal{B}^\infty_{\mathfrak{X}}$.

Proof. Let $\mathcal{K} = \{(Z, (V_i)_{i=0}^{k-1}) \in \mathcal{B}^\infty_{S_{\mathfrak{X}}} \times (\mathcal{B}^\infty_{B_{\mathfrak{X}}})^k : (V_i)_{i=0}^{k-1} \in (\mathcal{B}^\infty_{B_{\mathfrak{X}}}(Z))^k_\circ\}$. Then $\mathcal{K}$ is a closed subset of $\mathcal{B}^\infty_{\mathfrak{X}} \times (\mathcal{B}^\infty_{\mathfrak{X}})^k$ and $\mathcal{F}^{\mathfrak{F}} = \mathrm{proj}_1[(\mathcal{B}^\infty_{\mathfrak{X}} \times \mathfrak{F}) \cap \mathcal{K}]$. ■

Proof of Theorem 2. Let $(e_n)_n$ be a normalized basis of $\mathfrak{X}$ with basis constant C. Choose $\Delta' = (\delta'_n)_n$ such that $0 < \delta'_n \le (4C)^{-1}\delta_n$ and $\sum_{j=n+1}^\infty \delta'_j \le \delta'_n$. By Lemma 19, $\mathcal{F}^{\mathfrak{F}}$ is an analytic subset of $\mathcal{B}^\infty_{B_{\mathfrak{X}}}$, and by Theorem 16 there is a block subsequence $Z = (z_n)_n$ such that either

$\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Z) \cap \mathcal{F}^{\mathfrak{F}} = \emptyset$, or player II has a winning strategy in Gowers' game $G_{\mathfrak{X}}(Z)$ for $(\mathcal{F}^{\mathfrak{F}})_{\Delta'}$. Let $Y = Z|_N$, where $N = \{(2n+1)k : n \in \mathbb{N}\}$. We claim that Y satisfies the conclusion of the theorem.

Indeed, if $\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Z) \cap \mathcal{F}^{\mathfrak{F}} = \emptyset$ then $\mathfrak{F} \cap (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Z'))^k_{\circ} = \emptyset$ for all $Z' \in \mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Z)$. In particular, $\mathfrak{F} \cap (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Z|_L))^k_{\circ} = \emptyset$ for all $L \in [\mathbb{N}]^{\infty}$, which by Lemma 18 gives that $\mathfrak{F} \cap (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Y))^k_{\perp} = \emptyset$.

So assume that player II has a winning strategy in Gowers' game $G_{\mathfrak{X}}(Z)$ for $(\mathcal{F}^{\mathfrak{F}})_{\Delta'}$. Since $Y = Z|_N$ the same holds for the game $G_{\mathfrak{X}}(Y)$. Fix $(U_i)_{i=0}^{k-1} \in (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(Y))^k$. We have to show that there exists $(V_i)_{i=0}^{k-1} \in (\mathcal{B}^{\infty}_{\mathfrak{X}})^k$ such that $V_i \preceq U_i$ and $(V_i)_{i=0}^{k-1} \in \mathfrak{F}_{\Delta}$. Consider a run of the game such that in the nth move player I plays U_i, where $n = i \pmod k$. Then player II succeeds in constructing a block sequence $V = (v_n)_n$ in $(\mathcal{F}^{\mathfrak{F}})_{\Delta'}$ such that $v_n \in U_i$ for all $n = i \pmod k$. Choose W in $\mathcal{F}^{\mathfrak{F}}$ with $\operatorname{dist}(V, W) \leq \Delta'$ and for each i, $W_i \preceq W|_{\mathbb{N}_{i \,(\mathrm{mod}\, k)}}$ such that $(W_i)_{i=0}^{k-1} \in (\mathcal{B}^{\infty}_{B_{\mathfrak{X}}}(W))^k_{\circ} \cap \mathfrak{F}$. Let $W = (w_n)_n$ and $W_i = (w^i_n)_n$. Then for each $i = 1, \ldots, k$ there is a block sequence $(F^i_n)_n$ of finite subsets of $\mathbb{N}_{i \,(\mathrm{mod}\, k)}$ and a sequence $(\lambda_j)_j$ of scalars such that $w^i_n = \sum_{j \in F^i_n} \lambda_j w_j$ for all i and n. We set $v^i_n = \sum_{j \in F^i_n} \lambda_j v_j$ and $V_i = (v^i_n)_n$. Then $V_i \preceq V|_{\mathbb{N}_{i \,(\mathrm{mod}\, k)}} \preceq U_i$ for all i. It remains to show that $(V_i)_{i=0}^{k-1} \in \mathfrak{F}_{\Delta}$. For this it suffices to see that $\operatorname{dist}(V_i, W_i) \leq \Delta$ for all i. Indeed, fix $0 \leq i \leq k-1$ and $n \in \mathbb{N}$. Since $\|w^i_n\| \leq 1$ and $\|w_j\| = 1$, we get $|\lambda_j| \leq 2C$ and therefore

$$\|v^i_n - w^i_n\| \leq \sum_{j \in F^i_n} |\lambda_j| \, \|v_j - w_j\| \leq 2C \sum_{j \in F^i_n} \delta'_j \leq 4C\delta'_n \leq \delta_n.$$

Hence $(U_i)_{i=0}^{k-1} \in (\mathfrak{F}_{\Delta})^{\uparrow}$. ■

6. Comments. 1. C. Rosendal [21] proves a Ramsey dichotomy between winning strategies in Gowers' game and winning strategies in the infinite asymptotic game. By appropriately modifying his argument, one can check that the proof in [21] works in the more general setting of a linear space $\mathfrak{X}$ of countable dimension over the field of reals provided that both games are restricted to a *countable* subset $\mathfrak{D}$ of $\mathfrak{X}$ with property ($\mathfrak{D}$1) stated in the introduction. This modification can be used to derive an alternative proof of Theorem 1.

2. Theorem 2 is actually an extension of the following fact concerning pairs of infinite subsets of $\mathbb{N}$. Given an analytic family $\mathfrak{F} \subseteq [\mathbb{N}]^{\infty} \times [\mathbb{N}]^{\infty}$ there is an infinite subset L of $\mathbb{N}$ such that either all *disjoint* pairs of infinite subsets of L belong to the complement of $\mathfrak{F}$, or for every $(L_1, L_2) \in [L]^{\infty} \times [L]^{\infty}$, there is $(L'_1, L'_2) \in \mathfrak{F}$ such that $L'_i \subseteq L_i$ for all $i = 1, 2$. To see this, consider the map $\Phi : M \to (M_0, M_1)$ where if $M = \{m_i\}_i$ is the increasing enumeration of L then $M_0 = \{m_i\}_{i \text{ even}}$ and $M_1 = \{m_i\}_{i \text{ odd}}$. Then apply Silver's theorem (see

[23]) for the family $\varPhi^{-1}(\mathfrak{F}^{\uparrow})$ where $\mathfrak{F}^{\uparrow} = \{(L, M) : \exists (L', M') \in \mathfrak{F}$ with $L' \subseteq L$ and $M' \subseteq M\}$. It is easy to see that keeping the "half" of the monochromatic set, the result follows. Also, applying K. Milliken's theorem [16], one can derive an analogue of the above result for pairs of block sequences of finite subsets of $\mathbb{N}$.

Acknowledgements. We would like to thank the referee for his (or her) suggestions which simplified the proof of Theorem 1.

This research was supported by PEBE 2007.

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Received October 26, 2009;
received in final form February 5, 2010 (7730)