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DYNAMICAL SYSTEMS AND ERGODIC THEORY

SRB-like Measures for C^0 Dynamics

by

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Summary. For any continuous map $f\colon M \to M$ on a compact manifold M, we define SRB-like (or observable) probabilities as a generalization of Sinai–Ruelle–Bowen (i.e. physical) measures. We prove that f always has observable measures, even if SRB measures do not exist. We prove that the definition of observability is optimal, provided that the purpose of the researcher is to describe the asymptotic statistics for Lebesgue almost all initial states. Precisely, the never empty set $\mathcal{O}$ of all observable measures is the minimal weak* compact set of Borel probabilities in M that contains the limits (in the weak* topology) of all convergent subsequences of the empirical probabilities $\{(1/n)\sum_{j=0}^{n-1}\delta_{f^j(x)}\}_{n\geq 1}$, for Lebesgue almost all $x \in M$. We prove that any isolated measure in $\mathcal{O}$ is SRB. Finally we conclude that if $\mathcal{O}$ is finite or countably infinite, then there exist (countably many) SRB measures such that the union of their basins covers M Lebesgue a.e.

1. Introduction. Let $f\colon M \to M$ be a continuous map of a compact, finite-dimensional manifold M. Let m be a Lebesgue measure normalized so that $m(M) = 1$, and not necessarily f-invariant. We denote by $\mathcal{P}$ the set of all Borel probability measures on M, provided with the weak* topology, and a metric structure inducing this topology.

For any point $x \in M$ we denote by $p\omega(x)$ the set of all Borel probabilities on M that are limits in the weak* topology of convergent subsequences of the sequence

$$\left\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\right\}_{n\in\mathbb{N}} \tag{1.1}$$

where δ_y is the Dirac delta probability measure supported at $y \in M$. We call the probabilities of the sequence (1.1) *empirical probabilities* of the orbit

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of x. We call $p\omega(x)$ the *limit set in* $\mathcal{P}$ corresponding to the forward orbit of $x \in M$.

The following definition of a physical measure is standard in ergodic theory:

DEFINITION 1.1. A probability measure μ is *physical* or *SRB* (Sinai–Ruelle–Bowen) if $\{\mu\} = p\omega(x)$ for a set $A(\mu)$ of points $x \in M$ that has positive Lebesgue measure. We call $A(\mu)$ the *basin of attraction* of μ.

The adjective "physical" is used for instance in [Y02]). Nevertheless, in calling such probabilities SRB measures, we follow [V98] and Chapter 11 of [BDV05]. This preference is based on three reasons, which are also our motivations:

1. Our scenario includes *all continuous systems.* C^0 generic maps f are not differentiable. So, no Lyapunov exponents necessarily exist, to be able to assume some kind of hyperbolicity. Thus, we cannot assume the existence of an unstable foliation with differentiable leaves. Therefore, we aim to study those systems for which the SRB measures as usually defined in the literature (related to an unstable foliation $\mathcal{F}$) do not exist. We recall a popularly required property for μ: the conditional measures μ_x of μ along the local leaves $\mathcal{F}_x$ of a hyperbolic unstable foliation $\mathcal{F}$ are absolutely continuous with respect to the internal Lebesgue measures of those leaves. But this assumption needs the existence of such a regular foliation $\mathcal{F}$. It is well known that the ergodic theory based on this absolute continuity condition does not work for generic C^1 systems (that are not $C^{1+\alpha}$): see [RY80, BH98, AB07]. So, it does not work for most C^0 systems.

2. In the modern differentiable ergodic theory, for $C^{1+\alpha}$ systems that have some hyperbolic behavior, one of the ultimate purposes of searching measures with absolute continuity properties with respect to Lebesgue measure is to find probabilities that satisfy Definition 1.1. Therefore, if the system is not $C^{1+\alpha}$, or is not hyperbolic-like, but nevertheless there exists some probability μ describing the asymptotic behavior of the sequence (1.1) for a Lebesgue positive set of initial states (i.e. μ satisfies Definition 1.1), then one of the initial purposes of research of Sinai, Ruelle and Bowen in [B71, BR75, R76, S72] is also achieved. Therefore, it makes sense (principally for C^0 systems) to call μ an SRB measure if it satisfies Definition 1.1.

3. The SRB-like property of some invariant measures describes (with an error that is smaller than ε for an arbitrary $\varepsilon > 0$) the behavior of the sequence (1.1) for every n large enough and for a Lebesgue positive set of initial states. This property can be achieved by considering *observable measures* that we introduce in Definition 1.2, instead of only those satisfying Definition 1.1. This new setting will describe the statistics (defined by the sequence (1.1) of empirical probabilities) for Lebesgue almost all initial states (see Theorem 1.5). The result is particularly interesting in the cases for which SRB measures do not exist (see for instance [K04] and some examples in Section 5 of this paper). So, in the following, we use the words physical and SRB as synonymous, and we apply them only to the probability measures that satisfy Definition 1.1. To generalize this notion, we call the measures introduced in Definition 1.2 below *observable* or *SRB-like* or *physical-like.* Under this agreement all SRB measures are SRB-like but not conversely (we provide examples in Section 5).

