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Functions Equivalent to Borel Measurable Ones

by

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Summary. Let X and Y be two Polish spaces. Functions $f, g : X \to Y$ are called equivalent if there exists a bijection φ from X onto itself such that $g \circ \varphi = f$. Using a theorem of J. Saint Raymond we characterize functions equivalent to Borel measurable ones. This characterization answers a question asked by M. Morayne and C. Ryll-Nardzewski

1. Introduction. Let X and Y be two sets. Following Szpilrajn ([12]), we say that functions $f, g : X \to Y$ are *equivalent* if there exists a bijection φ from X onto itself such that $g \circ \varphi = f$. It is straightforward that f and g are equivalent if and only if $|f^{-1}(\{y\})| = |g^{-1}(\{y\})|$ for every $y \in Y$.

Let X be a Polish (i.e. separable, completely metrizable) topological space and let $\mathcal{B} \subset \mathcal{P}(X)$ be the family of Borel subsets of X. For a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ we say that a function $g : X \to Y$ is $\sigma(\mathcal{B}, \mathcal{I})$-*measurable* if $g^{-1}(U)$ is an element of the smallest σ-algebra containing $\mathcal{B}$ and $\mathcal{I}$ for any open set $U \subset Y$. In [9] Morayne and Ryll-Nardzewski proved that if a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ contains a set of size continuum and admits a Borel base, which is true in particular for the ideal of meager sets and the ideal of Lebesgue measure null-subsets of the interval $[0, 1]$, then a given function $f : X \to \mathbb{R}$ is equivalent to some $\sigma(\mathcal{B}, \mathcal{I})$-measurable function $g : X \to \mathbb{R}$ if and only if $\{y \in \mathbb{R} : f^{-1}(\{y\}) \neq \emptyset\}$ contains a topological copy of the Cantor space or there exists $y \in \mathbb{R}$ such that $|f^{-1}(\{y\})| = 2^{\aleph_0}$. They also asked if there is a similar characterization of functions equivalent to Borel measurable ones.

We answer this question using a recent result of J. Saint Raymond ([11]). The result of Morayne and Ryll-Nardzewski was proved by Kysiak for some σ-ideals not admitting a Borel base, in particular for the ideal of Marczewski null sets and the ideal of completely Ramsey null sets ([7]). Kwiatkowska ([6])

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proved that a function $f:[0,1]\to[0,1]$ is equivalent to a continuous function if and only if it fulfills certain requirements related to the Darboux property alongside with the descriptive set-theoretical requirement that the set $\{x\in[0,1]:|f^{-1}(x)|=2^{\aleph_0}\}$ is analytic.

2. Notation. All spaces considered are separable and metrizable. The *Baire space*, $\mathcal{N}$, is defined as the infinite countable product of $\mathbb{N}$, equipped with the Tikhonov topology. Let X be a Polish space. A set $A\subset X$ is *analytic* if it is a continuous image of $\mathcal{N}$. A set $C\subset X$ is *coanalytic* if its complement $X\setminus C$ is analytic.

Given a subset $A\subset X\times Y$ we define $A_x=\{y\in Y:(x,y)\in A\}$ and $A^y=\{x\in X:(x,y)\in A\}$.

Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences of the natural numbers. A nonempty subset $T\subset\mathbb{N}^{<\mathbb{N}}$ is a *tree* if $\forall n\ \forall k\le n\ [(t_0,\dots,t_{n-1})\in T\Rightarrow(t_0,\dots,t_{k-1})\in T]$. The space $\mathrm{Tr}\subset 2^{\mathbb{N}^{<\mathbb{N}}}$ is the set of all trees endowed with the topology inherited from $2^{\mathbb{N}^{<\mathbb{N}}}$, it is homeomorphic to the Cantor set. Let T be a tree. We say that $(x_0,x_1,\dots)$ is a *branch* of T if for every $n\in\mathbb{N}$ we have $(x_0,x_1,\dots,x_n)\in T$. Denote by $\mathrm{WF}\subset\mathrm{Tr}$ the set of all *well-founded* trees (i.e. having no branches), and by $\mathrm{UB}\subset\mathrm{Tr}$ the set of all trees with exactly one branch.

