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REAL FUNCTIONS

On a Variant of the Gagliardo–Nirenberg Inequality Deduced from the Hardy Inequality

by

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Summary. We obtain new variants of weighted Gagliardo–Nirenberg interpolation inequalities in Orlicz spaces, as a consequence of weighted Hardy-type inequalities. The weights we consider need not be doubling.

1. Introduction. Gagliardo–Nirenberg interpolation inequalities have a long history and several mathematicians investigated their numerous variants. Their rudiments can be found in old papers of Landau (see e.g. [33]), and today this name is most often associated with the classical variant

$$\|\nabla^{(k)}u\|_{L^q(\Omega)} \le C_1\|u\|_{L^r(\Omega)}^{1-k/m}\|\nabla^{(m)}u\|_{L^p(\Omega)}^{k/m} + C_2\|u\|_{L^r(\Omega)}, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a domain with sufficiently smooth boundary, the function $u : \Omega \to \mathbb{R}$ belongs to an appropriate Sobolev space on Ω, $\frac{1}{q} = (1-\frac{k}{m})\frac{1}{r} + \frac{k}{m}\frac{1}{p}$, $0 < k < m$. For $\Omega = \mathbb{R}$ and $p = q = r = \infty$ this inequality was obtained by Kolmogorov [27] (with $C_2 = 0$), whereas Gagliardo [12] and Nirenberg [39] independently proved its extensions to the form (1.1). We refer to the book [36] for an extensive description of their historical evolution.

Since the Gagliardo–Nirenberg inequalities involve two differential operators: $\nabla^{(k)}u$ and $\nabla^{(m)}u$, they are more difficult to analyse than Hardy-type inequalities, which involve one differential operator only. This is one of the reasons why many problems concerning the validity of the Gagliardo–Nirenberg inequalities remain unsolved so far. For example, one asks about

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Orlicz-space generalizations of (1.1), also for Orlicz spaces $L^M(\mu)$ with a Radon measure μ, even nondoubling.

Interest in inequalities in the Orlicz-space setting arises from linear and nonlinear PDEs, and calculus of variations, which in turn come from mathematical physics. See e.g. [1, 2, 10, 13, 14, 46], where many motivations for investigating degenerate PDEs in Orlicz spaces can be found.

The purpose of this paper is to show that certain variants of weighted modular Hardy inequalities for $u \in C_0^\infty(\Omega)$:

$$\int_\Omega P(|\nabla\varphi|\,|u|)\,d\mu \le K_1 \int_\Omega P(A|\nabla u|)\,d\mu + K_2 \int_\Omega M(|u|)\,d\mu, \tag{1.2}$$

where $d\mu(x) = e^{-\varphi(x)}dx$ with φ locally Lipschitz, imply variants of Gagliardo–Nirenberg inequalities for modulars:

$$\int_\Omega M(|\nabla u|)\,d\mu \le L \int_\Omega P(|\nabla^{(2)}u|)\,d\mu + \int_\Omega Q(B|u|)\,d\mu,$$

and for norms:

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le L_1 \sqrt{\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)}} + L_2\|u\|_{L^Q(\Omega,\mu)}, \tag{1.3}$$

valid with general (u-independent) constants $A, K_1, K_2, L, L_1, L_2, B$. Our approach requires M to be an N-function satisfying the Δ_2-condition. The functions P and Q are tied with M by a Young-type inequality:

$$\frac{M(u)}{u^2}vw \le M(u) + P(v) + Q(w).$$

For details, see Theorems 3.1, 3.4, and 3.5.

As opposed to previous works of Gutierrez and Wheeden [15], and also Chua [8, 9], Bang and coauthors (see e.g. [3]), the measure μ considered here need not be doubling. This allows obtaining inequalities e.g. for measures with finite mass on unbounded domains, which have been excluded from investigation so far. In [8, 9, 15], Gagliardo–Nirenberg inequalities were deduced from *local* Poincaré inequalities. In the present article, we work with global inequalities only, and show how *global* Hardy-type inequalities result in Gagliardo–Nireberg inequalities.

Inequality (1.3), obtained here as a consequence of the Hardy inequality, extends our former results from [24]. In that paper, nondoubling measures were considered as well, but the conditions on M, P, Q were different. In particular, the case $M = P = Q$ was not allowed (see Remark 4.4 in [24]). This is rectified in the present approach.

It is our intention to focus on Hardy-type inequalities (1.2). They imply a wide range of Gagliardo–Nirenberg inequalities. Since (1.2) is valid for a vast class of admissible measures, possibly nondoubling, our approach yields Gagliardo–Nirenberg inequalities which are often new also in the L^p-setting.

For results concerning Gagliardo–Nirenberg inequalities in Orlicz spaces, we refer to [3], [20]–[24]. Other results on Gagliardo–Nirenberg inequalites presented in various contexts can be found e.g. in [6], [31], [35], [44].

2. Preliminaries

Notation. Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is an open domain. By $C_0^\infty(\Omega)$ we denote the smooth functions compactly supported in Ω, and we use the standard notation $W^{m,p}(\Omega)$ and $W^{m,p}_{\mathrm{loc}}(\Omega)$ for global and local variants of Sobolev spaces. The lower-case symbol c denotes a universal constant whose value is irrelevant. For important constants we use upper-case letters.

Orlicz spaces. Let us recall some basic information about Orlicz spaces, referring e.g. to [28, 42] for details.

