BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 58, No. 1, 2010

FUNCTIONAL ANALYSIS

The Dual of a Non-reflexive L-embedded Banach Space Contains l^∞ Isometrically

by

Hermann PFITZNER

Presented by Stanisław KWAPIEŃ

Summary. A Banach space is said to be L-embedded if it is complemented in its bidual in such a way that the norm between the two complementary subspaces is additive. We prove that the dual of a non-reflexive L-embedded Banach space contains l^∞ isometrically.

This note is an afterthought to a result of Dowling [2] according to which a dual Banach space contains an isometric copy of c_0 if it contains an asymptotic one. (For definitions see below.) It is known ([7] or [4, Th. IV.2.7]) that the dual of a non-reflexive L-embedded Banach space contains c_0 isomorphically. For a special class of L-embedded Banach spaces the construction of the c_0-copy has been improved so as to yield an asymptotic one ([8, Prop. 6]) and it turns out that this improvement is possible in the general case, which together with Dowling's result yields isometric copies of c_0 in the dual of an L-embedded Banach space. As in [7], we will prove a bit more by constructing the c_0-copy within the context of Pełczyński's property (V*), that is, the c_0-basis will be constructed so as to behave approximately like biorthogonal functionals on the basis of a given l^1-basis in X; see (3) and (4) below where in particular the value $\tilde{c}_J(x_n)$ in (3) is optimal. (For the definition and some basic results on Pełczyński's property (V*) see [4].)

Preliminaries. A projection P on a Banach space Z is called an *L-projection* if $\|Pz\| + \|z - Pz\| = \|z\|$ for all $z \in Z$. A Banach space X is called *L-embedded* (or an *L-summand in its bidual*) if it is the image of an

2010 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B03, 46B04, 46B26.
Key words and phrases: L-embedded Banach spaces, L-summand, isometric copies of c_0.

DOI: 10.4064/ba58-1-4

L-projection on its bidual. In this case we write $X^{**} = X \oplus_1 X_{\rm s}$. Among classical Banach spaces, the Hardy space H_0^1, L^1-spaces and, more generally, the preduals of von Neumann algebras or of JBW*-triples serve as examples of L-embedded spaces. A sequence (x_n) in a Banach space X is said to *span* c_0 *asymptotically isometrically* (or just to *span* c_0 *asymptotically*) if there is a null sequence (δ_n) in $[0,1[$ such that

$$\sup{(1-\delta_n)|\alpha_n|} \le \Big\|\sum \alpha_n x_n\Big\| \le \sup{(1+\delta_n)|\alpha_n|}$$

for all $(\alpha_n) \in c_0$. X is said to *contain* c_0 *asymptotically* if it contains such a sequence (x_n). Recall the routine fact that if (x_n^*) in X^* is equivalent to the canonical basis of c_0 then $\sum \alpha_n x_n^*$ makes sense for all $(\alpha_n) \in l^\infty$ in the w^*-topology of X^*, and by lower w^*-semicontinuity of the norm an estimate $\|\sum \alpha_n x_n^*\| \le M \sup|\alpha_n|$ that holds for all $(\alpha_n) \in c_0$ extends to all $(\alpha_n) \in l^\infty$. The Banach spaces we consider in this note are real or complex; the set $\mathbb{N}$ starts at 1.

To a bounded sequence (x_n) in a Banach space X we associate its *James constant*

$$c_J(x_n) = \sup c_m \quad \text{where} \quad c_m = \inf_{\sum_{n\ge m}|\alpha_n|=1} \Big\|\sum_{n\ge m} \alpha_n x_n\Big\|$$