We will need the following lemma which is a known generalization of the Lusin Unicity Theorem ([1, 18.11]). We include a proof since we have been unable to find a direct reference. Our method of proof is similar to a method used by Z. Koslova in [3].

LEMMA 1. *For any Borel subset $B\subset X\times Y$ of the product of Polish spaces X,Y, the sets $B(n)=\{x\in X:|B_x|=n\}$ are coanalytic for $n=0,1,\dots,\aleph_0$.*

Proof. Being a complement of the projection of B onto X, the set $B(0)$ is coanalytic. The Lusin Unicity Theorem guarantees that $B(1)$ is coanalytic. Let $2\le n<\aleph_0$ and let $U_1,U_2,\dots$ be a basis of Y. Put $V_i=X\times U_i$, $i=1,2,\dots$. Since

$$x\in B(n)\ \Leftrightarrow\ \exists k_1,\dots,k_n\ \Big[(\forall i\ne j\ V_{k_i}\cap V_{k_j}=\emptyset)\\ \wedge(\forall i\le n\ |(B\cap V_{k_i})_x|=1)\wedge\Big(\Big(B\setminus\bigcup_{i=1}^{n}V_{k_i}\Big)_x=\emptyset\Big)\Big],$$

the set $B(n)$ is coanalytic.

Consider now the case $n=\aleph_0$. Let $f:F\to B$ be a continuous bijection defined on a closed set $F\subset\mathcal{N}$ (cf. [1, 13.7]). Put $G=\{(x,y,z):f(z)=(x,y)\}\subset X\times(Y\times\mathcal{N})$. For a given $x\in X$ the formula $y\mapsto(y,f^{-1}(x,y))$ defines a bijection between B_x and G_x. Hence it is sufficient to show that

$G(\aleph_0) = \{x \in X : |G_x| = \aleph_0\}$ is coanalytic. Note that G is closed and in particular all vertical sections of G are closed. Hence we may assume that B_x is closed for every $x \in X$ (we may replace B with G). The section B_x has infinitely many isolated points iff

$$\forall m \in \mathbb{N}\ \exists k_1, \ldots, k_m\ [(\forall i \neq j\ V_{k_i} \cap V_{k_j} = \emptyset) \wedge (\forall i = 1, \ldots, m\ |(B \cap V_{k_i})_x| = 1)].$$

Hence the lemma follows from the observation that

$$x \in B(\aleph_0) \ \Leftrightarrow\ B_x \text{ is countable and has infinitely many isolated points}$$

(the set $\{x \in X : |B_x| \leq \aleph_0\}$ is coanalytic, cf. [1, Theorem 29.19]). ∎

3. Complete pairs of disjoint coanalytic sets. In this section we deal with pairs of disjoint coanalytic sets. For short we call them just *pairs*. Following Louveau and Saint Raymond (see [8]) we introduce the notion of a reduction of pairs and the notion of a complete pair (called in [8] *une pair reductrice pour Γ*).

Let X and Y be Polish spaces and let (A, B) and (C, D) be pairs in X and Y respectively. We say that a function $r : X \to Y$ *reduces* (A, B) to (C, D) if for any $x \in X$,

$$x \in A \Leftrightarrow f(x) \in C \quad \text{and} \quad x \in B \Leftrightarrow f(x) \in D.$$

We say that r is a *reduction* of (A, B) to (C, D).