Suppose μ is a positive Radon measure on $\mathbb{R}_+$ and let $M : [0,\infty) \to [0,\infty)$ be an *N-function*, i.e. a continuous convex increasing function satisfying $\lim_{\lambda\to 0} M(\lambda)/\lambda = \lim_{\lambda\to\infty} \lambda/M(\lambda) = 0$. The symbol M^* denotes the N-function complementary to M, i.e. $M^*(y) = \sup_{x>0}[xy - M(x)]$, in particular we have Young's inequality: $xy \le M(x) + M^*(y)$ for $x, y \ge 0$.

Given two functions M_1 and M_2, we write $M_1 \asymp M_2$ if there exist two constants c_1, c_2 such that $c_1 M_2(\lambda) \le M_1(\lambda) \le c_2 M_2(\lambda)$ for every $\lambda > 0$ (or for every λ in the indicated range).

Let μ be a nonnegative Borel measure on Ω. The weighted Orlicz space $L^M(\Omega,\mu)$ we deal with is by definition

$$L^M(\Omega,\mu) = \Big\{ f : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega M(|f(x)|/K)\, d\mu(x) \le 1 \text{ for some } K > 0 \Big\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^M(\Omega,\mu)} = \inf \Big\{ K > 0 : \int_\Omega M(|f(x)|/K)\, d\mu(x) \le 1 \Big\}.$$

It is a Banach space. For $M(\lambda) = \lambda^p$, we have $L^M(\Omega,\mu) = L^p_\mu(\Omega)$, the classical L^p space.

The function M is said to satisfy the Δ_2-*condition* if for some constant $c > 0$ and every $\lambda > 0$, we have $M(2\lambda) \le cM(\lambda)$. In the class of differentiable convex functions the Δ_2-condition is equivalent to

$$\lambda M'(\lambda) \le DM(\lambda), \tag{2.1}$$

with the constant D independent of λ (see e.g. [28, Theorem 4.1]). One also considers the condition

$$dM(\lambda) \le \lambda M'(\lambda). \tag{2.2}$$

It holds with some $d > 1$ when the dual function, M^*, satisfies the Δ_2-condition (see e.g. [28, Theorem 4.3] or [23, Proposition 4.1]). The optimal constants in (2.2) and (2.1) are called the *Simonenko lower* and *upper indices* of M (see e.g. [5], [43]) and will be denoted by d_M and D_M respectively.

If M is an N-function such that both (2.1) and (2.2) are satisfied, then

$$M(a\lambda) \le \max(a^{d_M}, a^{D_M})M(\lambda) =: \bar{c}(a)M(\lambda) \tag{2.3}$$

for all $\lambda, a > 0$ (see e.g. [25, Lemma 4.1, iii)]).

We will need the property $\int_\Omega M(f(x)/\|f\|_{L^M(\Omega,\mu)})\, d\mu(x) \le 1$. When M satisfies the Δ_2-condition, this becomes an equality.

The assumptions. We will consider the following assumptions:

- **(M)** $M : [0,\infty) \to [0,\infty)$ is an N-function of class $C^1((0,\infty))$, satisfying the Δ_2-condition and such that $M'(\lambda)/\lambda$ is bounded near zero;
- **(μ)** $\mu(dx) = \exp(-\varphi(x))dx$ is a Radon measure on Ω, where $\varphi : \Omega \to \mathbb{R}$ belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega)$;
- **(Y)** P and Q are two real nonnegative and nondecreasing measurable functions on $[0,\infty)$ with $P(0) = Q(0) = 0$, such that for any $u, v, w > 0$ the following Young-type inequality holds:

$$\frac{M(u)}{u^2} vw \le M(u) + P(v) + Q(w). \tag{2.4}$$

The inequality (2.4) is satisfied for example when the following condition holds (see [21, Cor. 4.9]):

- **(MF)** M is an N-function satisfying the Δ_2-condition and such that $M(\lambda)/\lambda^2$ is nondecreasing, $P(\lambda) = M(F(\sqrt{\lambda}))$ and $Q(\lambda) = M(F^*(\sqrt{\lambda}))$, where F is another N-function.

Remark 2.1. (1) The choice of $F(\lambda) = \lambda^2/2$ in **(MF)** gives **(Y)** with $P = Q = M$.

(2) Suppose that $M(\lambda) = \lambda^p$, $p \ge 2$. Choose $F(\lambda) = \lambda^s/s$ with $s = 2q/p$ to obtain $P(\lambda) = C\lambda^q$, $Q(\lambda) = C\lambda^r$, where $2/p = 1/q + 1/r$ (the classical Gagliardo–Nirenberg triple).

(3) When $M(\lambda) = \lambda^p(\ln(2+\lambda))^\alpha, p \ge 2, \alpha > 0$, the choice of $F(\lambda) = \lambda^s(\ln(2+\lambda))^\mu$ with $s = 2q/p$, $\mu = (\beta-\alpha)/p$ results in $P(\lambda) \asymp \lambda^q(\ln(2+\lambda))^\beta$, $Q(\lambda) \asymp \lambda^r(\ln(2+\lambda))^\gamma$, where the parameters are related through $2/p = 1/q + 1/r$, $2\alpha/p = \beta/q + \gamma/r$ (the logarithmic Gagliardo–Nirenberg triple considered in [20] and [22]).

