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On Morphisms between Indecomposable Projective Modules over Special Biserial Algebras

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Summary. We investigate the categorical behaviour of morphisms between indecomposable projective modules over a special biserial algebra A over an algebraically closed field, which are associated to arrows of the Gabriel quiver of A.

1. Introduction and the main result. Let K be an algebraically closed field. Throughout the paper by an algebra we mean an associative basic finite-dimensional K-algebra. We denote by $\operatorname{mod} A$ the category of all finite-dimensional right A-modules. Further, we denote by rad_A the Jacobson radical of $\operatorname{mod} A$, generated by all non-isomorphisms between indecomposable modules in $\operatorname{mod} A$, and by $\operatorname{rad}_A^\infty$ the infinite radical of $\operatorname{mod} A$, which is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. For an algebra A, we consider its Auslander–Reiten quiver denoted by Γ_A. The vertices of Γ_A correspond to isomorphism classes $[X]$ of indecomposable A-modules X and there is an arrow $[X] \to [Y]$ between two vertices if and only if there is an irreducible morphism $f : X \to Y$, equivalently $f \in \operatorname{rad}_A \setminus \operatorname{rad}_A^2$. Throughout the paper we shall not distinguish between indecomposable modules in $\operatorname{mod} A$ and vertices of Γ_A. It is well known that Γ_A describes the quotient category $\operatorname{mod} A / \operatorname{rad}_A^\infty$. Let τ and τ^{-1} denote the Auslander–Reiten translations DTr and TrD in $\operatorname{mod} A$, respectively. By a component of Γ_A we shall mean a connected component of the translation quiver Γ_A. Following [23], we call a component $\mathcal{C}$ *generalized standard* if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$.

The class of special biserial algebras was introduced by Skowroński and Waschbüsch in [24]. An algebra A is said to be *special biserial* if there exists

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a finite bound quiver (Q_A, I_A) with $A \cong KQ_A/I_A$ such that:

(R1) Each vertex of Q_A is a source of at most two arrows and a target of at most two arrows.

(R2) For every arrow α in Q_A there exists at most one arrow β (respectively, γ) such that $\alpha\beta \notin I_A$ (respectively, $\gamma\alpha \notin I_A$).

A bound quiver (Q_A, I_A) satisfying (R1) and (R2) is called a *special biserial quiver*.

It was proved in [24] that all biserial representation-finite algebras are special biserial. Important examples of special biserial algebras are the Nakayama algebras and some blocks of group algebras with cyclic or dihedral defect groups (see [1], [8], [13]). Moreover, special biserial algebras occurred naturally in the Gelfand and Ponomarev description of the singular Harish-Chandra modules of the Lorentz group [14], the classification of restricted Lie algebras and infinitesimal groups with tame principal blocks ([11], [12]) and the classification of the finite-dimensional Hecke algebras of tame representation type [2].

In [25] Wald and Waschbüsch proved that special biserial algebras are of tame representation type and gave a classification of indecomposable finite-dimensional modules (see [4], [7] for alternative proofs). Then the tameness of biserial algebras was proved in [5] by Crawley-Boevey using geometric deformations to the class of special biserial algebras. Therefore, special biserial algebras play a prominent role and are often used as a test class for some general problems in the representation theory of finite-dimensional algebras. Nevertheless, the category of finite-dimensional modules over a special biserial algebra is often complicated and far from being well understood. For example, it was proved by Schröer in [19] that for any positive integer $n \geq 2$ there exists a special biserial algebra A of Krull–Gabriel dimension n. Moreover, Schröer proved in [20] that there are special biserial algebras A with arbitrarily complicated infinite radical of the module category $\operatorname{mod} A$. We also note that special biserial algebras have representation dimension at most 3 and finite finitistic dimension [9].

Let $A = KQ_A/I_A$ be an arbitrary algebra. Then $A \cong \operatorname{End}_A(A_A)$, where an isomorphism is given in the following way: to each element $a \in A$ we assign the morphism $f_a : A \to A$ which is left multiplication by a. Hence, we have a natural correspondence between an arrow $\alpha : x \to y$ in Q_A and the A-module homomorphism $f_\alpha : e_yA \to e_xA$ between indecomposable projective modules e_yA and e_xA, given by $f_\alpha(-) = \overline{\alpha} \cdot -$. Here, we denote by e_x the coset $\overline{\mathcal{E}_x} = \mathcal{E}_x + I_A$ of the trivial path $\mathcal{E}_x$ in (Q_A, I_A) at vertex x and by $\overline{\alpha}$ the coset $\alpha + I_A$ of an arrow α in Q_A. To simplify notation we shall write $P(z)$ for an indecomposable projective module e_zA, where z is a vertex in Q_A.

We are concerned with the general problem of describing properties of homomorphisms of type f_α between indecomposable projective A-modules lying in a common component of the Auslander–Reiten quiver Γ_A. The question of their structure, that is, how deep they emerge in the radical sequence $\mathrm{rad}_A \supseteq \mathrm{rad}_A^2 \supseteq \cdots \supseteq \mathrm{rad}_A^n \supseteq \cdots$ of the category $\mathrm{mod}\, A$, is of our special interest. Throughout the article we shall assume that A is a special biserial algebra of the form $A = KQ_A/I_A$ for a special biserial quiver (Q_A, I_A). We give a criterion for $f_\alpha \notin \mathrm{rad}_A^\infty$ in terms of walks in the bound quiver (Q_A, I_A). For this purpose we introduce some notation.