A pair (C, D) in a Polish space Y is *complete* (see [8, a definition before Theorem 9] and Definition 1 in [11]) if for any pair (A, B) in the Baire space there exists a continuous reduction of (A, B) to (C, D). We will need the following result of Saint Raymond:

THEOREM 2 ([11, Theorem 23]). *The pair* (WF, UB) *is complete.*

Note that since any Polish space X can be embedded into the Baire space by means of a Borel measurable and bijective function, if (C, D) is a complete pair and (A, B) is a pair in X, then there exists a Borel measurable function reducing (A, B) to (C, D) (it is the superposition of the Borel measurable embedding and the reduction). Hence we obtain

COROLLARY 3. *Let A,B be coanalytic and disjoint subsets of a Polish space Y. Then there exists a Borel function $f : Y \to \mathrm{Tr}$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$ and $x \in B$ if and only if $f(x) \in \mathrm{UB}$.*

Using a method of [2] one can prove that if every pair (A, B) in every Polish space can be reduced to (C, D) by a Borel measurable function, then the pair (C, D) is complete.

4. Construction. For a given countable family of disjoint, coanalytic subsets of a Polish space Y indexed by the numbers $\{0, \ldots, \aleph_0\}$, we will construct (Corollary 8) a Borel function on a Polish space X such that the coanalytic set with index $n \in \{0, \ldots, \aleph_0\}$ is equal to the set of points with all preimages of cardinality n. Furthermore, we will prove that the function may be chosen to be of the first Baire class.

LEMMA 4. *Let Y be a Polish space and let $B(0), B(1), \ldots, B(\aleph_0)$ be subsets of Y. The following conditions are equivalent:*

- (i) *For every uncountable Polish space X there exists a Borel measurable function $f : X \to Y$ such that $B(n) = \{y \in Y : |f^{-1}(y)| = n\}$ for every $n \in \{0, 1, \ldots, \aleph_0\}$.*
- (ii) *For every uncountable Polish space X there exists an uncountable Borel set $B \subset X \times Y$ such that $B(n) = \{y \in Y : |B^y| = n\}$ for every $n \in \{0, 1, \ldots, \aleph_0\}$.*
- (iii) *There exists an uncountable closed set $F \subset \mathcal{N} \times Y$ such that $B(n) = \{y \in Y : |F^y| = n\}$ for every $n \in \{0, 1, \ldots, \aleph_0\}$.*
- (iv) *There is a Polish space X and an uncountable Borel set $B \subset X \times Y$ such that $B(n) = \{y \in Y : |B^y| = n\}$ for every $n \in \{0, 1, \ldots, \aleph_0\}$.*

Proof. Since the graph of a Borel measurable function $f : X \to Y$ is a Borel subset of $X \times Y$, obviously (i) implies (ii). Now, let X be an uncountable Polish space and let $B \subset X \times Y$ be an uncountable Borel set. To show that (ii) implies (iii), fix a closed subset A of $\mathcal{N}$ and a continuous bijection $g : A \to B$ (cf. [1, 13.7]) and define $F = \{(z, y) \in \mathcal{N} \times Y : z \in A, y = \pi_Y \circ g(z)\}$, where $\pi_Y : X \times Y \to Y$ is the projection. Then F satisfies (iii), because for a given $y \in Y$ the formula $x \mapsto g^{-1}(x, y)$ defines a bijection between B^y and F^y. Clearly (iii) implies (iv). Finally, to prove (iv)⇒(i) assume that X and $B \subset X \times Y$ satisfy (iv) and X' is an arbitrary uncountable Polish space. Fix a Borel isomorphism $g : X' \to B$ (any two uncountable Borel subsets of Polish spaces are Borel isomorphic, cf. [1, 15.6]). Now we can define $f = \pi_Y \circ g$. The proof is complete. ∎

THEOREM 5. *Let $B(0), B(1), \ldots, B(\aleph_0)$ be subsets of a Polish space Y. If $B(0), B(1), \ldots, B(\aleph_0)$ are coanalytic, pairwise disjoint and $\bigcup_{n=0,\ldots,\aleph_0} B(n) \neq Y$ or $Y \setminus B(0)$ is uncountable then $B(0), B(1), \ldots, B(\aleph_0)$ satisfy conditions* (i)–(iv) *of Lemma* 4.