3. Main results. Our goal now is to show that certain Hardy-type inequalities imply Gagliardo–Nirenberg ones. Let us start with the following result.

THEOREM 3.1. *Let M, P, Q be three functions on $[0,\infty)$ satisfying* **(M)** *and* **(Y)**, *and let μ be a Radon measure on Ω satisfying* $(\boldsymbol{\mu})$. *Suppose that the following Hardy-type inequality holds true:*

(H) *for any $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega P(|\nabla\varphi|\,|u|)\,d\mu \le K\int_\Omega P(A|\nabla u|)\,d\mu \tag{3.1}$$

with positive constants K, A not depending on u.

Then:

(1) *there exist constants $L, B > 0$ such that for any $\theta > 0$ and any $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega M(|\nabla u|)\,d\mu \le L\int_\Omega P(\theta|\nabla^{(2)}u|)\,d\mu + \int_\Omega Q\left(\frac{B}{\theta}|u|\right)d\mu, \tag{3.2}$$

(2) *if additionally P and Q are N-functions, then for any $u \in C_0^\infty(\Omega)$,*

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le \tilde{L}\sqrt{\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)}}, \tag{3.3}$$

where $\tilde{L} = 2(L+2)\sqrt{B}$, and L and B are as in (3.2).

REMARK 3.2. Under the assumptions of Theorem 3.1, if either P or Q satisfies the Δ_2-condition, then

$$\int_\Omega M(|\nabla u|)\,d\mu \le L_1\int_\Omega P(|\nabla^{(2)}u|)\,d\mu + L_2\int_\Omega Q(|u|)\,d\mu, \tag{3.4}$$

with L_1, L_2 independent of u.

REMARK 3.3. By a standard regularization argument (see e.g. [34, Section 1.1.5]) and Lebesgue's Dominated Convergence Theorem we deduce that inequality (3.1) can be applied to any compactly supported $u \in W^{1,\infty}(\Omega)$ as well, with the same constant.

Proof of Theorem 3.1. We start with the following inequality (Lemma 3.1 of [24]), valid for any $u \in C_0^\infty(\Omega)$:

$$\begin{aligned} I &= \int_\Omega M(|\nabla u|)\,d\mu \\ &\le \alpha_n \int_{\Omega\cap\{\nabla u\neq 0\}} \frac{M(|\nabla u|)}{|\nabla u|^2}|\nabla^{(2)}u|\,|u|\,d\mu + \int_{\Omega\cap\{\nabla u\neq 0\}} \frac{M(|\nabla u|)}{|\nabla u|}|\nabla\varphi|\,|u|\,d\mu \\ &=: \alpha_n I_1 + I_2, \end{aligned} \tag{3.5}$$

where α_n depends on n only. In [24], the proof is given for $\Omega = \mathbb{R}^n$ and $\varphi \in C^1(\mathbb{R}^n)$, but it requires only minor alterations to cover the present case.

To estimate I_1 and I_2, we use the assumption (2.4) twice. One has, for any given $\epsilon, \theta > 0$,

$$
\begin{aligned}
(3.6)\qquad I_1 &= \epsilon \int\limits_{\Omega\cap\{\nabla u\neq 0\}} \left(\frac{M(|\nabla u|)}{|\nabla u|^2}\right)(\theta|\nabla^{(2)}u|)\left(\frac{|u|}{\theta\epsilon}\right) d\mu \\
&\le \epsilon \int\limits_{\Omega} M(|\nabla u|)\, d\mu + \epsilon \int\limits_{\Omega} P(\theta|\nabla^{(2)}u|)\, d\mu + \epsilon \int\limits_{\Omega} Q\left(\frac{|u|}{\theta\epsilon}\right) d\mu,
\end{aligned}
$$

$$
\begin{aligned}
(3.7)\qquad I_2 &= A\epsilon \int\limits_{\Omega\cap\{\nabla u\neq 0\}} \left(\frac{M(|\nabla u|)}{|\nabla u|^2}\right)\left(|\nabla\varphi|\frac{\theta}{A}|\nabla u|\right)\left(\frac{|u|}{\theta\epsilon}\right) d\mu \\
&\le A\epsilon \int\limits_{\Omega} M(|\nabla u|)\, d\mu + A\epsilon \int\limits_{\Omega} P\left(|\nabla\varphi|\frac{\theta}{A}|\nabla u|\right) d\mu + A\epsilon \int\limits_{\Omega} Q\left(\frac{|u|}{\theta\epsilon}\right) d\mu.
\end{aligned}
$$

To estimate the central term on the right hand side of (3.7), we apply (3.1) to the function $(\theta/A)f(x)$ where $f(x) = |\nabla u(x)|$ (see Remark 3.3). Whenever $\nabla u \neq 0$, one has

$$
\frac{\partial}{\partial x_i} f(x) = \left\langle \frac{\nabla u(x)}{|\nabla u(x)|}, \nabla \frac{\partial}{\partial x_i} u(x) \right\rangle,
$$

and so

$$
\begin{aligned}
|\nabla f(x)|^2 &= \sum_{i=1}^{n} \left| \left\langle \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\partial}{\partial x_i}(\nabla u(x)) \right\rangle \right|^2 \\
&\le \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i}(\nabla u(x)) \right\|^2 = \|\nabla^{(2)}u(x)\|^2.
\end{aligned}
$$