For a quiver $Q = (Q_1, Q_0, s, t)$ we denote by Q_0 the set of vertices in Q, by Q_1 the set of arrows in Q, and by $s, t : Q_1 \to Q_0$ two maps which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Let L be a quiver whose underlying graph is of the form $1 \frac{\quad}{} 2 \frac{\quad}{} \cdots \frac{\quad}{} r+1$, with r a non-negative integer. Fix an orientation of arrows of L. Then for an arbitrary quiver Q, a quiver homomorphism $\omega : L \to Q$ is called a *walk of length* r in Q from $\omega(1)$ to $\omega(r+1)$. These are the *starting point* $s(\omega)$ and the *ending point* $t(\omega)$ of ω, respectively. If L is of the form $1 \to 2 \to \cdots \to r+1$ then a quiver homomorphism $\omega : L \to Q$ is called a *path* (equivalently, an *oriented walk*). Recall that each associative basic algebra has a presentation as a path algebra KQ/I of a bound quiver (Q, I) where I is an ideal in KQ generated by relations, that is, elements of KQ of the form $\varrho = \sum_{i=1}^m \lambda_i \omega_i$ where λ_i are scalars, ω_i are paths in Q of length at least 2 with a common starting point and a common ending point. If $m = 1$, a relation ϱ is called a *zero-relation* or a *monomial* relation. If $m \geq 2$, a relation ϱ will be called *multinomial*. Further, we shall denote by $\mathrm{Gen}(I)$ a set of relations generating the ideal I which satisfies the following conditions:

(a) if $\sum_{i=1}^m \lambda_i \varrho_i \in \mathrm{Gen}(I)$, where each λ_i is a non-zero element of K, then $\sum_{j \in S} \lambda_j \varrho_j \notin \mathrm{Gen}(I)$ for any proper subset $S \subset \{1, \ldots, m\}$;
(b) if $\varrho \in \mathrm{Gen}(I)$, then $\beta\varrho, \varrho\beta \notin \mathrm{Gen}(I)$ for any arrow β.

Suppose ω is a non-zero walk in (Q, I). We shall denote by $\mathcal{X}_\omega$ the set of all non-zero walks in (Q, I) which start with ω, that is, all walks of the form $\omega\omega'$ where ω' is a walk such that $s(\omega') = t(\omega)$. Hence, for each walk ω we obtain the cardinality of $\mathcal{X}_\omega$ as an invariant.

We shall say that an arrow α is an *initiating arrow* in (Q, I) (or α *initiates* a relation) if there exists a relation $\varrho \in \mathrm{Gen}(I)$ such that $\varrho = \lambda_1 \alpha \varrho_1 + \lambda_2 \varrho_2 + \cdots + \lambda_m \varrho_m$ for some $m \geq 1$. Note that if an arrow $\alpha : x \to y$ in a special biserial quiver (Q, I) is not initiating then $\overline{\alpha} P(y) \cong P(y)$ and $P(y)$ is a direct summand the Jacobson radical $\mathrm{rad}\, P(x)$ of $P(x)$. Consequently, $f_\alpha : P(y) \to P(x)$ is an irreducible homomorphism. Conversely, if $f_\alpha : P(y) \to P(x)$ is irreducible then $\overline{\alpha} P(y)$ is a direct summand of $\mathrm{rad}\, P(x)$ and α is not an initiating arrow in (Q, I). Therefore, it is natural to ask when f_α does not belong to rad_A^∞ for an initiating arrow α of (Q, I).

The following theorem is the main result of the paper.

THEOREM. *Let $A = KQ_A/I_A$ be a special biserial algebra and $\alpha : x \to y$ an initiating arrow in (Q_A, I_A). The following statements are equivalent:*

(i) $f_\alpha : P(y) \to P(x) \notin \mathrm{rad}_A^\infty(P(y), P(x))$,
(ii) *$\mathcal{X}_\varrho$ is a finite set for any path ϱ in Q_A such that $\alpha\varrho \in \mathrm{Gen}(I_A)$ or $\lambda\alpha\varrho + \lambda'\varrho' \in \mathrm{Gen}(I_A)$ for some non-zero $\lambda, \lambda' \in K$ and a path ϱ' in Q_A.*

For background on the representation theory applied here we refer to [3], [21], [22].

2. Preliminary results. The aim of this section is to present all facts and notation applied in the proof of the main theorem.

A special biserial algebra $A = KQ_A/I_A$ is called a *string algebra* if there is a generating set of I_A formed by paths. There is a full classification of finite-dimensional indecomposable right modules over a string algebra A (see [7], [25]). For every indecomposable module $X \in \mathrm{mod}\, A$ we have two possibilities. The first is when X is induced by a walk ω that cannot be written as $\omega_1\alpha\alpha^{-1}\omega_2$ or $\omega_1\beta^{-1}\beta\omega_2$ for walks ω_1, ω_2, and ω does not contain a subwalk of the form υ or υ^{-1} with $\upsilon \in I_A$. In this case we say that X is a *string module* and denote it by $X(\omega)$. The second possibility is that X is induced by a primitive closed walk ν, an integer $n \geq 1$ and a non-zero element $\lambda \in K$. Recall that a closed walk ν in a bound quiver (Q_A, I_A) is called *primitive* if it is not of the form μ^i for any integer $i \geq 2$ and ν^j for any $j \geq 1$ is a non-zero walk in (Q_A, I_A). In this case we say that X is a *band module* and denote it by $X(\nu, n, \lambda)$.