If $B(0), B(1), \ldots, B(\aleph_0) \subset Y$ satisfy one of the conditions of Lemma 4 *then they are coanalytic, pairwise disjoint and $Y \setminus B(0)$ is uncountable or $\bigcup_{n=0,\ldots,\aleph_0} B(n) \neq Y$.*

Proof. Let $B(0), B(1), \ldots, B(\aleph_0)$ be pairwise disjoint coanalytic subsets of a Polish space Y. We will prove that condition (iv) of Lemma 4 is fulfilled. To this end we define $B(2^{\aleph_0}) = Y \setminus \bigcup_{n=0,\ldots,\aleph_0} B(n)$. The set $B(2^{\aleph_0})$ is analytic and if it is nonempty, then there exists a continuous surjection $s : \mathcal{N} \to B(2^{\aleph_0})$.

CLAIM 6. *If $B(0)$ and $B(1)$ are two disjoint coanalytic subsets of a Polish space Y then there exists a Borel set $B \subset \mathcal{N} \times Y$ such that $B(0) \subset \{y \in Y : B^y = \emptyset\}$, $B(1) \subset \{y \in Y : B^y$ contains exactly one element$\}$.*

Proof of the Claim. Due to Corollary 3 there exists a Borel map $r : Y \to \mathrm{Tr}$ reducing the pair $(B(0), B(1))$ to $(\mathrm{WF}, \mathrm{UB})$. Put $B = \{(x, y) : x$ is an infinite branch of $r(y)\}$. ∎

By the Claim there exist Borel sets $B_n \subset \mathcal{N} \times Y$ such that $\bigcup_{i \le n} B(i) \subset \{y \in Y : (B_n)^y = \emptyset\}$ and $\bigcup_{n < i \le \aleph_0} B(i) \subset \{y \in Y : (B_n)^y$ contains exactly one element$\}$, $n = 0, 1, 2, \ldots$.

If $B(2^{\aleph_0}) \neq \emptyset$ then we define $X = (\mathcal{N} \times \mathcal{N}) \oplus (\mathbb{N} \times \mathcal{N})$ and $B = \{(x, z, y) : s(x) = y\} \oplus \bigcup_{n \in \mathbb{N}} \{n\} \times B_n$. If $B(2^{\aleph_0}) = \emptyset$ then we define $X = \mathbb{N} \times \mathcal{N}$ and $B = \bigcup_{n \in \mathbb{N}} \{n\} \times B_n$. The set B is uncountable and has all the required properties.

Now, let Y and $B(0), B(1), \ldots, B(\aleph_0) \subset Y$ satisfy condition (iv) of Lemma 4 for a Polish space X and $B \subset X \times Y$. The sets $B(0), B(1), \ldots, B(\aleph_0)$ are pairwise disjoint and by Lemma 1 they are coanalytic. Since B is uncountable, $Y \setminus B(0)$ is uncountable or $\bigcup_{n=0,\ldots,\aleph_0} B(n) \neq Y$. ∎

In the next corollary we will need the following

PROPOSITION 7. *Let $f : X \to Y$ be a Borel function between Polish spaces X and Y. Then there exist maps of the first Baire class $g : X \to Y$ and $h : X \to X$ such that $g = f \circ h$.*

Proof. Let ϕ be a bijection of the first Baire class between X and the graph of f (see [5, Corollaire, p. 212]). Put $h = \pi_X \circ \phi$, $g = \pi_Y \circ \phi$, where π_X and π_Y are the projections from $X \times Y$ onto X and Y, respectively. ∎

COROLLARY 8. *Let X and Y be Polish spaces. For a function $f : X \to Y$ the following conditions are equivalent*

(i) *f is equivalent to a function of the first Baire class,*
(ii) *f is equivalent to a Borel measurable map,*
(iii) *$|f^{-1}(y)| \in \{0, 1, \ldots, \aleph_0, 2^{\aleph_0}\}$ for every $y \in Y$ and $\{y \in Y : |f^{-1}(y)| = n\}$ is coanalytic for $n = 0, \ldots, \aleph_0$.*

REMARK. Equivalence of (i) and (ii) was proved by Szpilrajn ([12, Section 3.1]).