Since P is nondecreasing, from **(H)** we get

$$
\int\limits_{\Omega} P\left(|\nabla\varphi|\frac{\theta}{A}|\nabla u|\right) d\mu \le K \int\limits_{\Omega} P(\theta|\nabla^{(2)}u|)\, d\mu.
$$

Using this fact, summing up estimates (3.6) and (3.7) we obtain

$$
I \le \epsilon(\alpha_n + A)I + \epsilon(\alpha_n + KA) \int\limits_{\Omega} P(\theta|\nabla^{(2)}u|)\, d\mu + \epsilon(\alpha_n + A) \int\limits_{\Omega} Q\left(\frac{|u|}{\theta\epsilon}\right) d\mu.
$$

Choosing $\epsilon = \frac{1}{2(\alpha_n + A)}$ gives, after rearranging,

$$
I \le (K+1) \int\limits_{\Omega} P(\theta|\nabla^{(2)}u|)\, d\mu + \int\limits_{\Omega} Q\left(\frac{2(\alpha_n + A)}{\theta}|u|\right) d\mu.
$$

This proves statement **(1)**.

To prove **(2)** we take an arbitrary $u \in C_0^\infty(\Omega)$ and apply (3.2) to

$$
\widetilde{u} := \frac{u}{a+b}, \quad \text{where} \quad a := \|\theta\nabla^{(2)}u\|_{L^P(\Omega,\mu)}, \quad b := \left\| \frac{B}{\theta} u(x) \right\|_{L^Q(\Omega,\mu)}.
$$

We can assume that neither a nor b is zero, because otherwise $\nabla u(x) = 0$ a.e. and (3.3) follows trivially. Inequality (3.2) for $\widetilde{u}$ reads

$$\begin{aligned}\int_{\Omega} M(|\nabla \widetilde{u}|)\,d\mu &\le L\int_{\Omega} P\biggl(\frac{|\theta\nabla^{(2)}u|}{a+b}\biggr)\,d\mu + \int_{\Omega} Q\biggl(\frac{|(B/\theta)u|}{a+b}\biggr)\,d\mu\\ &\le L\int_{\Omega} P\biggl(\frac{|\theta\nabla^{(2)}u|}{a}\biggr)\,d\mu + \int_{\Omega} Q\biggl(\frac{|(B/\theta)u|}{b}\biggr)\,d\mu\\ &= L\int_{\Omega} P\biggl(\frac{|\theta\nabla^{(2)}u|}{\|\theta\nabla^{(2)}u\|_{L^P(\Omega,\mu)}}\biggr)\,d\mu + \int Q\biggl(\frac{|(B/\theta)u|}{\|(B/\theta)u\|_{L^Q(\Omega,\mu)}}\biggr)\,d\mu.\\ &\le L+1.\end{aligned}$$

In the last inequality we have used the property $\int_{\Omega} R(w/\|w\|_{L^R(\Omega,\mu)})\,d\mu \le 1$ of modular functionals. Since for any $f \in L^M(\Omega,\mu)$ one has $\|f\|_{L^M(\Omega,\mu)} \le \int_{\Omega} M(|f|)\,d\mu + 1$ (see (9.4) and (9.20) of [28]), this gives $\|\nabla \widetilde{u}\|_{L^M(\Omega,\mu)} \le L+2$. Consequently,

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le (L+2)(a+b) = (L+2)\biggl(\theta\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)} + \frac{B}{\theta}\|u(x)\|_{L^Q(\Omega,\mu)}\biggr).$$

Minimizing the right hand side with respect to θ gives the result. ■

Our next theorem covers the case when the Hardy inequality **(H)** does not hold, but it does when an extra term, depending on u, is added to the right hand side.

THEOREM 3.4. *Suppose that the assumptions of Theorem* 3.1 *are satisfied, and the following Hardy-type inequality holds true:*

(H1) *for any* $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega} P(|\nabla\varphi|\,|u|)\,d\mu \le K_1\int_{\Omega} P(A|\nabla u|)\,d\mu + K_2\int_{\Omega} M(|u|)\,d\mu, \tag{3.8}$$

with positive constants K_1, K_2, A *not depending on* u.

Then:

(1) *there exist constants* $L, B > 0$ *such that for any* $\theta \in (0,1]$ *and any* $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega} M(|\nabla u|)\,d\mu \le L\int_{\Omega} P(\theta|\nabla^{(2)}u|)\,d\mu + \int_{\Omega} Q\biggl(\frac{B}{\theta}|u|\biggr)\,d\mu, \tag{3.9}$$

(2) *if* P *and* Q *are* N*-functions, then for any* $u \in C_0^\infty(\Omega)$,

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le L_1\sqrt{\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)}} + L_2\|u\|_{L^Q(\Omega,\mu)}, \tag{3.10}$$

where $L_1 = 2(L+2)\sqrt{B}$, $L_2 = 2(L+2)B$, *and* L *and* B *are as in* (3.9).