(the sequence (c_m) is increasing). If (x_n) is equivalent to the canonical basis of l^1 then $c_J(x_n) > 0$; more specifically, $c_J(x_n) > 0$ if and only if there is an integer m such that $(x_n)_{n\ge m}$ is equivalent to the canonical basis of l^1. (Roughly speaking, the number $c_J(x_n)$ may be thought of as the "approximately best l^1-basis constant" of (x_n); more precisely, there is a null sequence (τ_m) in $[0,1[$ (determined by $c_m = (1-\tau_m)c_J(x_n)$) such that $\|\sum_{n=m}^\infty \alpha_n x_n\| \ge (1-\tau_m)c_J(x_n)\sum_{n=m}^\infty |\alpha_n|$ for all $(\alpha_n) \in l^1$ and $m \in \mathbb{N}$, and $c_J(x_n)$ cannot be replaced by a strictly greater constant.) It is immediate from the definition of the James constant of an l^1-sequence (x_n) that there are pairwise disjoint finite sets $A_l \subset \mathbb{N}$ and a sequence (λ_n) of scalars such that $\sum_{k\in A_l}|\lambda_k| = 1$ and $\tilde z_l \to c_J(x_n)$ where $\tilde z_l = \sum_{k\in A_l}\lambda_k x_k$. James' l^1-distortion theorem states that an appropriate subsequence of the sequence (z_l) defined by $z_l = \tilde z_l/\|\tilde z_l\|$ spans l^1 almost isometrically in the sense that

$$(1-2^{-m})\sum_{l=m}^\infty |\alpha_l| \le \Big\|\sum_{l=m}^\infty \alpha_l z_l\Big\| \le (1+2^{-m})\sum_{l=m}^\infty |\alpha_l| \tag{1}$$

for all $m \in \mathbb{N}$ and all $(\alpha_n) \in l^1$. We will need the fact that if (z_l) spans l^1 almost isometrically in an L-embedded space X and if $x^{**} \in X^{**}$ is a w^*-accumulation point of the z_l then $x^{**} \in X_{\rm s}$ and $\|x^{**}\| = 1$. This follows from the proof of [8, Lem. 1] (or from a more elementary argument proving that $\mathrm{dist}(x^{**}, X) = \|x_{\rm s}\| = 1$ where $x^{**} = x + x_{\rm s}$).

If one passes to a subsequence (x_{n_k}) of (x_n) then $c_J(x_{n_k}) \geq c_J(x_n)$; hence it makes sense to define

$$\tilde{c}_J(x_n) = \sup_{n_k} c_J(x_{n_k}).$$

The standard reference for L-embedded Banach spaces is the monograph [4, Chap. IV]. For general Banach space theory and undefined notation we refer to [1], [5], or [6].

The main result of this note is

THEOREM 1. *Let X be an L-embedded Banach space and let (x_n) be equivalent to the canonical basis of l^1. Then there is a sequence (x_n^*) in X^* that generates l^∞ isometrically, more precisely*

$$\left\| \sum \alpha_n x_n^* \right\| = \sup |\alpha_n| \quad \textit{for all } (\alpha_n) \in l^\infty, \tag{2}$$

and there is a strictly increasing sequence (p_n) in $\mathbb{N}$ such that

$$\lim |x_n^*(x_{p_n})| = \tilde{c}_J(x_m), \tag{3}$$

$$x_n^*(x_{p_l}) = 0 \quad \textit{if } l < n. \tag{4}$$

In particular, the dual of a non-reflexive L-embedded Banach space contains an isometric copy of l^∞.

In order to prove the theorem we first state and prove Dowling's result in a way which fits our purpose.

PROPOSITION 2. *Let (ε_n) be a null sequence in $[0, 1[$, let (N_n) be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ and let (y_n^*) in the dual of a Banach space Y span c_0 such that*

$$\left\| \sum \alpha_n y_n^* \right\| \leq \sup (1 + \varepsilon_n)|\alpha_n| \quad \textit{and} \quad \|y_n^*\| \to 1 \tag{5}$$

for all $(\alpha_n) \in c_0$. Then the elements

$$x_n^* = \sum_{k \in N_n} \frac{y_k^*}{1 + \varepsilon_k} \tag{6}$$

generate l^∞ isometrically (as in (2)).

Proof. Clearly, $\|x_n^*\| \leq 1$ for all $n \in \mathbb{N}$ by the first half of (5). For the inverse inequality we have

$$\|x_n^*\| \geq \left\| 2\, \frac{y_m^*}{1 + \varepsilon_m} \right\| - \left\| \frac{y_m^*}{1 + \varepsilon_m} - \sum_{k \in N_n,\, k \neq m} \frac{y_k^*}{1 + \varepsilon_k} \right\| \geq 2\, \frac{\|y_m^*\|}{1 + \varepsilon_m} - 1$$

for all $m \in N_n$, hence $\|x_n^*\| \geq 1$ by the second half of (5), which proves $\|x_n^*\| = 1$.

