The Dual of a Non-reflexive L-embedded Banach Space Contains $l^\infty$ Isometrically

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Summary. A Banach space is said to be L-embedded if it is complemented in its bidual in such a way that the norm between the two complementary subspaces is additive. We prove that the dual of a non-reflexive L-embedded Banach space contains $l^\infty$ isometrically.

This note is an afterthought to a result of Dowling [2] according to which a dual Banach space contains an isometric copy of $c_0$ if it contains an asymptotic one. (For definitions see below.) It is known ([7] or [4, Th. IV.2.7]) that the dual of a non-reflexive L-embedded Banach space contains $c_0$ isomorphically. For a special class of L-embedded Banach spaces the construction of the $c_0$-copy has been improved so as to yield an asymptotic one ([8, Prop. 6]) and it turns out that this improvement is possible in the general case, which together with Dowling’s result yields isometric copies of $c_0$ in the dual of an L-embedded Banach space. As in [7], we will prove a bit more by constructing the $c_0$-copy within the context of Pełczyński’s property (V$^*$), that is, the $c_0$-basis will be constructed so as to behave approximately like biorthogonal functionals on the basis of a given $l^1$-basis in $X$; see [3] and [4] below where in particular the value $\tilde{c}_{\sl{f}}(x_n)$ in (3) is optimal. (For the definition and some basic results on Pełczyński’s property (V$^*$) see [4].)

Preliminaries. A projection $P$ on a Banach space $Z$ is called an L-projection if $\|Pz\| + \|z - Pz\| = \|z\|$ for all $z \in Z$. A Banach space $X$ is called L-embedded (or an L-summand in its bidual) if it is the image of an

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L-projection on its bidual. In this case we write $X^{**} = X \oplus_1 X_s$. Among classical Banach spaces, the Hardy space $H^1_0$, $L^1$-spaces and, more generally, the preduals of von Neumann algebras or of JBW*-triples serve as examples of L-embedded spaces. A sequence $(x_n)$ in a Banach space $X$ is said to span $c_0$ asymptotically isometrically (or just to span $c_0$ asymptotically) if there is a null sequence $(\delta_n)$ in $[0, 1[$ such that

$$
\sup (1 - \delta_n) |\alpha_n| \leq \left\| \sum \alpha_n x_n \right\| \leq \sup (1 + \delta_n) |\alpha_n|
$$

for all $(\alpha_n) \in c_0$. $X$ is said to contain $c_0$ asymptotically if it contains such a sequence $(x_n)$. Recall the routine fact that if $(x_n^*)$ in $X^*$ is equivalent to the canonical basis of $c_0$ then $\sum \alpha_n x_n^*$ makes sense for all $(\alpha_n) \in l^\infty$ in the $w^*$-topology of $X^*$, and by lower $w^*$-semicontinuity of the norm an estimate $\| \sum \alpha_n x_n^* \| \leq M \sup |\alpha_n|$ that holds for all $(\alpha_n) \in c_0$ extends to all $(\alpha_n) \in l^\infty$. The Banach spaces we consider in this note are real or complex; the set $\mathbb{N}$ starts at 1.

To a bounded sequence $(x_n)$ in a Banach space $X$ we associate its James constant

$$
c_J(x_n) = \sup c_m \quad \text{where} \quad c_m = \inf \frac{\left\| \sum_{n \geq m} \alpha_n x_n \right\|}{\sum_{n \geq m} |\alpha_n|}
$$