One of the major problems of the ergodic theory of dynamical systems is to find SRB measures. They are mostly studied for those systems that are $C^{1+\alpha}$ and show some kind of hyperbolicity ([PS82], [PS04], [V98], [BDV05]). One of the reasons for searching those measures is that they describe the asymptotic behavior of the sequence (1.1) for a Lebesgue positive set of initial states, that is, for a set that is not negligible from the viewpoint of the observer. By means of SRB measures the statistics of orbits is described, i.e. the time-mean of the future evolution of the system is predicted for a Lebesgue positive set of initial states. Nevertheless, it is unknown if most differentiable systems have SRB measures ([P99]). Some interesting continuous systems do not (see Example 5.5). In [K98], Keller considers an SRB-like property of a measure, even if the sequence (1.1) is not convergent. In fact, he takes those measures μ that belong to the set $p\omega(x)$ for a Lebesgue positive set of initial states $x \in M$, regardless of whether $p\omega(x)$ coincides with $\{\mu\}$ or not. Precisely, Keller considers those measures μ for which $\operatorname{dist}(\mu, p\omega(x)) = 0$ for a Lebesgue positive set of points $x \in M$. But, as he also remarks in his definition, that kind of weak-SRB measures may not exist. We now introduce the following notion, which generalizes the notion of observability of Keller, and the notion of SRB measures in Definition 1.1:

Definition 1.2. A probability measure $\mu \in \mathcal{P}$ is *observable* or *SRB-like* or *physical-like* if for any $\varepsilon > 0$ the set $A_\varepsilon(\mu) = \{x \in M : \operatorname{dist}(p\omega(x), \mu) < \varepsilon\}$ has positive Lebesgue measure. We call $A_\varepsilon(\mu)$ the *basin of* ε*-attraction* of μ. We denote by $\mathcal{O}$ the set of observable measures.

From Definitions 1.1 and 1.2 it is immediate that every SRB measure is observable. But not every observable measure is SRB (we provide examples in Section 5). It is standard to check that any observable measure is f-invariant. (In fact, if $\mathcal{P}_f \subset \mathcal{P}$ denotes the weak* compact set of f-invariant probabilities, since $p\omega(x) \subset \mathcal{P}_f$ for all x, we conclude that $\mu \in \overline{\mathcal{P}_f} = \mathcal{P}_f$ for all $\mu \in \mathcal{O}$.) For the experimenter, observable measures as defined in 1.2 should have the same relevance as SRB measures defined in 1.1. In fact, the basin of ε-attraction $A_\varepsilon(\mu)$ has positive Lebesgue measure *for all* $\varepsilon > 0$. The ε-approximation holds in the space $\mathcal{P}$ of probabilities, but it can be easily translated (through the functional operator induced by the probability μ in the space $C^0(M, \mathbb{R})$) to an ε-approximation (in time-mean) towards an "attractor" in the ambient manifold M.

Precisely, if μ is observable and $x \in A_\varepsilon(\mu)$ then, with a frequency that is asymptotically bounded away from zero, the iterates $f^n(x)$ will ε-approach the support of μ. Note that also for an SRB measure μ this ε-approximation to the support of μ holds in the ambient manifold M for all $\varepsilon > 0$, but it is not true in general if $\varepsilon = 0$. Namely, assuming that there exists an SRB measure μ, the empirical probabilities (defined in (1.1) for Lebesgue almost all orbits in the basin of μ) approximate μ. But in general they differ from μ after any finite time $n \geq 1$ of observation. If the experimenter aims to observe the orbits during a finite time n, then Definition 1.2 of observability ensures him a 2ε-approximation to the "attractor", for any given $\varepsilon > 0$, while Definition 1.1 of physical measures ensures him an ε-approximation. None guarantees a null error, and both guarantee an error smaller than $\varepsilon > 0$ for arbitrarily small values of $\varepsilon > 0$. Thus, the practical meaning of both definitions is similar.