Proof. **(1)** We start with inequality (3.5) and repeat the arguments in the proof of Theorem 3.1 up to (3.7). Now, instead of (3.1), we apply (3.8) to the function $f(x) = (\theta/A)|\nabla u(x)|$ (we use an argument similar to that in Remark 3.3). Since we have assumed $\theta \le 1$, we get

$$\int_\Omega P\bigg(|\nabla\varphi|\frac{\theta}{A}|\nabla u|\bigg)\,d\mu \le K_1\int_\Omega P(\theta|\nabla^{(2)}u|)\,d\mu + \bar{K}_2\int_\Omega M(|\nabla u|)\,d\mu,$$

where $\bar{K}_2 = \bar{c}(1/A)K_2$ with $\bar{c}(\cdot)$ coming from (2.3). This, (3.6) and (3.7) lead to

$$\begin{aligned} I &\le \epsilon(\alpha_n + A + \bar{K}_2A)I + \epsilon(\alpha_n + K_1A)\int_\Omega P(\theta|\nabla^{(2)}u|)\,d\mu \\ &\quad + \epsilon(\alpha_n + A)\int_\Omega Q\bigg(\frac{|u|}{\theta\epsilon}\bigg)\,d\mu. \end{aligned}$$

The choice of $\epsilon = (2(\alpha_n + A + A\bar{K}_2))^{-1}$ implies

$$I \le L\int_\Omega P(\theta|\nabla^{(2)}u|)\,d\mu + \int_\Omega Q\bigg(\frac{B}{\theta}|u|\bigg)\,d\mu, \tag{3.11}$$

where $L = K_1 + 1$, $B = 2(\alpha_n + A + A\bar{c}(1/A)K_2)$. This proves **(1)**.

(2) To prove the second part we observe that arguments similar to those in the proof of the second part of Theorem 3.1 lead to the inequality

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le \tilde{L}\bigg(\theta\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)} + \frac{B}{\theta}\|u\|_{L^Q(\Omega,\mu)}\bigg),$$

where $\tilde{L} = L + 2$, $L = K_1 + 1$ is the constant from (3.11), and $\theta \in (0,1]$ is arbitrary. Minimization of the inequality $a \le \tilde{L}(\theta b + \frac{1}{\theta}c)$ with respect to $\theta \in (0,1]$ gives the desired result. Indeed, when $\bar{\theta} := \sqrt{c/b} \in (0,1)$, we get $a \le 2\tilde{L}\sqrt{bc}$, while in the remaining case $c \ge b$ we have $a \le 2\tilde{L}c$ (choose $\theta = 1$). In either case, the inequality $a \le 2\tilde{L}(\sqrt{bc} + c)$ is satisfied. This completes the proof of the theorem. ∎

Since condition **(Y)** is satisfied for $P = Q = M$ (see Remark 2.1), as an immediate consequence we obtain the following:

THEOREM 3.5. *Suppose that M is an N-function satisfying condition* **(M)**, *and let μ be a Radon measure on $[0,\infty)$ satisfying* **(μ)**. *Moreover, assume that $M(\lambda)/\lambda^2$ is nondecreasing. If for every $u \in C_0^\infty(\Omega)$ the following Hardy-type inequality holds true:*

$$\int_\Omega M(|\nabla\varphi|\,|u|)\,d\mu \le K_1\int_\Omega M(|\nabla u|)\,d\mu + K_2\int_\Omega M(|u|)\,d\mu, \tag{3.12}$$

then:

(1) *there exist positive constants* C_1, C_2 *such that for any* $\theta \in (0,1]$ *and any* $u \in C_0^\infty(\Omega)$,

$$\int_\Omega M(|\nabla u|)\,d\mu \le C_1 \int_\Omega M(\theta|\nabla^{(2)}u|)\,d\mu + C_2 \int_\Omega M(|u|/\theta)\,d\mu, \tag{3.13}$$

(2) *there exist positive constants* $\tilde{C}_1, \tilde{C}_2$ *such that for any* $u \in C_0^\infty(\Omega)$,

$$\|\nabla u\|_{L^M(\Omega,\mu)} \le \tilde{C}_1 \sqrt{\|\nabla^{(2)}u\|_{L^M(\Omega,\mu)}\|u\|_{L^M(\Omega,\mu)}} + \tilde{C}_2\|u\|_{L^M(\Omega,\mu)}. \tag{3.14}$$

Remark 3.6 (open question). In Theorem 3.1, the sole purpose of assuming $M \in \Delta_2$ is to derive (3.5). We do not know whether one can extend (3.2), (3.3) to functions M for which the Δ_2-condition is not satisfied. Some results concerning the Gagliardo–Nirenberg inequalities (3.2), (3.3) do hold true without imposing the Δ_2-condition on M, but for a restricted family of measures (see e.g. [3], [23]).

4. Discussion and examples. Three theorems from the previous section reduce the question about the validity of the Gagliardo–Nirenberg inequality for given N-functions and measures to a question about the validity of Hardy-type inequalities. We will discuss this now.