(the sequence $(c_m)$ is increasing). If $(x_n)$ is equivalent to the canonical basis of $l^1$ then $c_J(x_n) > 0$; more specifically, $c_J(x_n) > 0$ if and only if there is an integer $m$ such that $(x_n)_{n \geq m}$ is equivalent to the canonical basis of $l^1$. (Roughly speaking, the number $c_J(x_n)$ may be thought of as the “approximately best $l^1$-basis constant” of $(x_n)$; more precisely, there is a null sequence $(\tau_n)$ in $[0, 1[$ (determined by $c_m = (1 - \tau_m) c_J(x_n)$) such that $\| \sum_{n=m}^{\infty} \alpha_n x_n \| \geq (1 - \tau_m) c_J(x_n) \sum_{n=m}^{\infty} |\alpha_n|$ for all $(\alpha_n) \in l^1$ and $m \in \mathbb{N}$, and $c_J(x_n)$ cannot be replaced by a strictly greater constant.) It is immediate from the definition of the James constant of an $l^1$-sequence $(x_n)$ that there are pairwise disjoint finite sets $A_l \subset \mathbb{N}$ and a sequence $(\lambda_k)$ of scalars such that $\sum_{k \in A_l} |\lambda_k| = 1$ and $\tilde{z}_l \to c_J(x_n)$ where $\tilde{z}_l = \sum_{k \in A_l} \lambda_k x_k$. James’ $l^1$-distortion theorem states that an appropriate subsequence of the sequence $(z_l)$ defined by $z_l = \tilde{z}_l/\|\tilde{z}_l\|$ spans $l^1$ almost isometrically in the sense that

$$
(1 - 2^{-m}) \sum_{l=m}^{\infty} |\alpha_l| \leq \left\| \sum_{l=m}^{\infty} \alpha_l z_l \right\| \leq (1 + 2^{-m}) \sum_{l=m}^{\infty} |\alpha_l|
$$

for all $m \in \mathbb{N}$ and all $(\alpha_n) \in l^1$. We will need the fact that if $(z_l)$ spans $l^1$ almost isometrically in an L-embedded space $X$ and if $x^{**} \in X^{**}$ is a $w^*$-accumulation point of the $z_l$ then $x^{**} \in X_s$ and $\|x^{**}\| = 1$. This follows from the proof of [8 Lem. 1] (or from a more elementary argument proving that $\text{dist}(x^{**}, X) = \|x_s\| = 1$ where $x^{**} = x + x_s$).
If one passes to a subsequence \((x_{n_k})\) of \((x_n)\) then \(c_J(x_{n_k}) \geq c_J(x_n)\); hence it makes sense to define
\[
\tilde{c}_J(x_n) = \sup_{n_k} c_J(x_{n_k}).
\]
The standard reference for L-embedded Banach spaces is the monograph [4, Chap. IV]. For general Banach space theory and undefined notation we refer to [1], [5], or [6].

The main result of this note is

**Theorem 1.** Let \(X\) be an L-embedded Banach space and let \((x_n)\) be equivalent to the canonical basis of \(l^1\). Then there is a sequence \((x^*_n)\) in \(X^*\) that generates \(l^\infty\) isometrically, more precisely
\[
(2) \quad \left\| \sum \alpha_n x^*_n \right\| = \sup |\alpha_n| \quad \text{for all } (\alpha_n) \in l^\infty,
\]
and there is a strictly increasing sequence \((p_n)\) in \(\mathbb{N}\) such that
\[
(3) \quad \lim \left| x^*_n(x_{p_n}) \right| = \tilde{c}_J(x_m),
\]
\[
(4) \quad x^*_n(x_{p_l}) = 0 \quad \text{if } l < n.
\]
In particular, the dual of a non-reflexive L-embedded Banach space contains an isometric copy of \(l^\infty\).

In order to prove the theorem we first state and prove Dowling’s result in a way which fits our purpose.

**Proposition 2.** Let \((\varepsilon_n)\) be a null sequence in \([0,1]\), let \((N_n)\) be a sequence of pairwise disjoint infinite subsets of \(\mathbb{N}\) and let \((y^*_n)\) in the dual of a Banach space \(Y\) span \(c_0\) such that
\[
(5) \quad \left\| \sum \alpha_n y^*_n \right\| \leq \sup (1 + \varepsilon_n)|\alpha_n| \quad \text{and } \left\| y^*_n \right\| \to 1
\]
for all \((\alpha_n) \in c_0\). Then the elements
\[
(6) \quad x^*_n = \sum_{k \in N_n} \frac{y^*_k}{1 + \varepsilon_k}
\]
generate \(l^\infty\) isometrically (as in [2]).