Statement of the results

Theorem 1.3 (Existence of observable measures). *For every continuous map f, the space $\mathcal{O}$ of all observable measures for f is nonempty and weak* compact.*

We prove this theorem in Section 3. It states that Definition 1.2 is weak enough to ensure the existence of observable measures for any continuous f. When we consider the set $\mathcal{P}_f$ of all invariant measures we also obtain existing probabilities describing all the weak* limit sets $p\omega(x)$ of Lebesgue almost all initial states $x \in M$. Nevertheless, $\mathcal{P}_f$ is less economic. In fact, along Section 5, we exhibit paradigmatic systems for which most invariant measures are not observable. We also show that observable measures (just as SRB measures defined in 1.1) are not necessarily ergodic. So, ergodic measures are not necessarily suitable to describe the asymptotic behavior of orbits in some Lebesgue positive sets. In fact, there exist examples (we will provide one in Section 5) for which the set of points $x \in M$ such that $p\omega(x)$ is an ergodic probability has zero Lebesgue measure.

In Definition 1.1, the set $A(\mu) = \{x \in X : p\omega(x) = \{\mu\}\}$ is called the basin of attraction of an SRB-measure μ. Inspired by that definition we introduce the following:

Definition 1.4. The *basin of attraction* $A(\mathcal{K})$ of a nonempty weak* compact subset $\mathcal{K}$ of probabilities is

$$A(\mathcal{K}) := \{x \in M : p\omega(x) \subset \mathcal{K}\}.$$

We are interested in those compact sets $\mathcal{K} \subset \mathcal{P}$ having basin $A(\mathcal{K})$ with full Lebesgue measure. We are also interested in not adding unnecessary probabilities to the set $\mathcal{K}$. The following result states that the optimal choice is the nonempty compact set of observable measures defined in 1.2.

THEOREM 1.5 (Full optimal attraction of $\mathcal{O}$). *The set $\mathcal{O}$ of observable measures for f is the minimal weak* compact subset of $\mathcal{P}$ whose basin of attraction has total Lebesgue measure. In other words, $\mathcal{O}$ is minimally weak* compact containing, for Lebesgue almost all initial states, the limits of all convergent subsequences of* (1.1).

We prove this theorem in Section 3.

Let us state the relations between the cardinality of $\mathcal{O}$ and the existence of SRB measures according to Definition 1.1.

THEOREM 1.6 (Finite set of observable measures). *$\mathcal{O}$ is finite if and only if there exist finitely many SRB measures such that the union of their basins of attraction covers M Lebesgue a.e. In this case $\mathcal{O}$ is the set of SRB measures.*

We prove this theorem in Section 4.

THEOREM 1.7 (Countable set of observable measures). *If $\mathcal{O}$ is countably infinite, then there exist countably infinitely many SRB measures such that their basins of attraction cover M Lebesgue a.e. In this case $\mathcal{O}$ is the weak* closure of the set of SRB measures.*

We prove this theorem in Section 4.

For systems preserving the Lebesgue measure the main question is their ergodicity, and most results of this work translate, for those systems, as conditions equivalent to being ergodic. The proof of the following result is standard in view of Theorem 1.5:

REMARK 1.8 (Observability and ergodicity). If $f : M \to M$ preserves the Lebesgue measure m, then the following assertions are equivalent:

1. f is ergodic with respect to m.
2. There exists a unique observable measure μ for f.
3. There exists a unique SRB measure ν for f and it attracts Lebesgue a.e. point $x \in M$.

Moreover, if the assertions above are satisfied, then $m = \mu = \nu$.

The ergodicity of many maps that preserve the Lebesgue measure is also an open question ([PS04], [B-W03]). Due to Remark 1.8 this property is equivalent to *unique observability.*

2. The convex-like property of $p\omega(x)$. For each $x \in M$ we have defined the nonempty compact set $p\omega(x) \subset \mathcal{P}_f$ composed of the limits of all convergent subsequences of the empirical probabilities in (1.1). For further uses we state the following property of $p\omega$-limit sets:

THEOREM 2.1 (Convex-like property). *For every point $x \in M$:*

1. *If $\mu, \nu \in p\omega(x)$ then for each real number $0 \leq \lambda \leq 1$ there exists a measure $\mu_\lambda \in p\omega(x)$ such that* $\operatorname{dist}(\mu_\lambda, \mu) = \lambda \operatorname{dist}(\nu, \mu)$.
2. *$p\omega(x)$ either has a single element or is uncountable.*

Proof. Statement 2 is an immediate consequence of 1. To prove 1 it is enough to exhibit, in the case $\mu \neq \nu$, a convergent subsequence of (1.1) whose limit μ_λ satisfies 1. It is an easy exercise to observe that the existence of such a convergent sequence follows (just take $\varepsilon = 1/n$) from Lemma 2.2 below. ∎