For the first type of module, the following algorithm for computing Auslander–Reiten sequences was given by Skowroński and Waschbüsch [24]. If $\omega = \delta_{1,s_1} \ldots \delta_{1,1}\delta_{2,1}^{-1} \ldots \delta_{2,s_2}^{-1} \ldots \delta_{r-1,s_{r-1}} \ldots \delta_{r-1,1}\delta_{r,1}^{-1} \ldots \delta_{r,s_r}^{-1}$ is a walk in the bound quiver (Q_A, I_A), where $\delta_{j,t}$ is an arrow in Q_A and $\delta_{1,s_1} \ldots \delta_{1,1}$ or $\delta_{r,1}^{-1} \ldots \delta_{r,s_r}^{-1}$ may be trivial, then we set

$$\omega_R = \omega\delta_{r,s_r+1}^{-1}\delta_{r+1,s_{r+1}} \ldots \delta_{r+1,1},$$

where $\delta_{r+1,s_{r+1}} \ldots \delta_{r+1,1}$ is a maximal non-zero path in (Q_A, I_A), provided such a walk exists. If the walk $\delta_{r,s_r+1}^{-1}\delta_{r+1,s_{r+1}} \ldots \delta_{r+1,1}$ does not exist then

$$\omega_R = \delta_{1,s_1} \ldots \delta_{1,1}\delta_{2,1}^{-1} \ldots \delta_{2,s_2}^{-1} \ldots \delta_{r-1,s_{r-1}} \ldots \delta_{r-1,2}.$$

Using the same rules on the other end of the walk ω we obtain ω_L. The composition of these constructions gives us the walks ω_{RL} and ω_{LR}, respectively. Moreover, if ω_R and ω_L are non-zero we have $\omega_{RL} = \omega_{LR}$. Then, by [24], for a non-injective string module $X(\omega)$, there is an Auslander–Reiten sequence in $\operatorname{mod} A$ of the form

$$0 \to X(\omega) \to X(\omega_R) \oplus X(\omega_L) \to X(\omega_{RL}) \to 0.$$

We shall write briefly ω_{R^2} (ω_{L^2}) instead of $(\omega_R)_R$ ($(\omega_L)_L$, respectively), and analogously ω_{R^i} (ω_{L^i}, respectively) will mean the above operations applied i times. We shall also write $\omega_{R^{-i}} = \eta$ provided $\eta_{R^i} = \omega$, for any positive integer i.

Note that for a special biserial algebra A the algebra $A/\operatorname{soc}(R)$, where $\operatorname{soc}(R)$ is the socle of the direct sum R of all indecomposable projective-injective modules which are not serial, is a string algebra. Moreover, each indecomposable projective-injective A-module X occurs in an Auslander–Reiten sequence of the form

$$0 \to \operatorname{rad}(X) \to X \oplus \operatorname{rad}(X)/\operatorname{soc}(X) \to X/\operatorname{soc}(X) \to 0,$$

where rad denotes the Jacobson radical of a module. This allows us to use the Skowroński–Waschbüsch algorithm for any special biserial algebra.

Let ω be a non-zero walk in a special biserial quiver (Q_A, I_A). Then the set $\mathcal{X}_\omega$ can be equipped with a partial order $\preceq$ in the following way. For $\eta_1, \eta_2 \in \mathcal{X}_\omega$ we write $\eta_1 \preceq \eta_2$ if and only if there exists a non-negative integer i such that $\eta_2 = (\eta_1)_{R^i}$. Note that if $\delta \in \mathcal{X}_\omega$ and $\delta_{R^l} \in \mathcal{X}_\omega$, where $l \geq 2$, then $\delta_{R^i} \in \mathcal{X}_\omega$ for all $1 \leq i \leq l-1$.

The following lemma will play an important role in the proof of the Theorem.

LEMMA 2.1. *Let $A = KQ_A/I_A$ be a path algebra of a special biserial quiver (Q_A, I_A) and ω be a non-zero walk in (Q_A, I_A). Then:*

(a) *$\mathcal{X}_\omega$ contains a unique minimal and a unique maximal element.*
(b) *If $\mathcal{X}_\omega$ is a finite set then it is well ordered.*
(c) *If $\mathcal{X}_\omega$ contains a finite chain $\mathcal{L}$ then $\mathcal{X}_\omega = \mathcal{L}$.*

Proof. Assume that $\omega = \omega'\beta^{-1}\alpha_1 \ldots \alpha_n$, where, for $1 \leq i \leq n$, α_i is an arrow in Q_A, and if $\omega'\beta^{-1}$ is a non-trivial walk then β is an arrow such that $s(\beta) = s(\alpha_1)$. Without loss of generality we may assume that $\omega'\beta^{-1}$ is a non-trivial walk.

(a) Let η' be a maximal non-zero path in (Q_A, I_A) which belongs to $\mathcal{X}_{\alpha_1 \ldots \alpha_n}$. Consider $\eta = \omega'\beta^{-1}\eta' \in \mathcal{X}_\omega$. From the Skowroński–Waschbüsch algorithm we find that $\eta = \omega'_R$ and $\omega'_{R^i} \notin \mathcal{X}_\omega$ for any integer $i \leq 0$. Moreover, if a path v satisfies $v_{R^{-1}} \notin \mathcal{X}_\omega$ and $v \in \mathcal{X}_\omega$, then $v = \eta$. Hence η is a minimal element in $\mathcal{X}_\omega$. Similarly, we show the existence of a maximal element in $\mathcal{X}_\omega$. If $u \in \mathcal{X}_\omega$ is such that $u_R \notin \mathcal{X}_\omega$ then $u = \omega'\beta^{-1}\alpha_1 \ldots \alpha_n(u')^{-1}$, where u' is a maximal non-zero (maybe trivial) path which does not contain $\alpha_1 \ldots \alpha_n$ as a subpath and $t(u') = t(\alpha_n)$.