**Proof.** Clearly, \(\|x^*_n\| \leq 1\) for all \(n \in \mathbb{N}\) by the first half of (5). For the inverse inequality we have
\[
\|x^*_n\| \geq \left\| \frac{2}{1 + \varepsilon_m} \right\| - \left\| \sum_{k \in N_n, k \neq m} \frac{y^*_k}{1 + \varepsilon_k} \right\| \geq \frac{2}{1 + \varepsilon_m} - 1
\]
for all \(m \in N_n\), hence \(\|x^*_n\| \geq 1\) by the second half of (5), which proves \(\|x^*_n\| = 1\).
Similarly we show (2): First, “≤” of (2) follows from the first half of (5); second, by the inequality just shown we have
\[ \left\| \sum \alpha_n x_n^* \right\| \geq 2|\alpha_m| - \left\| \alpha_m x_m^* - \sum_{n \neq m} \alpha_n x_n^* \right\| \geq 2|\alpha_m| - \sup |\alpha_n| \]
for all \( m \in \mathbb{N} \), giving “≥” of (2). \( \blacksquare \)

Proof of the Theorem. Let \((\delta_n)\) be a sequence in \([0,1]\) converging to 0. Suppose \((x_n)\) is an \(l^1\)-basis and write \( \tilde{c} = \tilde{c}_J(x_n) \) for short.

Observation. Given \( \tau > 0 \) there is a subsequence \((x_{n_k})\) of \((x_n)\) such that \( |\tilde{c} - c_J(x_{n_k})| < \tau \). By James’ \(l^1\)-distortion theorem, as described above, there are pairwise disjoint finite sets \( A_l \subset \{ n_k : k \in \mathbb{N} \} \) and a sequence \((\lambda_n)\) of scalars such that (1) holds with \( \tilde{z}_l, z_l \) as above; furthermore \( \|\tilde{z}_l\| \to c_J(x_{n_k}) \), whence the existence of \( l' \) such that \( |\tilde{c} - \|\tilde{z}_{l'}\|| < \tau \).

By induction over \( n \in \mathbb{N} \) we will construct finite sequences \((y_i^{(n)*})_{i=1}^n\) in \(X^*\), a sequence \((\tilde{y}_n)\) in \(X\), pairwise disjoint finite sets \(C_n \subset \mathbb{N}\) and a scalar sequence \((\mu_n)\) such that, with the notation \(y_n = \tilde{y}_n/\|\tilde{y}_n\|\),
\[
\sum_{k \in C_n} |\mu_k| = 1, \quad \tilde{y}_n = \sum_{k \in C_n} \mu_k x_k, \quad |\tilde{c} - \|\tilde{y}_n\|| < \delta_n,
\]
\[
|y_i^{(n)*}(y_i)| > 1 - \delta_i \quad \forall i \leq n, \tag{8}
\]
\[
y_i^{(n)*}(y_l) = 0 \quad \forall l < i \leq n, \tag{9}
\]
\[
y_i^{(n)*}(x_p) = 0 \quad \forall p \in C_l, \forall l < i \leq n, \tag{10}
\]
\[
\left\| \sum_{i=1}^m \alpha_i y_i^{(n)*} \right\| \leq \max_{i \leq m} (1 + (1 - 2^{-n})\delta_i) |\alpha_i| \quad \forall m \leq n, \alpha_i \text{ scalars}. \tag{11}
\]

For \( n = 1 \) we use the observation above with \( \tau = \delta_1 \) and choose \( l_1 \) such that \( \|\tilde{z}_{l_1}\| - \| \tilde{c} \| < \delta_1 \). Then we choose \( y_i^{(1)*} \) such that \( \|y_i^{(1)*}\| = 1 \) and \( y_i^{(1)*}(z_{l_1}) = \|z_{l_1}\| \). It remains to set \( C_1 = A_{l_1}, \mu_k = \lambda_k \) for \( k \in C_1 \) and \( \tilde{y}_1 = \tilde{z}_{l_1} \).