LEMMA 2.2. *For all $x \in M$ and $n \geq 1$ define $\mu_n = n^{-1} \sum_{j=0}^{n-1} \delta_{f^j(x)}$. Assume that there exist two weak* convergent subsequences $\mu_{m_j} \to \mu$, $\mu_{n_j} \to \nu$. Then for all $0 \leq \lambda \leq 1$, $\varepsilon > 0$ and $K > 0$ there exists a natural number $h = h(\varepsilon, K) > K$ such that $|\operatorname{dist}(\mu_h, \mu) - \lambda \operatorname{dist}(\nu, \mu)| \leq \varepsilon$.*

Proof. First let us choose m_j and then n_j such that

$$m_j > K; \quad \frac{1}{m_j} < \frac{\varepsilon}{4}; \quad \operatorname{dist}(\mu, \mu_{m_j}) < \frac{\varepsilon}{4}; \quad n_j > m_j; \quad \operatorname{dist}(\nu, \mu_{n_j}) < \frac{\varepsilon}{4}.$$

We will consider the following distance in $\mathcal{P}$:

$$\operatorname{dist}(\rho, \delta) = \sum_{i=1}^{\infty} \frac{1}{2^i} \Big| \int g_i \, d\rho - \int g_i \, d\delta \Big| \quad \text{for } \rho, \delta \in \mathcal{P},$$

where $\{g_i\}_{i \in \mathbb{N}}$ is a countable dense subset of $C^0(M, [0,1])$. Note from (1.1) that $|\int g \, d\mu_n - \int g \, d\mu_{n+1}| \leq (1/n)\|g\|$ for all $g \in C(M, [0,1])$ and $n \geq 1$. Then in particular for $n = m_j + k$, we obtain

$$\operatorname{dist}(\mu_{m_j+k}, \mu_{m_j+k+1}) \leq \frac{1}{m_j} < \frac{\varepsilon}{4} \quad \text{for all } k \geq 0. \tag{2.1}$$

Now let us choose a natural number $0 \leq k \leq n_j - m_j$ such that

$$|\operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) - \lambda \operatorname{dist}(\mu_{m_j}, \mu_{n_j})| < \varepsilon/4 \quad \text{for the given } \lambda \in [0,1].$$

Such a k exists because inequality (2.1) is satisfied for all $k \geq 0$ and moreover if $k = 0$ then $\operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) = 0$ and if $k = n_j - m_j$ then $\operatorname{dist}(\mu_{m_j}, \mu_{m_j+k}) = \operatorname{dist}(\mu_{m_j}, \mu_{n_j})$. Now renaming $h = m_j + k$, applying the triangular property and tying together the inequalities above, we deduce

$$|\mathrm{dist}(\mu_h,\mu)-\lambda\,\mathrm{dist}(\nu,\mu)|\le|\mathrm{dist}(\mu_h,\mu)-\mathrm{dist}(\mu_h,\mu_{m_j})| \\ +|\mathrm{dist}(\mu_h,\mu_{m_j})-\lambda\,\mathrm{dist}(\mu_{m_j},\mu_{n_j})|+\lambda|\mathrm{dist}(\mu_{m_j},\mu_{n_j})-\mathrm{dist}(\mu_{m_j},\nu)| \\ +\lambda|\mathrm{dist}(\mu_{m_j},\nu)-\mathrm{dist}(\mu,\nu)|<\varepsilon. \ \blacksquare$$

3. Proofs of Theorems 1.3 and 1.5. From the beginning we have a fixed metric in the space $\mathcal{P}$ of all Borel probability measures in M, inducing its weak* topology. We denote by $\mathcal{B}_\varepsilon(\mu)$ the open ball in $\mathcal{P}$, with that metric, centered at $\mu\in\mathcal{P}$ and with radius $\varepsilon>0$.

Proof of Theorem 1.3. Let us prove that $\mathcal{O}$ is compact. The complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ is the set of all probability measures μ (not necessarily f-invariant) such that for some $\varepsilon=\varepsilon(\mu)>0$ the set $\{x\in M : p\omega(x)\cap \mathcal{B}_\varepsilon(\mu)\neq\emptyset\}$ has zero Lebesgue measure. Therefore $\mathcal{O}^c$ is open in $\mathcal{P}$, and so $\mathcal{O}$ is closed. As $\mathcal{P}$ is compact we deduce that $\mathcal{O}$ is compact as desired.