(b) Let $v \in \mathcal{X}_\omega$ and η be a minimal element of $\mathcal{X}_\omega$. Recall that for two representations $M = (M_x, M_\alpha)$, $M' = (M'_x, M'_\alpha)$ of (Q_A, I_A) a homomorphism $h : M \to M'$ is a family $h = (h_x)_{x \in Q_0}$ of K-linear maps $(h_x : M_x \to M'_x)_{x \in Q_0}$ such that for each arrow $\alpha : a \to b$ we have $M'_\alpha h_a = h_b M_\alpha$. Note that for a walk μ of length r such that $\mu(i_1) = \cdots = \mu(i_l) = z$ for some vertex $z \in Q_A$ and $1 \le l \le r+1$, we have $X(\mu)_{\mu(i_j)} = K$ for any $1 \le j \le l$, and $\bigoplus_{j=1}^{l} X(\mu)_{\mu(i_j)} = X(\mu)_z$. Take $f : X(\eta) \to X(v)$ with $f_{\eta(1)} \neq 0$. Such an f exists since $v = \omega v' = \omega'\beta^{-1}\alpha_1 \ldots \alpha_n v'$ and $\eta = \omega'\beta^{-1}\alpha_1 \ldots \alpha_r$ where $\alpha_1, \ldots, \alpha_r$ are arrows in Q_A and $\alpha_1 \ldots \alpha_r$ is a maximal non-zero oriented walk in $\mathcal{X}_{\alpha_1 \ldots \alpha_n}$. Suppose that $f = f''f'$ for some A-module M and homomorphisms $f'' : M \to X(v)$, $f' : X(\eta) \to M$. Without loss of generality we may assume that $M = X(\mu)$ is an indecomposable string module (the image of a morphism between string modules is a string module). Assume $\omega \in \mathcal{X}_\delta$ for some arrow $\delta \in Q_A$ and $\mu \notin \mathcal{X}_\delta$. Then μ or μ^{-1} contains an arrow γ such that $t(\gamma) = s(\delta)$ or $s(\gamma) = s(\delta)$. Since $f_{\eta(1)} \neq 0$ we have $f'_{\eta(1)} = f'_{v(1)} \neq 0$ and $f''_{\eta(1)} = f''_{v(1)} \neq 0$.

Assume that $M_\gamma : M_{s(\gamma)} \to M_{\eta(1)}$, where $M_{\eta(1)}$ is a one-dimensional subspace of $M_{t(\gamma)}$, is non-zero. Then for $f''_{\eta(1)} : M_{\eta(1)} \to X(v)_{v(1)}$ we have $0 \neq f''_{\eta(1)} M_\gamma = X(v)_\gamma f''_{s(\gamma)} = 0$, because $X(v)_\gamma : X(v)_{s(\gamma)} \to X(v)_{v(1)}$ is zero, a contradiction. Suppose $M_\gamma : M_{\eta(1)} \to M_{t(\gamma)}$ is non-zero for a one-dimensional subspace $M_{\eta(1)}$ of $M_{s(\gamma)}$. Then for $f'_{\eta(1)} : X(\eta)_{\eta(1)} \to M_{\eta(1)}$ we have $0 \neq M_\gamma f'_{\eta(1)} = f'_{t(\gamma)} X(\eta)_\gamma = 0$, because $X(\eta)_\gamma : X(\eta)_{\eta(1)} \to X(\eta)_{t(\gamma)}$ is zero, a contradiction. Hence $\mu \in \mathcal{X}_\delta$ and $\mu = \delta\mu'$ for some walk μ'. Since by assumption $\omega \in \mathcal{X}_\delta$, there exists a walk ω'' such that $\omega = \delta\omega''$ and we repeat the above considerations for ω'' instead for ω and μ' instead for μ. By induction we find that $\mu \in \mathcal{X}_\omega$. We use dual arguments for $\omega \in \mathcal{X}_{\delta^{-1}}$, where δ is an arrow in Q_A.

Let now $\mathcal{X}_\omega$ be a finite set. Then there exists $i \ge 0$ such that $\eta_{R^i} \in \mathcal{X}_\omega$ and $\eta_{R^{i+1}} \notin \mathcal{X}_\omega$. Suppose that $v \in \mathcal{X}_\omega$ is not $\preceq$-related with η. Note that any morphism $f : X(\eta) \to X(v)$ with $f_{\eta(1)} = f_{v(1)} \neq 0$ factorizes through the middle term E of an Auslander–Reiten sequence which starts in $X(\eta_{R^i})$. This leads to a contradiction since no direct summand M of E belongs to $\mathcal{X}_\omega$. Observe that for any morphism $f \in \mathrm{Hom}_A(M, X(v))$ we have $f_{v(1)} = 0$. Hence the relation $\preceq$ is connected and $\mathcal{X}_\omega$ has the form

$$\eta \preceq \eta_R \preceq \eta_{R^2} \preceq \cdots \preceq \eta_{R^j}$$

for some integer $j \ge 0$.

(c) If $\mathcal{X}_\omega$ contains a finite chain $\mathcal{L}$, then by repeating the arguments from (b) we get the claim.

Analogously we provide the proof for $\omega = \omega'\alpha\beta_1^{-1} \ldots \beta_m^{-1}$, where for any $1 \le i \le m$, β_i is an arrow and if $\omega'\alpha$ is a non-trivial walk then α denotes an arrow such that $t(\alpha) = t(\beta_1)$. ■

3. Proof of the Theorem. We have two cases to consider: α does not initiate any multinomial relations and α initiates a multinomial relation.