Proof. The equivalence of (ii) and (iii) follows immediately from Theorem 5 and Lemma 4. The implication (i)⇒(ii) is obvious and (ii)⇒(i) follows from Proposition 7. ∎

Note that Corollary 8 gives a full answer to the Morayne and Ryll-Nardzewski's question.

5. Functions equivalent to bimeasurable or continuous ones. Let X, Y be Polish spaces. Following [10] we will call a function $f : X \to Y$ *bimeasurable* if it is Borel and $f[B]$ is Borel for every $B \in \mathcal{B}(X)$. In the previous section, for a given function $f : X \to Y$ we described conditions which guarantee that f is equivalent to a Borel measurable function. We may consider a more general problem of characterizing those functions which are equivalent to a function from a given class. In this section we consider the problem for two classes of functions:

- Theorem 10 indicates conditions which ensure that $f : X \to Y$ is equivalent to a bimeasurable function; in Proposition 11 we consider the notion of *Borel equivalence* of functions and prove that it is connected with the notion of bimeasurability.
- In Theorem 13 we assume that $X = \mathcal{N}$ and formulate necessary and sufficient conditions for f to be equivalent to a continuous function.

First, we recall

THEOREM 9 (Purves [10]). *A Borel function $f : X \to Y$ is bimeasurable if and only if for all but countably many $y \in Y$ the fiber $f^{-1}(y)$ is countable.*

THEOREM 10. *A function $f : X \to Y$ is equivalent to a bimeasurable one if and only if for all but countably many $y \in Y$ the fiber $f^{-1}(y)$ is countable, the cardinalities of all fibers of f belong to the set $\{0, 1, \ldots, \aleph_0, 2^{\aleph_0}\}$ and the sets $\{y \in Y : |f^{-1}(y)| = n\}$ are Borel for $n = 0, \ldots, \aleph_0$.*

Proof. Necessity follows directly from Theorem 9 and the fact that every uncountable Borel set has cardinality $2^{\aleph_0}$. The above condition is also sufficient. By Corollary 8 the function f is equivalent to some Borel measurable function g. By Theorem 9 the function g is bimeasurable. ∎

We say that $f, g : X \to Y$ are *Borel equivalent* if there is a Borel automorphism $h : X \to X$ such that $f = g \circ h$. Directly from the definition, for every pair of Borel equivalent functions, if one of them is Borel, then so is the other. The converse is in general false—Proposition 11 gives a wide class of examples of pairs of Borel functions which are equivalent, but not Borel equivalent.

For fixed spaces Y, X and a subset $F \subset Y \times X$, we call a Borel function $u : \pi_Y[F] \to X$ a *Borel uniformization* of F if $(y, u(y)) \in F$ for every $y \in \pi_Y[F]$.

OBSERVATION. (a) If $f, g : X \to Y$ are Borel equivalent, $Y_0 \subset Y$ and the set $F = \{(y, x) \in Y_0 \times X : f(x) = y\}$ has a Borel uniformization, then $G = \{(y, x) \in Y_0 \times X : g(x) = y\}$ has one as well (take the superposition of any uniformization of F and an equivalence of f and g).

(b) For every uncountable Borel set B:

- There exists a Borel function $\psi : B \to \mathcal{N}$ with all fibers uncountable such that $\{(y,x) \in \mathcal{N} \times B : \psi(x) = y\}$ has a Borel uniformization u. Indeed, if $f : B \to \mathcal{N} \times \mathcal{N}$ is a Borel isomorphism and $s : \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ is given by $s(x) = (x, y_0)$ for some $y_0 \in \mathcal{N}$, then we define
$$\psi = \pi_1 \circ f, \qquad u = f^{-1} \circ s,$$
where $\pi_i : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is the projection on the ith coordinate ($i = 1, 2$).
- There exists a Borel function $\phi : B \to \mathcal{N}$ with all fibers uncountable such that $\{(y,x) \in \mathcal{N} \times B : \phi(x) = y\}$ does not have a Borel uniformization. Indeed, let $A \subset \mathcal{N} \times \mathcal{N}$ be a Borel set without a Borel uniformization and with all vertical sections uncountable (cf. [1, Exercise 18.17, one solution on p. 360 and another one on p. 365 in notes to 35.1]) and let $f : B \to A$ be a Borel isomorphism. We define
$$\phi = \pi_1 \circ f.$$
Then for every Borel uniformization $u : \mathcal{N} \to B$ of the set $\{(y,x) \in \mathcal{N} \times B : \phi(x) = y\}$, the superposition $\pi_2 \circ f \circ u : \mathcal{N} \to \mathcal{N}$ would be a Borel uniformization of A, a contradiction.