We now prove that $\mathcal{O}$ is not empty. For contradiction, assume $\mathcal{O}^c=\mathcal{P}$. Then for every $\mu\in\mathcal{P}$ there exists some $\varepsilon=\varepsilon(\mu)>0$ such that the set $A=\{x\in M : p\omega(x)\subset(\mathcal{B}_\varepsilon(\mu))^c\}$ has total Lebesgue probability. As $\mathcal{P}$ is compact, let us consider a finite covering of $\mathcal{P}$ with such open balls $\mathcal{B}_\varepsilon(\mu)$, say $\mathcal{B}_1,\dots,\mathcal{B}_k$, and the respective sets $A_1,\dots,A_k$ defined as above. As $m(A_i)=1$ for all $i=1,\dots,k$, the intersection $B=\bigcap_{i=1}^k A_i$ is not empty. By construction, for all $x\in B$ the $p\omega$-limit set of x is contained in the complement of $\mathcal{B}_i$ for all $i=1,\dots,k$, and so it would not be contained in $\mathcal{P}$; that is the contradiction ending the proof. ∎

Proof of Theorem 1.5. Recall Definition 1.4 of the basin of attraction $A(\mathcal{K})$ of any weak* compact and nonempty set $\mathcal{K}$ of probabilities. We must prove the following two assertions:

1. $m(A(\mathcal{O}))=1$, where m is the Lebesgue measure.
2. $\mathcal{O}$ is minimal among all compact sets $\mathcal{K}\subset\mathcal{P}$ with this property.

Define the following family $\aleph$ of sets of probabilities:

$$\aleph=\{\mathcal{K}\subset\mathcal{P} : \mathcal{K}\text{ is compact and } m(A(\mathcal{K}))=1\}.$$

Thus $\aleph$ is composed of all weak* compact sets $\mathcal{K}$ of probabilities such that $p\omega(x)\subset\mathcal{K}$ for Lebesgue almost every point $x\in M$. The family $\aleph$ is not empty since it contains the set $\mathcal{P}_f$ of all invariant probabilities. So, to prove Theorem 1.5, we must prove that $\mathcal{O}\in\aleph$ and $\mathcal{O}=\bigcap_{\mathcal{K}\in\aleph}\mathcal{K}$.

Let us first prove that $\mathcal{O}\subset\mathcal{K}$ for all $\mathcal{K}\in\aleph$. This is equivalent to proving that if $\mathcal{K}\in\aleph$ and $\mu\notin\mathcal{K}$, then $\mu\notin\mathcal{O}$.

If $\mu\notin\mathcal{K}$ set $\varepsilon=\mathrm{dist}(\mu,\mathcal{K})>0$. For all $x\in A(\mathcal{K})$ the set $p\omega(x)\subset\mathcal{K}$ is disjoint from the ball $\mathcal{B}_\varepsilon(\mu)$. But Lebesgue almost all points are in $A(\mathcal{K})$, because $\mathcal{K}\in\aleph$. Therefore $p\omega(x)\cap\mathcal{B}_\varepsilon(\mu)=\emptyset$ Lebesgue a.e. This last assertion, combined with Definition 1.2 and the compactness of $p\omega(x)$, implies that $\mu \notin \mathcal{O}$, as desired.

Now let us prove that $m(A(\mathcal{O})) = 1$. By Theorem 1.3 the set $\mathcal{O}$ is compact and nonempty. So, $\mathrm{dist}(\mu, \mathcal{O}) > 0$ for any $\mu \notin \mathcal{O}$. Observe that the complement $\mathcal{O}^c$ of $\mathcal{O}$ in $\mathcal{P}$ can be written as the increasing union of compact sets $\mathcal{K}_n$ (not in $\aleph$), as follows:

$$\mathcal{O}^c = \bigcup_{n=1}^{\infty} \mathcal{K}_n, \qquad \mathcal{K}_n = \{\mu \in \mathcal{P} : \mathrm{dist}(\mu, \mathcal{O}) \geq 1/n\} \subset \mathcal{K}_{n+1}. \tag{3.1}$$

Let us consider the sequence $A'_n = A'(\mathcal{K}_n)$ of sets in M, where

$$A'(\mathcal{K}) := \{x \in M : p\omega(x) \cap \mathcal{K} \neq \emptyset\}. \tag{3.2}$$

Define $A'_\infty = A'(\mathcal{O}^c)$. We deduce from (3.1) and (3.2) that

$$A'_\infty = \bigcup_{n=1}^{\infty} A'_n, \qquad m(A'_n) \to m(A'_\infty) = m(A'(\mathcal{O}^c)).$$

To end the proof we must show that $m(A'_n) = 0$ for all $n \in \mathbb{N}$. In fact, $A'_n = A'(\mathcal{K}_n)$ and $\mathcal{K}_n$ is compact and contained in $\mathcal{O}^c$. By Definition 1.2 there exists a finite covering of $\mathcal{K}_n$ by open balls $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that

$$m(A'(\mathcal{B}_i)) = 0 \quad \text{for all } i = 1, \ldots, k. \tag{3.3}$$

By (3.2) the finite collection of sets $A'(\mathcal{B}_i)$ with $i = 1, \ldots, k$ covers A'_n. Therefore (3.3) implies $m(A'_n) = 0$, ending the proof. ∎