Similarly we show (2): First, "$\le$" of (2) follows from the first half of (5); second, by the inequality just shown we have

$$\Big\|\sum \alpha_n x_n^*\Big\| \ge 2|\alpha_m| - \Big\|\alpha_m x_m^* - \sum_{n\ne m}\alpha_n x_n^*\Big\| \ge 2|\alpha_m| - \sup|\alpha_n|$$

for all $m\in\mathbb{N}$, giving "$\ge$" of (2). ∎

Proof of the Theorem. Let (δ_n) be a sequence in $]0,1[$ converging to 0. Suppose (x_n) is an l^1-basis and write $\tilde c = \tilde c_J(x_n)$ for short.

Observation. Given $\tau > 0$ there is a subsequence (x_{n_k}) of (x_n) such that $|\tilde c - c_J(x_{n_k})| < \tau$. By James' l^1-distortion theorem, as described above, there are pairwise disjoint finite sets $A_l \subset \{n_k \mid k\in\mathbb{N}\}$ and a sequence (λ_n) of scalars such that (1) holds with λ_n, $\tilde z_l$, z_l as above; furthermore $\|\tilde z_l\| \to c_J(x_{n_k})$, whence the existence of l' such that $|\tilde c - \|\tilde z_{l'}\|\,| < \tau$.

By induction over $n\in\mathbb{N}$ we will construct finite sequences $(y_i^{(n)*})_{i=1}^n$ in X^*, a sequence $(\tilde y_n)$ in X, pairwise disjoint finite sets $C_n\subset\mathbb{N}$ and a scalar sequence (μ_n) such that, with the notation $y_n = \tilde y_n/\|\tilde y_n\|$,

$$\text{(7)}\qquad \sum_{k\in C_n}|\mu_k| = 1,\qquad \tilde y_n = \sum_{k\in C_n}\mu_k x_k,\qquad |\tilde c - \|\tilde y_n\|\,| < \delta_n,$$

$$\text{(8)}\qquad |y_i^{(n)*}(y_i)| > 1-\delta_i \quad \forall i\le n,$$

$$\text{(9)}\qquad y_i^{(n)*}(y_l) = 0 \qquad \forall l < i \le n,$$

$$\text{(10)}\qquad y_i^{(n)*}(x_p) = 0 \qquad \forall p\in C_l,\ \forall l < i\le n,$$

$$\text{(11)}\qquad \Big\|\sum_{i=1}^m \alpha_i y_i^{(n)*}\Big\| \le \max_{i\le m}(1+(1-2^{-n})\delta_i)|\alpha_i| \qquad \forall m\le n,\ \alpha_i \text{ scalars.}$$

For $n=1$ we use the observation above with $\tau=\delta_1$ and choose l_1 such that $|\,\|\tilde z_{l_1}\| - \tilde c| < \delta_1$. Then we choose $y_1^{(1)*}$ such that $\|y_1^{(1)*}\| = 1$ and $y_1^{(1)*}(z_{l_1}) = \|z_{l_1}\|$. It remains to set $C_1 = A_{l_1}$, $\mu_k = \lambda_k$ for $k\in C_1$ and $\tilde y_1 = \tilde z_{l_1}$.

For the induction step $n\mapsto n+1$ we recall that $(P^*)_{|X^*}$ is an isometric isomorphism from X^* onto $X_{\mathrm{s}}^\perp$, that $X^{***} = X^\perp \oplus_\infty X_{\mathrm{s}}^\perp$ and that $(P^*x^*)_{|X} = (x^*)_{|X}$ for all $x^*\in X^*$. Let (z_l) be as in the observation above with $\tau = \delta_{n+1}$ and let $z_{\mathrm{s}}\in X^{**}$ be a w^*-accumulation point of the z_l. Then $z_{\mathrm{s}}\in X_{\mathrm{s}}$ and $\|z_{\mathrm{s}}\| = 1$ (as explained in the preliminaries). Choose $t\in\ker P^*\subset X^{***}$ such that $\|t\|=1$ and $t(z_{\mathrm{s}}) = \|z_{\mathrm{s}}\|$. Put