CASE 1: α does not initiate any multinomial relation. This is the case when $P(x)$ is a string module, say $P(x) = X(\eta^{-1}\alpha\varepsilon_1 \dots \varepsilon_n)$, where η is a path (maybe trivial) which starts at x and $\eta \notin \mathcal{X}_\alpha$ and $\varepsilon_1 \dots \varepsilon_n$ is the composition of arrows $\varepsilon_1, \dots, \varepsilon_n \in Q_A$ ($\varepsilon_1 \dots \varepsilon_n$ may be trivial). Note that $X(\varepsilon_1 \dots \varepsilon_n)$ is then a direct summand of $\operatorname{rad} P(x)$ and $X(\varepsilon_1 \dots \varepsilon_n)$ is equal to the image $\operatorname{Im} f_\alpha$ of $f_\alpha : P(y) \to P(x)$.

We start by showing the implication (ii)⇒(i).

(a) Assume that y is a source of exactly one arrow ε_1. Clearly, then $P(y) = X(\varepsilon_1 \dots \varepsilon_r)$ for some $r \geq n$. Since α is an initiating arrow in a special biserial quiver, there is a unique path $\varrho \in \mathcal{X}_{\varepsilon_1}$ for which $\alpha\varrho \in \operatorname{Gen}(I_A)$. But $P(x) = X(\eta^{-1}\alpha\varepsilon_1 \dots \varepsilon_n)$ implies that there exists in Q_A an arrow ε_{n+1} with $s(\varepsilon_{n+1}) = t(\varepsilon_n)$ such that $\varrho = \varepsilon_1 \dots \varepsilon_n\varepsilon_{n+1}$. Obviously, $\varrho \notin I_A$ and $n+1 \leq r$.

Consider now the set $\mathcal{X}_\varrho$. Since by assumption $\mathcal{X}_\varrho$ is finite, and by Lemma 2.1, $\varepsilon_1 \dots \varepsilon_r$ is a maximal non-zero path in $\mathcal{X}_\varrho$, there exists a positive integer a such that

$$X((\varepsilon_1 \dots \varepsilon_r)_{R^a}) = X(\varepsilon_1 \dots \varepsilon_n).$$

Thus we have the following sectional path in $\varGamma_A$:

$$P(y) = X(\varepsilon_1 \dots \varepsilon_r) \xrightarrow{f_1} X((\varepsilon_1 \dots \varepsilon_r)_R) \xrightarrow{f_2} \dots \xrightarrow{f_a} X(\varepsilon_1 \dots \varepsilon_n) \xrightarrow{f_{a+1}} P(x).$$

Then by [15, Theorem 13.3] we see that $f_{a+1} \dots f_1 \in \operatorname{rad}_A^{a+1} \setminus \operatorname{rad}_A^{a+2}$. We claim that $f_\alpha = f_{a+1} \dots f_1$ up to scalar multiplication. We will show that $f_\alpha : P(y) = X(\varepsilon_1 \dots \varepsilon_r) \to P(x)$ does not factorize through any module M different from $X((\varepsilon_1 \dots \varepsilon_r)_{R^i})$ for $1 \leq i \leq a$. Indeed, assume that $f_\alpha = gf$, where $f : P(y) \to M$ and $g : M \to P(x)$ for some such A-module M. Without loss of generality we may assume that M is a string module $X(\omega)$ for a walk ω in (Q_A, I_A). Since $\operatorname{Im} f = X(\varepsilon_1 \dots \varepsilon_j)$ for some $j \leq r$, M contains $X(\varepsilon_1 \dots \varepsilon_j)$ as a submodule. Further, M has $X(\varepsilon_1 \dots \varepsilon_n)$ as a factor module, because $\operatorname{Im} f_\alpha = \operatorname{Im} gf = X(\varepsilon_1 \dots \varepsilon_n)$. Observe that by definition $(f_\alpha)_y \neq 0$. But $(gf)_y \neq 0$ if and only if $\omega = \varepsilon_1 \dots \varepsilon_n\zeta$ for some non-zero walk ζ in (Q_A, I_A). Now the fact that $X(\varepsilon_1 \dots \varepsilon_n)$ is a factor module of M implies $\omega = \varepsilon_1 \dots \varepsilon_n\varepsilon_{n+1}\zeta'$ for some arrow ε_{n+1} and walk ζ' in (Q_A, I_A). Hence $\omega \in \mathcal{X}_\varrho$. Since $\mathcal{X}_\varrho$ is finite and $\varepsilon_1 \dots \varepsilon_r$ is a minimal element in $\mathcal{X}_\varrho$ we deduce that $X(\omega) = X((\varepsilon_1 \dots \varepsilon_r)_{R^i})$, for some $1 \leq i \leq a$, a contradiction. Thus $f_\alpha = f_{a+1} \dots f_1 h$ for some $h \in \operatorname{End}_A(P(x))$. But $\operatorname{End}_A(P(x)) \cong K$ and we conclude that f_α is equal to $f_{a+1} \dots f_1$ up to scalar multiplication.

(b) Assume now that y is a source of exactly two arrows which we denote by ε_1, δ_1, and suppose $\alpha\delta_1 \in I_A$.

(1) Suppose that $P(y)$ is a string module of the form

$$X(\nu) = X(\delta_p^{-1} \dots \delta_1^{-1} \varepsilon_1 \dots \varepsilon_r)$$

for some $p, r \geq 1$. Then either $r = n$, which is equivalent to the fact that $\alpha\delta_1$ is the unique relation initiated by α, or $r \geq n+1$, equivalently $\alpha\varepsilon_1 \dots \varepsilon_{n+1} \in \mathrm{Gen}(I_A)$ (because $P(x) = X(\eta^{-1}\alpha\varepsilon_1 \dots \varepsilon_n)$).