For the induction step \( n \to n + 1 \) we recall that \((P^*)|X^*\) is an isometric isomorphism from \(X^*\) onto \(X_s^\perp\), that \(X^{***} = \bigperp \bigoplus X_s^\perp\) and that \((P^* x^*)|X = (x^*)|X\) for all \(x^* \in X^*\). Let \((z_i)\) be as in the observation above with \( \tau = \delta_{n+1} \) and let \( z_s \in X^{**} \) be a \(w^*\)-accumulation point of the \( z_l \). Then \( z_s \in X_s \) and \( \|z_s\| = 1 \) (as explained in the preliminaries). Choose \( t \in \ker P^* \subset X^{***} \) such that \( \|t\| = 1 \) and \( t(z_s) = \|z_s\| \). Put
\[
E = \text{lin}(\{P^* y_i^{(m)*} \mid i \leq m \leq n\} \cup \{t\}) \subset X^{***},
\]
\[
F = \text{lin}(\{z_s\} \cup \{x_p \mid p \in \bigcup_{l \leq n} C_l\}) \subset X^{**}
\]
and choose \( \eta > 0 \) such that
\[
(1 + \eta)(1 + (1 - 2^{-n})\delta_i) < 1 + (1 - 2^{-(n+1)})\delta_i \quad \text{and} \quad \eta < (1 - 2^{-(n+1)})\delta_{n+1}
\]
for all \( i \leq n \). The principle of local reflexivity provides an operator \( R : E \to X^* \) such that

\[
(12) \quad (1 - \eta)\|e^{**}\| \leq \|Re^{**}\| \leq (1 + \eta)\|e^{**}\|,
\]

\[
(13) \quad f^{**}(Re^{**}) = e^{**}(f^{**}),
\]

for all \( e^{**} \in E \) and \( f^{**} \in F \).

We define \( y_i^{(n+1)*} = R(P^*y_i^{(n)*}) \) for \( i \leq n \) and \( y_{n+1}^{(n+1)*} = Rt \) and obtain (11) \( n+1 \) (with \( \alpha_i = 0 \) if \( m < i \leq n + 1 \)) by

\[
\left\| \sum_{i=1}^{n+1} \alpha_i y_i^{(n+1)*} \right\| \leq (1 + \eta) \left( \left\| \sum_{i=1}^{n} \alpha_i P^* y_i^{(n)*} \right\| + \alpha_{n+1} t \right) \]

\[
= (1 + \eta) \max \left( \left\| \sum_{i=1}^{n} \alpha_i P^* y_i^{(n)*} \right\|, \|\alpha_{n+1} t\| \right) \]

\[
= (1 + \eta) \max \left( \left\| \sum_{i=1}^{n} \alpha_i y_i^{(n)*} \right\|, \|\alpha_{n+1} t\| \right) \]

\[\leq \max_{i \leq n+1} (1 + (1 - 2^{-(n+1)}) |\delta_i|) |\alpha_i| \]

Since \( z_s \) is a \( w^* \)-cluster point of \( (z_l) \) we have

\[
|y_{n+1}^{(n+1)*}(z_l)| > |z_s(y_{n+1}^{(n+1)*})| - \delta_{n+1} \leq |t(z_s)| - \delta_{n+1} = 1 - \delta_{n+1}
\]

for infinitely many \( l \); furthermore, an \( l_{n+1} \) can be chosen among those \( l \) so as to obtain \( |\tilde{z}_{l_{n+1}} - \tilde{c}| < \delta_{n+1} \). Set \( C_{n+1} = A_{l_{n+1}}, \tilde{y}_{n+1} = \tilde{z}_{l_{n+1}}, \mu_k = \lambda_k \) for \( k \in C_{n+1} \). Then (17) \( n+1 \) holds and (8) \( n+1 \) holds for \( i = n + 1 \). For \( i \leq n \), (8) \( n+1 \) follows from

\[
y_i^{(n+1)*}(y_i) = (P^*y_i^{(n)*})(y_i) = y_i^{(n)*}(y_i) \leq 1 - \delta_i.
\]