4. Proofs of Theorems 1.6 and 1.7

Lemma 4.1. *If an observable or SRB-like measure μ is isolated in the set $\mathcal{O}$ of observable measures, then it is an SRB measure.*

Proof. Recall that we denote by $\mathcal{B}_\varepsilon(\mu)$ the open ball in $\mathcal{P}$ centered at $\mu \in \mathcal{P}$ and with radius $\varepsilon > 0$. Since μ is isolated in $\mathcal{O}$, there exists $\varepsilon_0 > 0$ such that the set $\overline{\mathcal{B}_{\varepsilon_0}(\mu)} \setminus \{\mu\}$ is disjoint from $\mathcal{O}$. By Definition 1.2, $m(A) > 0$, where $A := A_{\varepsilon_0}(\mu) = \{x \in M : \mathrm{dist}(p\omega(x), \mu) < \varepsilon_0\}$.

By Definition 1.1, to prove that μ is SRB it is enough to prove that for m-almost all $x \in A$, the limit set $p\omega(x)$ of the sequence (1.1) of empirical probabilities is $\{\mu\}$. In fact, fix an arbitrary $0 < \varepsilon < \varepsilon_0$. The compact set $\overline{\mathcal{B}_{\varepsilon_0}(\mu)} \setminus \mathcal{B}_\varepsilon(\mu)$ is disjoint from $\mathcal{O}$, so it can be covered by a finite number of open balls $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that $m(A_i) = 0$ for all $i = 1, \ldots, k$, where $A_i := \{x \in M : p\omega(x) \cap \mathcal{B}_i \neq \emptyset\}$. Thus, for m-a.e. $x \in A$ the limit set $p\omega(x)$ intersects $\mathcal{B}_\varepsilon(\mu)$ but it does not intersect $\mathcal{B}_{\varepsilon_0}(\mu) \setminus \mathcal{B}_\varepsilon(\mu)$. From Theorem 2.1 we obtain $p\omega(x) \subset \mathcal{B}_\varepsilon(\mu)$ for Lebesgue almost all $x \in A$. Taking the values $\varepsilon_n = 1/n$, for all $n \geq 1$, we deduce that $p\omega(x) = \{\mu\}$ m-a.e. $x \in A$, as desired. ∎

Proof of Theorem 1.6. Denote by SRB the (a priori maybe empty) set of all SRB measures, according to Definition 1.1. By Definition 1.2, SRB $\subset \mathcal{O}$. If $\mathcal{O}$ is finite, then all its measures are isolated, and by Lemma 4.1, they are all SRB measures. Therefore SRB $= \mathcal{O}$ is finite. Applying Theorem 1.5 which states the full attraction property of $\mathcal{O}$, we obtain $m(A(\mathrm{SRB})) = 1$ where $A(\mathrm{SRB}) = \bigcup_{\mu \in \mathrm{SRB}} A(\mu)$, $A(\mu)$ being the basin of attraction of the SRB measure μ. Therefore, we deduce that if $\mathcal{O}$ is finite there exist finitely many SRB measures such that the union of their basins covers Lebesgue almost all points $x \in M$, as desired.

Now, let us prove the converse. Assume that SRB is finite and the union of the basins of attraction of all measures in SRB covers Lebesgue a.e. $x \in M$. By the minimality property of $\mathcal{O}$ stated in Theorem 1.5, $\mathcal{O} \subset$ SRB. On the other hand, we have SRB $\subset \mathcal{O}$. We conclude $\mathcal{O} =$ SRB, and thus $\mathcal{O}$ is finite, as desired. ∎

To prove Theorem 1.7 we need the following lemma (which in fact holds in any compact metric space $\mathcal{P}$).

LEMMA 4.2. *If the compact subset $\mathcal{O} \subset \mathcal{P}$ is countably infinite, then the subset $\mathcal{S}$ of its isolated points is nonempty, countably infinite and $\overline{\mathcal{S}} = \mathcal{O}$. Therefore,* $\mathrm{dist}(\nu, \mathcal{O}) = \mathrm{dist}(\nu, \mathcal{S})$ *for all $\nu \in \mathcal{P}$.*

Proof. The set $\mathcal{O} \subset \mathcal{P}$ is nonempty and compact by Theorem 1.3. Assume for contradiction that $\mathcal{S}$ is empty. Then $\mathcal{O}$ is perfect, i.e. every measure of $\mathcal{O}$ is an accumulation point. The set $\mathcal{P}$ of all Borel probabilities in M is a Polish space, since it is metric and compact. As nonempty perfect sets in a Polish space always have the cardinality of the continuum [K95], we deduce that $\mathcal{O}$ cannot be countably infinite, contradicting the hypothesis.