Assume $r = n$. Then $\mathrm{Im}\, f_\alpha = X(\varepsilon_1 \dots \varepsilon_n)$ and there is the following sectional path in Γ_A:

$$P(y) = X(\nu) \xrightarrow{f_1} X(\nu_L) \xrightarrow{f_2} \dots \xrightarrow{f_b} X(\nu_{L^b}) = X(\varepsilon_1 \dots \varepsilon_n) \xrightarrow{f_{b+1}} P(x),$$

for $b \geq 1$. Again by [15, Theorem 13.3] we have $f_{b+1} \dots f_1 \in \mathrm{rad}_A^{b+1} \setminus \mathrm{rad}_A^{b+2}$. We claim that $f_\alpha = f_{b+1} \dots f_1$ up to scalar multiplication. Note that f_α does not factorize through any module different from $X(\nu_{L^i})$ where $1 \leq i \leq b$. Suppose $f_\alpha = gf$ for some string module $M = X(\omega)$, $f : P(y) \to M$ and $g : M \to P(x)$. Then M has a submodule of the form $\mathrm{Im}\, f = X(\delta_l^{-1} \dots \delta_1^{-1} \varepsilon_1 \dots \varepsilon_k)$ with $l \leq p$, $k \leq n$. If $k < n$ then for a vertex $z = t(\varepsilon_{k+1})$ we have $f_z = 0$ and hence $(gf)_z = 0$, which implies $f_\alpha \neq gf$. Therefore, we conclude that $k = n$. Moreover, since $\mathcal{X}_{\delta_1}$ is finite there exists an integer $j \geq 1$ such that $X((\delta_1 \dots \delta_p)_{R^j}) = X((\delta_p^{-1} \dots \delta_1^{-1})_{L^j}) = X(\delta_l^{-1} \dots \delta_1^{-1})$ and hence $X(\delta_l^{-1} \dots \delta_1^{-1} \varepsilon_1 \dots \varepsilon_k) = X(\nu_{L^j})$ for some j. The fact that $X(\varepsilon_1 \dots \varepsilon_n)$ is a factor module of M and $\varepsilon_1 \dots \varepsilon_n = \varepsilon_1 \dots \varepsilon_r$ is a maximal non-zero path in $\mathcal{X}_{\varepsilon_1}$, implies that $\omega = \omega' \delta_1^{-1} \varepsilon_1 \dots \varepsilon_n$ for some walk ω' in (Q_A, I_A). But then $\delta_1 \omega'^{-1} \in \mathcal{X}_{\delta_1}$ and we conclude that $X(\omega) = X(\nu_{L^m})$ for some $m \geq 1$. Thus there is no proper factorization of f_α through an A-module different from $X(\nu_{L^i})$, $i \in \{1, \dots, b\}$. Hence $f_\alpha = f_{b+1} \dots f_1$ up to scalar multiplication since $\mathrm{End}_A(P(x)) \cong K$.

Consider now $r \geq n+1$. Then we have $\alpha\delta_1, \alpha\varepsilon_1 \dots \varepsilon_{n+1} \in \mathrm{Gen}(I_A)$. Since by assumption $\mathcal{X}_{\delta_1}$ and $\mathcal{X}_{\varepsilon_1 \dots \varepsilon_{n+1}}$ are finite sets, there are positive integers a and b such that

$$X(\nu_{R^a L^b}) = X(\varepsilon_1 \dots \varepsilon_n).$$

Thus $f_\alpha \in \mathrm{rad}_A^{a+b+1}(P(y), P(x))$. We claim that $f_\alpha \notin \mathrm{rad}_A^{a+b+2}(P(y), P(x))$. Suppose that f_α has a non-trivial factorization $f_\alpha = gf$ for $f : P(y) \to M$, $g : M \to P(x)$ and a string module $M = X(\omega)$ different from $X(\nu_{R^i L^j})$ for $1 \leq i \leq a$, $1 \leq j \leq b$. As above, M contains a submodule $\mathrm{Im}\, f = X(\delta_l^{-1} \dots \delta_1^{-1} \varepsilon_1 \dots \varepsilon_k)$ for $l \leq p$ and $k \leq r$, and has a factor module $X(\varepsilon_1 \dots \varepsilon_n)$. If $k < n$, then for the vertex $z = t(\varepsilon_{k+1})$ we have $f_z = 0$ and hence $(gf)_z = 0$, which implies $f_\alpha \neq gf$, a contradiction. Therefore, $k \geq n$. Since $X(\varepsilon_1 \dots \varepsilon_n)$ is a factor module of M, we have $\omega = \omega' \delta_1^{-1} \varepsilon_1 \dots \varepsilon_n$ or $\omega = \omega_1 \delta_1^{-1} \varepsilon_1 \dots \varepsilon_n \varepsilon_{n+1} \omega_2$ for some walks $\omega', \omega_1, \omega_2$ and an arrow ε_{n+1} in (Q_A, I_A) . Using the same arguments as above we infer that in both cases $\omega = \nu_{R^i L^j}$ with $1 \leq i \leq a$, $1 \leq j \leq b$. Clearly, there is no cycle in Γ_A

having only $X(\nu_{R^iL^j})$ with $i \in \{1,\dots,a\}$, $j \in \{1,\dots,b\}$ as vertices. Thus $f_\alpha \in \operatorname{rad}_A^{a+b+1} \setminus \operatorname{rad}_A^{a+b+2}$.