Condition (10) \( n+1 \) holds for \( i = n + 1 \) by

\[
y_{n+1}^{(n+1)*}(x_p) = (Rt)(x_p) = t(x_p) = 0 \quad \forall p \in C_l, \forall l < n + 1
\]

and it holds for \( i < n + 1 \) by

\[
y_i^{(n+1)*}(x_p) = (P^*y_i^{(n)*})(x_p) = y_i^{(n)*}(x_p) \leq 0 \quad \forall p \in C_l, \forall l < i.
\]

Condition (9) \( n+1 \) follows from (10) \( n+1 \). This ends the induction.

Now we define \( y_i^* = \frac{1}{1+\delta_i} \lim_{n \in U} y_i^{(n)*} \) for all \( i \in \mathbb{N} \) where \( U \) is a fixed nontrivial ultrafilter on \( \mathbb{N} \) and where the limit is understood in the \( w^* \)-topo-
logy of $X^*$. Then by $w^*$-lower semicontinuity of the norm and by (11),
\[ \left\| \sum \alpha_i y_i^* \right\| \leq \sup (1 + \delta_i) \frac{|\alpha_i|}{1 + \delta_i} = \sup |\alpha_i| \]
for all $(\alpha_i) \in l^\infty$. In particular, $\|y_i^*\| \leq 1$, hence $\|y_i^*\| \to 1$ by (8) and $(y_i^*)$ satisfies (5) for $\varepsilon_n = 0$.

Let $(N_n)$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $(i_n)$ increases strictly where $i_n = \min N_n$. By the proposition the sequence defined by
\[ x_n^* = \sum_{i \in N_n} y_i^* \]
generates $l^\infty$ isometrically and we have
\[ |x_n^*(y_{i_n})| \geq \frac{1 - \delta_{i_n}}{1 + \delta_{i_n}}. \]

By construction of the $y_i$ there is, for each $n \in \mathbb{N}$, an index $p_n \in C_{i_n}$ such that
\[ (1 + \delta_{i_n})|x_n^*(x_{p_n})| \geq (1 - \delta_{i_n}) \|y_{i_n}\| \geq (1 - \delta_{i_n})(\tilde{c} - \delta_{i_n}), \]
which will yield “$\geq$” of (3). In order to show “$\leq$” of (3) suppose to the contrary that $\sum_{j \neq m}|x_{n_j}^*(x_{p_{nm}})| < \kappa/2$ for all $m$. Then given $(\alpha_m)$ and $\theta_m$ such that $\theta_m \alpha_m = |\alpha_m|$ we obtain
\begin{align*}
(1 + \delta_{i_n}) |x_n^*(x_{p_n})| &\geq (1 - \delta_{i_n}) \|y_{i_n}\| \\
&\geq (1 - \delta_{i_n})(\tilde{c} - \delta_{i_n}),
\end{align*}
which yields the contradiction $c_J(x_{p_{nm}}) > \tilde{c}$ and thus shows “$\leq$” and all of (3), whereas (4) follows from (10) via $y_i^*(x_p) = 0$ for $p \in C_l$, $l < i$.

The last assertion of the theorem is immediate from the fact that non-reflexive L-embedded spaces contain $l^1$ isomorphically [4, IV.2.3] .