$$E = \mathrm{lin}(\{P^*y_i^{(m)*} \mid i\le m\le n\}\cup\{t\}) \subset X^{***},$$
$$F = \mathrm{lin}\Big(\{z_{\mathrm{s}}\}\cup\Big\{x_p \,\Big|\, p\in\bigcup_{l\le n} C_l\Big\}\Big) \subset X^{**}$$

and choose $\eta > 0$ such that

$$(1+\eta)(1+(1-2^{-n})\delta_i) < 1+(1-2^{-(n+1)})\delta_i \quad\text{and}\quad \eta < (1-2^{-(n+1)})\delta_{n+1}$$

for all $i \le n$. The principle of local reflexivity provides an operator $R: E \to X^*$ such that

$$(1-\eta)\|e^{***}\| \le \|Re^{***}\| \le (1+\eta)\|e^{***}\|, \tag{12}$$
$$f^{**}(Re^{***}) = e^{***}(f^{**}), \tag{13}$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define $y_i^{(n+1)*} = R(P^* y_i^{(n)*})$ for $i \le n$ and $y_{n+1}^{(n+1)*} = Rt$ and obtain $(11)_{n+1}$ (with $\alpha_i = 0$ if $m < i \le n+1$) by

$$\begin{aligned}
\Big\|\sum_{i=1}^{n+1} \alpha_i y_i^{(n+1)*}\Big\| &\overset{(12)}{\le} (1+\eta)\Big\|\Big(\sum_{i=1}^{n} \alpha_i P^* y_i^{(n)*}\Big) + \alpha_{n+1} t\Big\| \\
&= (1+\eta)\max\Big(\Big\|\sum_{i=1}^{n} \alpha_i P^* y_i^{(n)*}\Big\|, \|\alpha_{n+1} t\|\Big) \\
&= (1+\eta)\max\Big(\Big\|\sum_{i=1}^{n} \alpha_i y_i^{(n)*}\Big\|, \|\alpha_{n+1} t\|\Big) \\
&\overset{(11)}{\le} (1+\eta)\max\big(\max_{i\le n}(1+(1-2^{-n})\,\delta_i)|\alpha_i|, |\alpha_{n+1}|\big) \\
&\le \max_{i\le n+1}(1+(1-2^{-(n+1)})\,\delta_i)|\alpha_i|.
\end{aligned}$$

Since z_{s} is a w^*-cluster point of (z_l) we have

$$\begin{aligned}
|y_{n+1}^{(n+1)*}(z_l)| &> |z_{\mathrm{s}}(y_{n+1}^{(n+1)*})| - \delta_{n+1} \\
&\overset{(13)}{=} |t(z_{\mathrm{s}})| - \delta_{n+1} = 1 - \delta_{n+1}
\end{aligned}$$

for infinitely many l; furthermore, an l_{n+1} can be chosen among those l so as to obtain $|\,\|\tilde{z}_{l_{n+1}}\| - \tilde{c}| < \delta_{n+1}$. Set $C_{n+1} = A_{l_{n+1}}$, $\tilde{y}_{n+1} = \tilde{z}_{l_{n+1}}$, $\mu_k = \lambda_k$ for $k \in C_{n+1}$. Then $(7)_{n+1}$ holds and $(8)_{n+1}$ holds for $i = n+1$. For $i \le n$, $(8)_{n+1}$ follows from

$$y_i^{(n+1)*}(y_i) = (P^* y_i^{(n)*})(y_i) = y_i^{(n)*}(y_i) \overset{(8)}{>} 1 - \delta_i.$$

Condition $(10)_{n+1}$ holds for $i = n+1$ by

$$y_{n+1}^{(n+1)*}(x_p) = (Rt)(x_p) \overset{(13)}{=} t(x_p) = 0 \quad \forall p \in C_l, \forall l < n+1$$

and it holds for $i < n+1$ by

$$y_i^{(n+1)*}(x_p) = (P^* y_i^{(n)*})(x_p) = y_i^{(n)*}(x_p) \overset{(10)}{=} 0 \quad \forall p \in C_l, \forall l < i.$$

Condition $(9)_{n+1}$ follows from $(10)_{n+1}$. This ends the induction.