4.1. The scope of Theorem 3.1

4.1.1. *The case* $\Omega = \mathbb{R}_+$, *condition* **(H)**. A necessary and sufficient condition for Radon measures μ, ν to obey the inequality

$$\int_{\mathbb{R}_+} \left|\int_0^x f(t)\,dt\right|^p d\nu(x) \le C \int_{\mathbb{R}_+} |f(x)|^p\,d\mu(x) \tag{4.1}$$

was given by Muckenhoupt (see [37] or [34, Section 1.3, Theorem 1]):

$$\sup_{r>0} \nu(r,\infty)^{1/p} \left(\int_0^r \left(\frac{d\mu^*}{dx}\right)^{-1/(p-1)} dx\right)^{(p-1)/p} < \infty, \tag{4.2}$$

where μ^* is the absolutely continuous part of μ. Since for $u \in C_0^\infty(0,\infty)$ one has $u(x) = \int_0^x u'(t)\,dt$, it follows that whenever ν, μ obey (4.2), then

$$\int_{\mathbb{R}_+} |u(x)|^p\,d\nu(x) \le C \int_{\mathbb{R}_+} |u'(x)|^p\,d\mu(x) \tag{4.3}$$

for all $u \in C_0^\infty(\mathbb{R}_+)$. Observe that in the particular case of the measures $d\nu(x) = |\varphi'(x)|^p \cdot \exp(-\varphi(x))dx$, $d\mu(x) = \exp(-\varphi(x))dx$, inequality (4.3) is nothing but our condition **(H)** for $P(\lambda) = \lambda^p$. In this case, condition (4.2) reads

$$\sup_{r>0}\Big(\int_r^\infty |\varphi'(x)|^p \exp(-\varphi(x))dx\Big)^{1/p}\Big(\int_0^r \exp(-\varphi(x))^{-1/(p-1)}dx\Big)^{1-1/p} < \infty, \tag{4.4}$$

and so if (4.4) holds true, then **(H)** is true for $P(\lambda)=\lambda^p$, $\Omega=\mathbb{R}_+$. Therefore we obtain:

THEOREM 4.1. *Let $p>1$ be given, and let $\mu(dx)=e^{-\varphi(x)}dx$ be a Radon measure on $[0,\infty)$ satisfying* **(μ)**. *Suppose that* (4.4) *holds true. Next, let M be an N-function satisfying condition* **(M)**, *and let Q be another N-function such that* **(Y)** *is satisfied for M, $P=P(\lambda)=\lambda^p$, and Q. Then for any $u\in C_0^\infty(\mathbb{R}_+)$,*

$$\int_{\mathbb{R}_+} M(|u'|)\,d\mu \le K_1\int_{\mathbb{R}_+}|u''|^p\,d\mu + K_2\int_{\mathbb{R}_+} Q(|u|)\,d\mu, \tag{4.5}$$

and also

$$\|u'\|^2_{L^M(\mathbb{R}_+,\mu)} \le K\|u''\|_{L^p(\mathbb{R}_+,\mu)}\|u\|_{L^Q(\mathbb{R}_+,\mu)}, \tag{4.6}$$

where the constants K_1, K_2, K do not depend on u.

We illustrate this case with two examples.

EXAMPLE 4.2 (classical Hardy inequality). Consider the classical Hardy inequality, which involves power weights [16], [17]:

$$\int_{\mathbb{R}_+} |u(t)|^p t^{\alpha-p}\,dt \le C\int_{\mathbb{R}_+}|u'(t)|^p t^\alpha\,dt, \tag{4.7}$$

where $C=(p/|\alpha-p+1|)^p$, $\alpha\neq p-1$. In this case we have $\mu(dt)=\exp(-\varphi(t))dt$, where $\varphi(t)=-\alpha\ln t$. In particular $\varphi'(t)=-\alpha/t$, and (4.7) reads

$$\int_{\mathbb{R}_+}(|\varphi'(t)|\,|u(t)|)^p t^\alpha\,dt \le \Big(\frac{p|\alpha|}{|\alpha-p+1|}\Big)^p\int_{\mathbb{R}_+}|u'(t)|^p t^\alpha\,dt.$$

EXAMPLE 4.3 (Hardy inequality and power-exponential weights). We now consider measures on $(0,\infty)$ with power-exponential-type densities:

$$\mu(dx) = x^\alpha e^{-x^\beta}\,dx = \exp(-\varphi(x))dx, \qquad \alpha\ge 0,\ \beta>0,\ \varphi(x) = -\ln x^\alpha + x^\beta. \tag{4.8}$$

This class of measures contains in particular the exponential distribution ($\alpha=0$, $\beta=1$) and the Gaussian distribution ($\alpha=0$, $\beta=2$).

As $|\varphi'(x)| \asymp 1/x$ for x small and $|\varphi'(x)| \asymp x^{\beta-1}$ for x close to ∞, the Muckenhoupt condition (4.4) for the measure (4.8) is equivalent to

$$\sup_{r>0} A(r)^{1/p} B(r)^{1-1/p} < \infty, \tag{4.9}$$

where

$$B(r) := \int_0^r \frac{e^{x^\beta/(p-1)}}{x^{\alpha/(p-1)}}\, dx,$$

$$A(r) := \Big(\int_r^1 x^{-p+\alpha} e^{-x^\beta}\, dx\Big)\chi_{\{r<1\}} + \Big(\int_r^\infty x^{(\beta-1)p+\alpha} e^{-x^\beta}\, dx\Big)\chi_{\{r\geq 1\}}.$$

We observe that $A(r)$ is finite for all choices of $\alpha \geq 0$, $\beta > 0$, whereas $B(r)$ is finite if and only if $\alpha < p-1$. Both functions A, B are locally bounded and continuous near 0 and ∞. Moreover, for r close to 0, we have

$$A(r)^{1/p} B(r)^{1-1/p} \asymp (r^{-1+\alpha/p+1/p} + C)\cdot(r^{-\alpha/p+1-1/p}) < \text{const}.$$

Therefore (4.9) holds true if and only if $\limsup_{r\to\infty} A(r)B(r)^{p-1} < \infty$.