PROPOSITION 11. *Let $f : X \to Y$ be a Borel function. Then the following conditions are equivalent:*

(i) *every Borel map from X to Y equivalent to f is also Borel equivalent to f,*
(ii) *f is bimeasurable.*

Proof. (ii)⇒(i). Let $g : X \to Y$ be a Borel map equivalent to f and let F, G be the graphs of f, g, respectively. Then the sets
$$F(n) = \{y : |f^{-1}(y)| = n\}, \qquad G(n) = \{y : |g^{-1}(y)| = n\}$$
are Borel, $F(n) = G(n)$ ($n = 1, 2, \ldots, \aleph_0, 2^{\aleph_0}$) and $F(2^{\aleph_0}) = G(2^{\aleph_0}) = \{y_j : j \in J\}$ for a countable set J.

Let $n \le \aleph_0$. For every $y \in F(n)$ the horizontal section F^y is of cardinality n. By the Lusin–Novikov Theorem ([1, 18.10]) there are Borel functions $f_n^i : F(n) \to X$, $1 \le i \le n$, such that $\bigcup_{i=1}^n \mathrm{Graph}(f_n^i) = F \cap (X \times F(n))$. Maps g_n^i are defined in the same way. Let $A_n^i = f_n^i[F(n)]$, $B_n^i = g_n^i[G(n)]$, $1 \le n \le \aleph_0$, $1 \le i \le n$, and let $A_j = f^{-1}(y_j)$, $B_j = g^{-1}(y_j)$, $j \in J$. Now, let $h_n^i : A_n^i \to B_n^i$, $h_j : A_j \to B_j$ be Borel isomorphisms. Finally, we define the required Borel automorphism $h : X \to X$ to be h_n^i on A_n^i, $1 \le i \le n \le \aleph_0$, and h_j on A_j, $j \in J$.

(i)⇒(ii). Assume, aiming for a contradiction, that f is not bimeasurable. By Theorem 9 the set $\{y \in Y : |f^{-1}(y)| = 2^{\aleph_0}\}$ is uncountable. Since it is analytic (cf. [1, Theorem 29.19]), it contains a copy N of the Baire space. Let $B = f^{-1}[N]$. If the set $F = \{(y, x) \in N \times B : f(x) = y\}$ does not have a Borel uniformization, then we define $g : X \to Y$ to be ψ on B and f on $X \setminus B$. If the set F has a Borel uniformization, then we define g to be ϕ on B and f on $X \setminus B$ (ϕ, ψ are defined in Observation (b) preceding the proposition).

The fibers of f and g have the same cardinality, hence the functions are equivalent. However, according to Observation (a) applied to the set $N \subset Y$, they are not Borel equivalent. ∎

Let X, Y be Polish spaces and $f : X \to Y$ be a Borel function that is not bimeasurable. The construction in the proof of the implication (i)⇒(ii), after a minor modification, gives a family of $2^{\aleph_0}$ Borel functions which are pairwise Borel inequivalent, but equivalent to f.