(2) If $P(y)$ is not a string module then $f_\alpha : P(y) \to P(x)$ factorizes through $P(y)/\operatorname{soc}P(y) = X(\nu)$ where ν is a non-zero walk in (Q_A, I_A). We repeat the above arguments to show that $X(\varepsilon_1 \dots \varepsilon_n) = X(\nu_{R^aL^b})$ for some $a, b \geq 1$ and that f_α factorizes only through modules of the form $X(\nu_{R^iL^j})$, $1 \leq i \leq a$, $1 \leq j \leq b$.

Now we show the implication (i)⇒(ii).

Assume $f_\alpha \notin \operatorname{rad}_A^\infty(P(y), P(x))$ and $P(y)$ is a string module $X(\delta_p^{-1} \dots \delta_1^{-1}\varepsilon_1 \dots \varepsilon_r)$ with the above notation and $\delta_p^{-1} \dots \delta_1^{-1}$ may be a trivial walk in (Q_A, I_A). We denote by $f_{\alpha|}$ left multiplication by $\overline{\alpha}$ defined on a factor module $X(\varepsilon_1 \dots \varepsilon_r)$ of $P(y)$. The composition $f_{\alpha|}g$ for a canonical epimorphism $g : P(y) \to X(\varepsilon_1 \dots \varepsilon_r)$ is equal to f_α. Hence, $f_{\alpha|} \notin \operatorname{rad}_A^\infty(X(\varepsilon_1 \dots \varepsilon_r), P(x))$. Then, for $f_{\alpha|} = \sum_{i=1}^{j} \lambda_i f_i$ with non-zero $\lambda_i \in K$, $j \geq 1$, there exists $i \in \{1,\dots,j\}$ and a positive integer m such that

$$f_i \in \operatorname{rad}_A^m \setminus \operatorname{rad}_A^{m+1}.$$

Consider $f_i : X(\varepsilon_1 \dots \varepsilon_r) \to P(x)$. If $X((\varepsilon_1 \dots \varepsilon_r)_L) \neq 0$, then there is an arrow β in Q_A such that $t(\beta) = y$ and $\beta\varepsilon_1 \dots \varepsilon_r \notin I_A$. Hence $\alpha\varepsilon_1 \in I_A$ and f_i factorizes through a simple module $S(y)$ at a vertex y. Moreover, $X((\varepsilon_1 \dots \varepsilon_r)_L) = X(\mu^{-1}\beta\varepsilon_1 \dots \varepsilon_r)$, where μ is a maximal non-zero path (may be trivial) which starts at $s(\beta)$ and $\mu \notin \mathcal{X}_\beta$. Observe that a non-zero morphism $h : X(\mu^{-1}\beta\varepsilon_1 \dots \varepsilon_r) \to P(x)$ factorizes through $S(y)$ provided $\mu^{-1}\beta\varepsilon_1 \dots \varepsilon_r = \delta_1 \dots \beta\varepsilon_1 \dots \varepsilon_r$ or $\mu^{-1}\beta\varepsilon_1 \dots \varepsilon_r = \varepsilon_1 \dots \beta\varepsilon_1 \dots \varepsilon_r$, a contradiction. Hence f_i does not factorize through $X((\varepsilon_1 \dots \varepsilon_r)_L)$. Similarly, f_i does not factorize through $X((\varepsilon_1 \dots \varepsilon_r)_{R^kL})$ for any $1 \leq k \leq m-1$. Therefore, f_i is a composition $f_i^m \dots f_i^2 f_i^1$ of irreducible morphisms $f_i^k : X((\varepsilon_1 \dots \varepsilon_r)_{R^{k-1}}) \to X((\varepsilon_1 \dots \varepsilon_r)_{R^k})$ for $1 \leq k \leq m-1$ and $f_i^m : X(\varepsilon_1 \dots \varepsilon_n) \to P(x)$, because $X(\varepsilon_1 \dots \varepsilon_n)$ is a direct summand of $\operatorname{rad} P(x)$. Thus $X((\varepsilon_1 \dots \varepsilon_n)_{R^{-1}}) = X((\varepsilon_1 \dots \varepsilon_r)_{R^{m-1}})$ is a maximal element of $\mathcal{X}_{\varepsilon_1 \dots \varepsilon_{n+1}}$. Applying now Lemma 2.1(a) we conclude that $X(\varepsilon_1 \dots \varepsilon_r)$ is a minimal element of $\mathcal{X}_{\varepsilon_1 \dots \varepsilon_{n+1}}$ and by Lemma 2.1(c), $\mathcal{X}_{\varepsilon_1 \dots \varepsilon_{n+1}}$ is finite. Analogously, we show that $\mathcal{X}_{\delta_1}$ is a finite set.

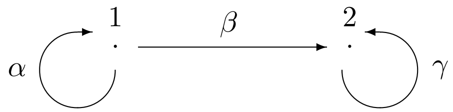
Assume now that $f_\alpha \notin \operatorname{rad}_A^\infty(P(y), P(x))$ and $P(y)$ is not a string A-module. Let $\varepsilon_1 \dots \varepsilon_r$ be one of two maximal non-zero paths starting at vertex y where, for $1 \leq i \leq r$, ε_i denotes an arrow. Then we repeat the above considerations with respect to the factor module $X(\varepsilon_1 \dots \varepsilon_{r-1})$ of $P(y)$.