By a direct application of the de l'Hospital rule we see that for $a \in \mathbb{R}$ and $b > 0$,

$$\lim_{r\to\infty} \frac{\int_r^\infty x^a e^{-x^b}\, dx}{r^{a+1-b} e^{-r^b}} = \lim_{r\to\infty} \frac{r^a e^{-r^b}}{b r^a e^{-r^b} - (a+1-b) r^{a-b} e^{-r^b}} = \frac{1}{b},$$

and for $a < 1$ and $C > 0$,

$$\lim_{r\to\infty} \frac{\int_0^r \frac{e^{Cx^b}}{x^a} dx}{e^{Cr^b} r^{-(a+b-1)}} = \lim_{r\to\infty} \frac{e^{Cr^b}}{r^a}\left[\frac{bCe^{Cr^b}}{r^a} - \frac{(a+b-1)e^{Cr^b}}{r^{a+b}}\right]^{-1} = \frac{1}{bC}.$$

Therefore, for r large, we have

$$A(r) \asymp r^{(\beta-1)(p-1)+\alpha} e^{-r^\beta} \quad \text{and} \quad B(r) \asymp r^{-\alpha/(p-1)-(\beta-1)} e^{r^\beta/(p-1)},$$

and so $A(r)B(r)^{p-1} \asymp$ const for large r. Consequently, (4.9) is satisfied whenever $0 \leq \alpha < p-1$ and $\beta > 0$.

We end up with the following theorem.

THEOREM 4.4. *Let $p > 1$ and let $\mu(dx) = x^\alpha e^{-x^\beta} dx$, where $\alpha \neq p-1$, $\beta = 0$, or $0 \leq \alpha < p-1$, $\beta > 0$. Suppose that M is an N-function satisfying condition* **(M)**, *and Q is another N-function such that*

$$\frac{M(t)}{t^2} rs \leq M(t) + cr^p + Q(s) \quad \text{for all } t, r, s > 0.$$

Then inequalities (4.5) *and* (4.6) *hold for any $u \in C_0^\infty(\mathbb{R}_+)$, with constants K_1, K_2, K independent of u.*

REMARK 4.5. As to the validity of Orlicz-space counterparts of (4.1), which would yield **(H)**, we refer to the papers of Bloom–Kerman [4], Lai [32], Heinig–Maligranda [19], Heinig–Lai [18], their references and also to the authors' paper [25, Section 3.3], where another type of sufficient condition for **(H)** to hold on $\mathbb{R}_+$ is given.

4.1.2. *Multidimensional Hardy inequalities*

(A) *Inequalities on bounded domains.* Multidimensional Hardy inequalities of the form

$$\int_\Omega \left(\frac{1}{\delta(x)}|u(x)|\right)^q \delta(x)^a\,dx \le C\int_\Omega |\nabla u(x)|^q\delta(x)^a\,dx, \qquad u\in C_0^1(\Omega), \tag{4.10}$$

where $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with sufficiently regular boundary, $a<q-1$, $1<q<\infty$, $\delta(x)=\operatorname{dist}(x,\partial\Omega)$, were first obtained by Nečas [38] (for bounded domains with Lipschitz boundary) and extended further by Kufner [29, Theorem 8.4] and Wannebo [47] to Hölder domains.

As a direct consequence of Theorem 3.1, we obtain Gagliardo–Nirenberg-type inequalities for L^p-spaces weighted by the distance from the boundary, which can be stated as follows.

THEOREM 4.6. *Suppose that Ω is a bounded Lipschitz domain and $q>1$, $a>q-1$. Then:*

(i) *if $p\ge 2$, $r>1$, $2/p=1/q+1/r$, then for every $u\in C_0^\infty(\Omega)$,*

$$\Big(\int_\Omega |\nabla u(x)|^p\delta(x)^a\,dx\Big)^{2/p} \le c\Big(\int_\Omega |\nabla^{(2)}u(x)|^q\delta(x)^a\,dx\Big)^{1/q} \cdot\Big(\int_\Omega |u(x)|^r\delta(x)^a\,dx\Big)^{1/r},$$

with a constant $c>0$ independent of u,

(ii) *if M and Q are N-functions such that*

$$\frac{M(u)}{u^2}vw \le M(u)+cv^q+Q(w)$$

and M satisfies condition **(M)**, *then for every $u\in C_0^\infty(\Omega)$,*

$$\int_\Omega M(|\nabla u(x)|)\delta(x)^a\,dx \le c\Big(\int_\Omega |\nabla^{(2)}u(x)|^q\delta(x)^a\,dx + \int_\Omega Q(|u(x)|)\delta(x)^a\,dx\Big),$$

and

$$\|\nabla u\|^2_{L^M(\Omega,\mu)} \le c\|\nabla^{(2)}u\|_{L^q(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)},$$

with constants independent of u.