Now we define $y_i^* = \frac{1}{1+\delta_i} \lim_{n\in\mathcal{U}} y_i^{(n)*}$ for all $i \in \mathbb{N}$ where $\mathcal{U}$ is a fixed nontrivial ultrafilter on $\mathbb{N}$ and where the limit is understood in the w^*-topology of X^*. Then by w^*-lower semicontinuity of the norm and by (11),

$$\Big\|\sum \alpha_i y_i^*\Big\| \le \sup\,(1+\delta_i)\frac{|\alpha_i|}{1+\delta_i} = \sup|\alpha_i|$$

for all $(\alpha_i) \in l^\infty$. In particular, $\|y_i^*\| \le 1$, hence $\|y_i^*\| \to 1$ by (8) and (y_i^*) satisfies (5) for $\varepsilon_n = 0$.

Let (N_n) be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ such that (i_n) increases strictly where $i_n = \min N_n$. By the proposition the sequence defined by

$$x_n^* = \sum_{i\in N_n} y_i^*$$

generates l^∞ isometrically and we have

$$|x_n^*(y_{i_n})| \overset{(9)}{=} |y_{i_n}^*(y_{i_n})| \overset{(8)}{\ge} \frac{1-\delta_{i_n}}{1+\delta_{i_n}}.$$

By construction of the y_i there is, for each $n\in\mathbb{N}$, an index $p_n \in C_{i_n}$ such that

$$(1+\delta_{i_n})|x_n^*(x_{p_n})| \ge (1-\delta_{i_n})\|\tilde{y}_{i_n}\| \overset{(7)}{\ge} (1-\delta_{i_n})(\tilde{c}-\delta_{i_n}),$$

which will yield "$\ge$" of (3). In order to show "$\le$" of (3) suppose to the contrary that $x_{n_m}^*(x_{p_{n_m}}) > \kappa + \tilde{c}$ for appropriate subsequences, all m and $\kappa > 0$. According to an extraction lemma of Simons [10] we may furthermore suppose that $\sum_{j\ne m}|x_{n_j}^*(x_{p_{n_m}})| < \kappa/2$ for all m. Then given (α_m) and θ_m such that $\theta_m\alpha_m = |\alpha_m|$ we obtain

$$\begin{aligned}(14)\qquad \Big\|\sum \alpha_m x_{p_{n_m}}\Big\| &\ge \Big(\sum_j \theta_j x_{n_j}^*\Big)\Big(\sum_m \alpha_m x_{p_{n_m}}\Big)\\ &\ge (\kappa+\tilde{c})\sum_m |\alpha_m| - \sum_m\sum_{j\ne m}|\alpha_m|\,|x_{n_j}^*(x_{p_{n_m}})|\\ &\ge (\kappa/2+\tilde{c})\sum_m|\alpha_m|,\end{aligned}$$

which yields the contradiction $c_J(x_{p_{n_m}}) > \tilde{c}$ and thus shows "$\le$" and all of (3), whereas (4) follows from (10) via $y_i^*(x_p) = 0$ for $p\in C_l$, $l<i$.

The last assertion of the theorem is immediate from the fact that non-reflexive L-embedded spaces contain l^1 isomorphically [4, IV.2.3]. ∎

Remarks. 1. It is not clear whether (4) can be obtained also for $l > n$. What can be said by Simons' extraction lemma (used in the proof) is that, under the assumptions of the theorem and given $\varepsilon > 0$, it is possible (after passing to appropriate subsequences) to deduce in addition to (4) that $\sum_{n=1}^{l-1}|x_n^*(x_{p_l})| = \sum_{n\ne l}|x_n^*(x_{p_l})| < \varepsilon$ for all l. In case $\tilde{c}_J(x_n) = 1 = \lim\|x_n\|$

(which happens when the x_n span l^1 almost isometrically) this can be improved to

$$\sum_{n\neq l}|x_n^*(x_{p_l})| = \Big(\sum_n |x_n^*(x_{p_l})|\Big) - |x_l^*(x_{p_l})| \le \|x_{p_l}\| - |x_l^*(x_{p_l})| \to 0. \tag{15}$$

One might also construct straightforward perturbations of the x_n^* in order to get (4) for $l \neq n$ but then it is not clear whether these perturbations can be arranged to span c_0 isometrically, not just almost isometrically.