Even more, the argument above also shows that if $\mathcal{O}$ is countably infinite, then it does not contain nonempty perfect subsets.

Let us prove now that the subset $\mathcal{S}$ of isolated measures of $\mathcal{O}$ is countably infinite. Assume for contradiction that $\mathcal{S}$ is finite. Then $\mathcal{O} \setminus \mathcal{S}$ is nonempty and compact, and by construction it has no isolated points. Therefore it is a nonempty perfect set, contradicting the above.

It is left to prove that $\mathrm{dist}(\nu, \mathcal{O}) = \mathrm{dist}(\nu, \mathcal{S})$ for all $\nu \in \mathcal{P}$. This assertion, if proved, implies in particular that $\mathrm{dist}(\mu, \mathcal{S}) = 0$ for all $\mu \in \mathcal{O}$, and therefore, recalling that $\mathcal{O}$ is compact, $\overline{\mathcal{S}} = \mathcal{O}$.

To prove that $\mathrm{dist}(\nu, \mathcal{O}) = \mathrm{dist}(\nu, \mathcal{S})$ for all $\nu \in \mathcal{P}$, first fix ν and take $\mu \in \mathcal{O}$ such that $\mathrm{dist}(\nu, \mathcal{O}) = \mathrm{dist}(\nu, \mu)$. Such a probability μ exists because $\mathcal{O}$ is compact. If $\mu \in \mathcal{S}$, then the asserted equality is trivial. If $\mu \in \mathcal{O} \setminus \mathcal{S}$, fix any $\varepsilon > 0$ and take $\mu' \in \mathcal{S} \cap \mathcal{B}_\varepsilon(\mu)$. Such a μ' exists because, if not, the nonempty set $\mathcal{B}_\varepsilon(\mu) \cap \mathcal{O}$ would be perfect, contradicting the above. Therefore, $\mathrm{dist}(\nu, \mathcal{S}) \le \mathrm{dist}(\nu, \mu') \le \mathrm{dist}(\nu, \mu) + \mathrm{dist}(\mu, \mu') = \mathrm{dist}(\nu, \mathcal{O}) + \mathrm{dist}(\mu, \mu')$.

So, $\operatorname{dist}(\nu, \mathcal{S}) < \operatorname{dist}(\nu, \mathcal{O}) + \varepsilon$. As this holds for all $\varepsilon > 0$, we conclude that $\operatorname{dist}(\nu, \mathcal{S}) \leq \operatorname{dist}(\nu, \mathcal{O})$. The opposite inequality is immediate, since $\mathcal{S} \subset \mathcal{O}$. ■

Proof of Theorem 1.7. Denote by $\mathcal{S}$ the set of isolated measures in $\mathcal{O}$. By Lemma 4.2, $\mathcal{S}$ is countably infinite. Thus, applying Lemma 4.1, μ is SRB for all $\mu \in \mathcal{S}$. Then there exist countably infinitely many SRB measures (those in $\mathcal{S}$ and possibly some others in $\mathcal{O} \setminus \mathcal{S}$). Denote by SRB the set of all SRB measures. By Lemma 4.2, $\mathcal{O} = \overline{\mathcal{S}} \subset \overline{\mathrm{SRB}} \subset \mathcal{O}$. So $\overline{\mathrm{SRB}} = \mathcal{O}$. It is only left to prove that the union of the basins of attractions $A(\mu_i)$ for all $\mu_i \in \mathrm{SRB}$ covers Lebesgue almost all points of M. Denote by m the Lebesgue measure. Theorem 1.5 yields $p\omega(x) \subset \mathcal{O}$ m-a.e. $x \in M$. Together with Theorem 2.1 and the hypothesis of countability of $\mathcal{O}$, this implies that for m-a.e. $x \in M$ the set $p\omega(x)$ has a unique element $\{\mu_x\} \subset \mathcal{O}$. Thus,

$$p\omega(x) = \{\mu_x\} \subset \mathcal{O} \quad m\text{-a.e. } x \in M. \tag{4.1}$$

We write $\mathcal{O} = \{\mu_i : i = 1, \ldots, n\}$, where $\mu_i \neq \mu_j$ if $i \neq j$. Define $A = \bigcup_{i \in \mathbb{N}} A(\mu_i)$, where $A(\mu_i) := \{x \in M : \mu_x = \mu_i\}$. Assertion (4.1) can be written as $m(A) = 1$. In addition, $A(\mu_i) \cap A(\mu_j) = \emptyset$ if $\mu_i \neq \mu_j$. So $1 = \sum_{i=1}^{\infty} m(A(\mu_i))$. By Definition 1.1, $\mathrm{SRB} = \{\mu_i \in \mathcal{O} : m(A(\mu_i)) > 0\}$. We conclude that $\sum_{\mu_i \in \mathrm{SRB}} m(A(\mu_i)) = \sum_{i=1}^{\infty} m(A(\mu_i)) = 1$, as desired. ■

5. Examples

Example 5.1. For every transitive $C^{1+\alpha}$ Anosov diffeomorphism the unique SRB measure μ is the unique observable measure. But there are also infinitely many other ergodic and nonergodic invariant probabilities that are not observable (for instance those supported on periodic orbits).