Proof. **(i)** Obviously, $M(\lambda)=\lambda^p$ satisfies condition **(M)**, and $P(\lambda)=\lambda^q$, $Q(\lambda)=\lambda^r$ satisfy **(Y)** due to Remark 2.1. Moreover, the measure $\mu(dx)=$

$\exp(-\varphi(x))dx$, where $\varphi(x) = -a \ln \delta(x)$, satisfies **($\boldsymbol{\mu}$)**. It is not hard to verify that $|\nabla\varphi(x)| \asymp 1/\delta(x)$ (note that $|\nabla\delta| \asymp \text{const}$). This together with (4.10) implies **(H)**. Now it suffices to apply Theorem 3.1. Part **(ii)** is proven similarly. ∎

Remark 4.7. Note that $\delta(x)^a$ is an A_q-weight when $-1 < a < q-1$ (see e.g. [45] for the definition of A_p-weights introduced by Muckenhoupt). Gagliardo–Nirenberg inequalities with A_p-weights for homogeneous spaces were earlier obtained in [9], [23] by different methods. Those results also covered the case $1 < p < 2$.

Remark 4.8. Counterparts of inequality (4.10) in Orlicz norms were obtained by Cianchi in [11].

(B) *Inequalities on* $\mathbb{R}^n$. The Hardy inequality on $\mathbb{R}^n$ with power weights (see e.g. [7], [34], [30, p. 70], and their references)

$$\big\||x|^{\alpha-1}|u|\big\|_{L^q} \le C\big\||x|^{\alpha}|\nabla u|\big\|_{L^q},$$

where $u \in C_0^\infty(\mathbb{R}^n)$, $1/q + (\alpha-1)/n > 0$, $q > 1$, gives rise to Gagliardo–Nirenberg inequalities on $\mathbb{R}^n$ with power weights $|x|^\alpha$ and N-functions $M, P = P(\lambda) = \lambda^q, Q$, satisfying **(Y)**. The result, obtained directly from Theorem 3.1, reads as follows.

Theorem 4.9. *Suppose that* $1/q + (\alpha-1)/n > 0$, $q > 1$. *Then:*

(i) *if* $p \ge 2$, $r > 1$, $2/p = 1/q + 1/r$, *then for every* $u \in C_0^\infty(\Omega)$,

$$\Big(\int_\Omega |\nabla u(x)|^p |x|^\alpha\,dx\Big)^{2/p} \le c\Big(\int_\Omega |\nabla^{(2)} u(x)|^q |x|^\alpha\,dx\Big)^{1/q} \cdot \Big(\int_\Omega |u(x)|^r |x|^\alpha\,dx\Big)^{1/r},$$

with a constant $c > 0$ *independent of* u,

(ii) *if* M *and* Q *are* N*-functions such that*

$$\frac{M(u)}{u^2}\,vw \le M(u) + cv^q + Q(w),$$

and M *satisfies condition* **(M)**, *then for every* $u \in C_0^\infty(\Omega)$,

$$\Big(\int_\Omega M(|\nabla u(x)|)|x|^\alpha\,dx\Big) \le c\Big(\int_\Omega |\nabla^{(2)} u(x)|^q |x|^\alpha\,dx + \int_\Omega Q(|u(x)|)|x|^\alpha\,dx\Big)$$

and

$$\|\nabla u\|^2_{L^M(\Omega,\mu)} \le c\|\nabla^{(2)} u\|_{L^q(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)},$$

with constants independent of u.

4.2. The scope of Theorem 3.4. We now discuss the validity of Theorem 3.4. To shorten the discussion, we only focus on condition **(H1)** in its assumptions. While the Hardy inequality **(H)** has been thoroughly investigated, condition **(H1)** has not attracted much attention so far.

Inequalities on bounded domains

(A) *Result by Oinarov.* Condition **(H1)** and its special variant (3.12) have drawn less attention than the 'classical' Hardy inequality **(H)**. Oinarov [40] considered the inequalities

$$\Big(\int_a^b |\omega u|^q\,dr\Big)^{1/q} \le C\Big(\int_a^b |\rho u'|^p\,dr + \int_a^b |vu|^p\,dr\Big)^{1/p} \tag{4.11}$$

for general weights ω, v, ρ and derived necessary and sufficient conditions for (4.11) to hold for all $u \in C_0^\infty(a,b)$. Our condition (3.12) with $\mu(dx) = e^{-\varphi(x)}dx$ is exactly (4.11), under the choice of $\omega(r) = \varphi'(r)e^{-\varphi(r)/p}$, $v(r) = \rho(r) = e^{-\varphi(r)/p}$, $p = q$.

(B) *Result by Cianchi.* The Orlicz-norms counterpart of **(H1)**,

$$\Big\|\frac{u}{d^{1+\alpha}}\Big\|_{L^B(G)} \le C\bigg(\Big\|\frac{u}{d^\alpha}\Big\|_{L^A(G)} + \Big\|\frac{\nabla u}{d^\alpha}\Big\|_{L^A(G)}\bigg), \tag{4.12}$$

has been established by Cianchi [11]. Here $G \subset \mathbb{R}^n$ is a sufficiently regular domain, A and B are N-functions related by a certain domination condition (in particular $A = B$ is possible), $d(x) = \mathrm{dist}(x, \partial G)$, and the measure considered is the Lebesgue measure. In our work, we need modular versions of (4.12); in general they do not come as its direct consequence. Note that in the case of homogeneous N-functions, (4.12) is an extension of (4.10).

(C) *Our approach. Inequalities on intervals.* In the forthcoming paper [26], the authors work out conditions concerning M, φ which ensure the validity of (3.12) on intervals $(a,b) \subset \mathbb{R}$, including the cases $a = -\infty$ and $b = \infty$. See also the paper [41] devoted solely to the Hardy and Gagliardo–Nirenberg inequalities for the Gaussian measure on $\mathbb{R}^n$.

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