Since in general L-embedded spaces do not contain l^1 isometrically (see below, last remark) it is not in general possible, in case all x_n have the same norm, to improve (3) and (4) so as to obtain $x_n^*(x_{p_l}) = \tilde{c}(x_m)$ if $l = n$ and $= 0$ if $l \neq n$.

2. As already alluded to in the introduction, the construction of c_0 in this paper bears much resemblance to the one of [7]. A different way to construct c_0 is contained in [9] but it seems unlikely that this construction can be improved to yield an isometric c_0-copy.

3. It follows from (3) (or rather from a reasoning similar to the one in (14)) that $c(x_{p_n}) \ge \tilde{c}(x_n)$, which means that in L-embedded spaces the sup in the definition of $\tilde{c}_J$ is attained by the James constant of an appropriate subsequence. For general Banach spaces this is not known, although it can be shown by a routine diagonal argument that each bounded sequence (x_n) admits a c_J-stable subsequence (x_{n_k}) (meaning that $\tilde{c}_J(x_{n_k}) = c_J(x_{n_k})$) whose James constant is arbitrarily near to $\tilde{c}(x_n)$.

4. Each normalized sequence (x_n) in an L-embedded Banach space that spans l^1 almost isometrically contains a subsequence each of whose w^*-accumulation points in the bidual attains its norm on the dual unit ball. To see this, let (x_n^*) and (x_{p_n}) be the sequences given by the theorem and by Simons' extraction lemma (see (15) above), let x_{s} be a w^*-accumulation point of the x_{p_n} and let $x^* = \sum x_n^*$; then $\|x^*\| = 1$ and on the one hand $\|x_{\mathrm{s}}\| = 1$ by [8] and on the other hand $x_{\mathrm{s}}(x^*) = \lim x^*(x_{p_n}) \overset{(15)}{=} \lim x_n^*(x_{p_n}) \overset{(3)}{=} c_J(x_n) = 1$.

It would be interesting to know whether this remark holds for the whole sequence (x_n) instead of only a subsequence (x_{p_n}). A kind of converse follows from [9, Rem. 2] for separable X: If $x_{\mathrm{s}} \in X_{\mathrm{s}}$ attains its norm on the dual unit ball then it does so on the sum of a wuC-series.

5. Let us finally note that the presence of isometric c_0-copies in X^* does not necessarily entail the presence of isometric copies of l^1 in X even if X is the dual of an M-embedded Banach space. This follows from [4, Cor. III.2.12], which states that there is an L-embedded Banach space which is the dual of an M-embedded space (to wit, the dual of c_0 with an equivalent norm) which is strictly convex and therefore does not contain l^1 isometrically although it contains, as do all non-reflexive L-embedded spaces, l^1 asymptotically ([8],

see [3] for the definition of asymptotic copies [1] and the difference from almost isometric ones).

References

[1] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, Berlin, 1984.

[2] P. N. Dowling, *Isometric copies of c_0 and ℓ^∞ in duals of Banach spaces*, J. Math. Anal. Appl. 244 (2000), 223–227.

[3] P. N. Dowling, W. B. Johnson, C. J. Lennard, and B. Turett, *The optimality of James's distortion theorems*, Proc. Amer. Math. Soc. 125 (1997), 167–174.

[4] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, 1993.

[5] W. B. Johnson and J. Lindenstrauss (eds.), *Handbook of the Geometry of Banach Spaces, Volumes 1 and 2*, North-Holland, 2001, 2003.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer, Berlin, 1977, 1979.

[7] H. Pfitzner, *L-summands in their biduals have Pełczyński's property* (V^*), Studia Math. 104 (1993), 91–98.

[8] —, *A note on asymptotically isometric copies of l^1 and c_0*, Proc. Amer. Math. Soc. 129 (2001), 1367–1373.

[9] —, *Separable L-embedded Banach spaces are unique preduals*, Bull. London Math. Soc. 39 (2007), 1039–1044.

[10] S. Simons, *On the Dunford–Pettis property and Banach spaces that contain c_0*, Math. Ann. 216 (1975), 225–231.

Hermann Pfitzner
Université d'Orléans
BP 6759
F-45067 Orléans Cedex 2, France
E-mail: hermann.pfitzner@univ-orleans.fr

Received January 8, 2010;
received in final form March 25, 2010 (7744)

---

[1] In the literature there is another notion of "asymptotic l^p" which is quite different from the one of this note.