Case 2: α initiates a multinomial relation $\lambda_1\omega_1 + \lambda_2\alpha\varepsilon_1 \dots \varepsilon_n$ for some non-trivial path ω_1 in (Q_A, I_A), $1 \leq n \leq r$ and non-zero $\lambda_1, \lambda_2 \in K$. Without loss of generality, we may assume (Q_A, I_A) is a presentation of the algebra A such that $\lambda_1 = 1$, $\lambda_2 = -1$. If α initiates a multinomial relation, say

$\alpha'\varepsilon'_1 \dots \varepsilon'_l - \alpha\varepsilon_1 \dots \varepsilon_n$ with $1 \le l$, $1 \le n \le r$, then for a string module $P(y) = X(\delta_p^{-1} \dots \delta_1^{-1}\varepsilon_1 \dots \varepsilon_r)$ the morphism $f_\alpha : P(y) \to P(x)$ is the composition $f_{\alpha|}g$ of $g : P(y) \to X(\varepsilon_1 \dots \varepsilon_r)$ and $f_{\alpha|} : X(\varepsilon_1 \dots \varepsilon_r) \to P(x)$. If $P(y)$ is not a string A-module and $\delta_1 \dots \delta_p, \varepsilon_1 \dots \varepsilon_r$ are maximal non-zero paths which start at y, then we take $X(\varepsilon_1 \dots \varepsilon_{r-1})$ instead of $X(\varepsilon_1 \dots \varepsilon_r)$. Since $\operatorname{rad} P(x) = X(\varepsilon_1 \dots \varepsilon_n \varepsilon_l'^{-1} \dots \varepsilon_1'^{-1})$ it is sufficient to study $\mathcal{X}_{\delta_1}$ for g (if y is a source of an arrow δ_1 different from ε_1) and $\mathcal{X}_{\varepsilon_1 \dots \varepsilon_n}$ for $f_{\alpha|}$. To show the equivalence (i)⇔(ii) in the second case, the arguments from Case 1 are now repeated.

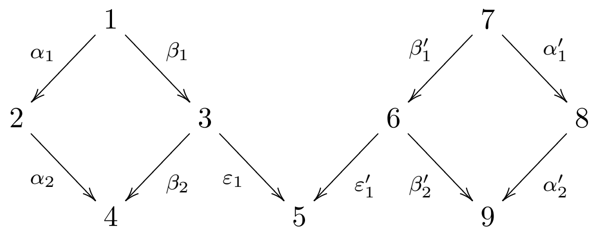
4. Examples. We end the paper with examples illustrating the Theorem.

EXAMPLE 4.1. Let $A = KQ/I$ where Q is of the form

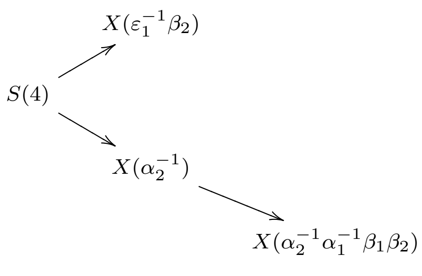


and $I = \langle \alpha^2, \gamma^2, \alpha\beta - \beta\gamma \rangle$. Consider the initiating arrow β. Then for $f_\beta : P(2) \to P(1)$ we have $\mathcal{X}_\gamma = \{\gamma, \gamma\beta^{-1}, \gamma\beta^{-1}\alpha\}$ and $f_\beta \notin \operatorname{rad}^\infty(P(2), P(1))$. Note that A is of finite representation type.

EXAMPLE 4.2. Let $A = KQ/I$ where Q is as follows:



and I is the two-sided ideal of KQ generated by $\beta_1\varepsilon_1, \beta_1'\varepsilon_1'$. We know by [18] that the Auslander–Reiten quiver Γ_A of A has a component $\mathcal{P}(A)$ which contains all indecomposable projective A-modules. Furthermore, it is a starting component, that is, there are no non-zero morphisms $f : X \to Y$ for indecomposable modules $X \notin \mathcal{P}(A)$ and $Y \in \mathcal{P}(A)$ (see [18]). There is the following walk in $\mathcal{P}(A)$:



where $S(4)$ denotes a simple module at vertex 4, $X(\alpha_2^{-1}\alpha_1^{-1}\beta_1\beta_2) = P(1)$ and $X(\varepsilon_1^{-1}\beta_2) = P(3)$. Consider $f_{\beta_1} : P(3) \to P(1)$. The set $\mathcal{X}_{\varepsilon_1}$ is infinite since it contains all walks of the form $\varepsilon_1\varepsilon_1'^{-1}\beta_2'(\alpha_2'^{-1}\alpha_1'^{-1}\beta_1'\beta_2')^r$ for any integer $r \geq 1$. Thus $f_{\beta_1} \in \mathrm{rad}_A^\infty$ and $\mathcal{P}(A)$ is not generalized standard. Analogously, $f_{\beta_1'} \in \mathrm{rad}_A^\infty$, where $f_{\beta_1'} : P(6) \to P(7)$. Moreover, for all remaining arrows $\delta : x \to y$ in Q, morphisms $f_\delta : P(y) \to P(x)$ between indecomposable projective A-modules $P(x)$ and $P(y)$, belong to $\mathrm{rad}_A \setminus \mathrm{rad}_A^2$.

Imagine now that Q is a finite quiver such that the number of arrows with a prescribed source or target is at most 2. The above theorem allows us to construct special biserial algebras A associated to a bound quiver (Q, I) such that the morphisms between indecomposable projective modules belong to an arbitrary given power of the radical rad_A. Obviously, if Q does not contain a subquiver of type $\widetilde{\mathbb{A}}_m$ then none of these morphisms belongs to rad_A^∞.

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