**Remarks.** 1. It is not clear whether (4) can be obtained also for $l > n$. What can be said by Simons’ extraction lemma (used in the proof) is that, under the assumptions of the theorem and given $\varepsilon > 0$, it is possible (after passing to appropriate subsequences) to deduce in addition to (4) that
\[ \sum_{n=1}^{l-1} |x_n^*(x_{p_l})| = \sum_{n \neq l} |x_n^*(x_{p_l})| < \varepsilon \]
for all $l$. In case $c_J(x_n) = 1 = \lim \|x_n\|$. 

(which happens when the $x_n$ span $l^1$ almost isometrically) this can be improved to

$$\sum_{n \neq l} |x_n^*(x_{p_l})| = \left( \sum_n |x_n^*(x_{p_l})| \right) - |x_l^*(x_{p_l})| \leq \|x_{p_l}\| - |x_l^*(x_{p_l})| \to 0. \quad (15)$$

One might also construct straightforward perturbations of the $x_n^*$ in order to get $L$ for $l \neq n$ but then it is not clear whether these perturbations can be arranged to span $c_0$ isometrically, not just almost isometrically.

Since in general $L$-embedded spaces do not contain $l^1$ isometrically (see below, last remark) it is not in general possible, in case all $x_n$ have the same norm, to improve $L$ and $(4)$ so as to obtain $x_n^*(x_{p_l}) = \tilde{c}(x_m)$ if $l = n$ and $= 0$ if $l \neq n$.

2. As already alluded to in the introduction, the construction of $c_0$ in this paper bears much resemblance to the one of [7]. A different way to construct $c_0$ is contained in [9] but it seems unlikely that this construction can be improved to yield an isometric $c_0$-copy.

3. It follows from $(3)$ (or rather from a reasoning similar to the one in $(14)$) that $c(x_{p_n}) \geq \tilde{c}(x_n)$, which means that in $L$-embedded spaces the sup in the definition of $c_J$ is attained by the James constant of an appropriate subsequence. For general Banach spaces this is not known, although it can be shown by a routine diagonal argument that each bounded sequence $(x_n)$ admits a $c_J$-stable subsequence $(x_{n_k})$ (meaning that $\tilde{c}_J(x_{n_k}) = c_J(x_{n_k})$) whose James constant is arbitrarily near to $\tilde{c}(x_n)$.

4. Each normalized sequence $(x_n)$ in an $L$-embedded Banach space that spans $l^1$ almost isometrically contains a subsequence each of whose $w^*$-accumulation points in the bidual attains its norm on the dual unit ball. To see this, let $(x_n^*)$ and $(x_{p_n})$ be the sequences given by the theorem and by Simons’ extraction lemma (see $(15)$ above), let $x_s$ be a $w^*$-accumulation point of the $x_{p_n}$ and let $x^* = \sum x_n^*$; then $\|x^*\| = 1$ and on the one hand $\|x_s\| = 1$ by [8] and on the other hand $x_s(x^*) = \lim x^*(x_{p_n}) = \lim x_n^*(x_{p_n}) = c_J(x_n) = 1$.

It would be interesting to know whether this remark holds for the whole sequence $(x_n)$ instead of only a subsequence $(x_{p_n})$. A kind of converse follows from [9, Rem. 2] for separable $X$: If $x_s \in X_s$ attains its norm on the dual unit ball then it does so on the sum of a wuC-series.

5. Let us finally note that the presence of isometric $c_0$-copies in $X^*$ does not necessarily entail the presence of isometric copies of $l^1$ in $X$ even if $X$ is the dual of an M-embedded Banach space. This follows from [11, Cor. III.2.12], which states that there is an $L$-embedded Banach space which is the dual of an M-embedded space (to wit, the dual of $c_0$ with an equivalent norm) which is strictly convex and therefore does not contain $l^1$ isometrically although it contains, as do all non-reflexive L-embedded spaces, $l^1$ asymptotically ([8],...
see [3] for the definition of asymptotic copies (1) and the difference from almost isometric ones).

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(1) In the literature there is another notion of “asymptotic $l^p$” which is quite different from the one of this note.