The rest of the section is devoted to the characterization of the functions which are equivalent to a continuous function from $\mathcal{N}$ to a Polish space Y. We call a nonempty subset B of a Polish space X *locally uncountable* if every nonempty relatively open subset of B is uncountable. We will need the following standard lemma which is a variation of [1, Exercise 7.15]:

LEMMA 12. *If $B \subset X$ is a locally uncountable Borel set, then there exists a continuous bijection $\psi : \mathcal{N} \to B$.*

REMARK. (a) Let B be a Borel and locally uncountable set. We fix a continuous bijection $\psi : \mathcal{N}^2 \to B$ given by Lemma 12 ($\mathcal{N}^2$ and $\mathcal{N}$are homeomorphic). Let $D_0, D_1, \ldots$ be Borel, pairwise disjoint, dense subsets of $\mathcal{N}$ such that $\mathcal{N} = \bigcup_{i=0}^{\infty} D_i$. Then the sets $B_i = \psi[D_i \times \mathcal{N}]$, $i = 0, 1, \ldots$, are Borel, pairwise disjoint, locally uncountable and dense in B.

(b) Let A be a locally uncountable set. For a given set B, if $A \subset B \subset \overline{A}$, then B is also locally uncountable.

THEOREM 13. *Let $f : \mathcal{N} \to Y$ be a Borel function. Then the following conditions are equivalent:*

(i) *there is a continuous $g : \mathcal{N} \to Y$ equivalent to f,*
(ii) *there is a continuous $g : \mathcal{N} \to Y$ Borel equivalent to f,*
(iii) *for every open set U in Y, the preimage $f^{-1}[U]$ is empty or uncountable.*

Proof. The implications (i)⇒(iii) and (ii)⇒(i) are easy, so we give a proof of (iii)⇒(ii) only. Let K be a compactification of $\mathcal{N}$ and let $G = \{(x, f(x)) : x \in \mathcal{N}\} \subset K \times Y$ be the graph of f. Let A be the set of points of G with a countable neighborhood in G. Then both A and $Z = \pi_Y[A] \setminus \{y \in Y : |f^{-1}(y)| = 2^{\aleph_0}\}$ are countable. Let $B = G \setminus (A \cup \pi_Y^{-1}[Z])$. Note that $|B^y| = |G^y| = |f^{-1}(y)|$ for $y \in Y \setminus Z$ and $|B^y| = 0$ for $y \in Z$. According to (iii) and because of compactness of K, for every Z there is a point $k_y \in K$ such that $p_y = (k_y, y) \in \overline{B}^{K\times Y} \subset K \times Y$.

The set B is locally uncountable and Borel, hence Remark (a) guarantees that there exist Borel subsets $B_0, B_1, \dots$ of B which are pairwise disjoint, locally uncountable, dense in B and such that $B = \bigcup_{n\in\mathbb{N}} B_n$.

Since the sets $C_i = \{p_y : i < |f^{-1}(y)|,\ y \in Z\}$ are countable, the sets $\widetilde{B}_i = (B_i \cup C_i) \times \{i\}$ are Borel in $K \times Y \times \mathbb{N}$. Moreover, by Remark (b), they are locally uncountable. Let $\psi_i : \mathcal{N} \to \widetilde{B}_i$ be continuous bijections given by Lemma 12. Let $\widetilde{B} = \bigcup_{i\in\mathbb{N}} \widetilde{B}_i$ and let $\psi : \mathcal{N} \times \mathbb{N} \to \widetilde{B}$ be defined by $\psi(x, i) = \psi_i(x)$, $i \in \mathbb{N}$. For every $y \in Z$ take a bijection $\phi_y : \{p_y\} \times \{i : i < |f^{-1}(y)|\} \to G \cap \pi_Y^{-1}(y)$. Then $\phi : \widetilde{B} \to G$ defined to be $\pi_{K\times Y}$ on $\pi_{K\times Y}^{-1}[B]$ and ϕ_y on the domain of ϕ_y is a Borel isomorphism. Fix a homeomorphism $k : \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ and let $\widetilde{f} : \mathcal{N} \to G$ be defined by $\widetilde{f}(x) = (x, f(x))$. Now it is sufficient to define $g : \mathcal{N} \to Y$ and $h : \mathcal{N} \to \mathcal{N}$ by

$$g = \pi_Y \circ \phi \circ \phi \circ k, \qquad h = k^{-1} \circ \psi^{-1} \circ \phi^{-1} \circ \widetilde{f}. \ \blacksquare$$

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