Example 5.2. In [HY95] Hu and Young study a class of diffeomorphisms f of the two-torus obtained from an Anosov diffeomorphism by weakening the unstable eigenvalue of df at a fixed point x_0 to become equal to 1. The stable eigenvalue remains strictly smaller than 1, and the uniform hyperbolicity outside a neighborhood of x_0 is preserved. The authors prove that f has a single SRB measure, which is the Dirac delta δ_{x_0} supported at x_0, and that its basin has total Lebesgue measure. Therefore, δ_{x_0} is the single observable measure for f, it is ergodic and there are infinitely many other ergodic and nonergodic invariant measures that are not observable.

Example 5.3. The diffeomorphism $f\colon [0,1]^2 \to [0,1]^2$, $f(x,y) = (x/2, y)$, has as $\mathcal{O}$ the set of Dirac delta measures $\delta_{(0,y)}$ for all $y \in [0,1]$. In this case $\mathcal{O}$ coincides with the set of all ergodic invariant measures for f. Note that, for instance, the one-dimensional Lebesgue measure on the interval $[0] \times [0,1]$ is invariant and not observable, and that there are no SRB measures as defined in 1.1. This example also shows that the set $\mathcal{O}$ of observable measures is not necessarily closed under convex combinations.

Example 5.4. Maps with infinitely many simultaneous hyperbolic sinks $\{x_i\}_{i\in\mathbb{N}}$ (constructed from Newhouse's theorem [N74]) have a space $\mathcal{O}$ of observable measures which contains δ_{x_i} for all $i \in \mathbb{N}$. Moreover, all of them are physical measures and isolated in $\mathcal{O}$. Also maps with infinitely many Hénon-like attractors (constructed by Colli in [C98]) have a space of observable measures that contains countably infinitely many isolated probabilities that are SRB measures supported on Hénon-like attractors.

Example 5.5. The following example (attributed to Bowen [T94, GK07] and earlier cited in [T82]) shows that even if the system is C^2 regular, the space of observable measures may be formed by the limit set of the non-convergent sequence (1.1) for Lebesgue almost all initial states. Consider a diffeomorphism f of a ball in $\mathbb{R}^2$ with two hyperbolic saddle points A and B such that a half-branch of the unstable global manifold $W^u_{\text{half}}(A) \setminus \{A\}$ is an embedded arc that coincides with a half-branch of the stable global manifold $W^s_{\text{half}}(B) \setminus \{B\}$, and conversely $W^u_{\text{half}}(B) \setminus \{B\} = W^s_{\text{half}}(A) \setminus \{A\}$. Take f such that there exists a source $C \in U$ where U is the topological open ball with boundary $W^u_{\text{half}}(A) \cup W^u_{\text{half}}(B)$. One can design f such that for all $x \in U$ the α-limit set is $\{C\}$ and the ω-limit set contains $\{A, B\}$. If the eigenvalues of the derivative of f at A and B are adequately chosen (as specified in [T94, GK07]), then the empirical sequence (1.1) for any $x \in U \setminus \{C\}$ is not convergent. It has at least two different subsequences that converge to different convex combinations of the Dirac deltas δ_A and δ_B. Applying Theorem 1.7 there exist uncountably many observable measures. In addition, as observable measures are invariant under f, due to the Poincaré Recurrence Theorem they are all supported on $\{A\} \cup \{B\}$. So, by Theorem 2.1 all the observable measures are convex combinations of δ_A and δ_B and form a segment in the space $\mathcal{M}$ of probabilities. This example shows that observable measures are not necessarily ergodic.

Finally, the eigenvalues of df at the saddles A and B can be suitably modified to obtain, instead of the result above, the convergence of the sequence (1.1) as stated in Lemma (i) on page 457 of [T82]. In fact, taking conservative saddles (and C^0 perturbing f outside small neighborhoods of the saddles A and B so the topological ω-limit set of the orbits in $U \setminus \{C\}$ still contains A and B), one can make the sequences (1.1) converge to a single measure $\mu = \lambda\delta_A + (1-\lambda)\delta_B$ (with a fixed constant $0 < \lambda < 1$) for all $x \in U \setminus \{C\}$. So μ is physical according to Definition 1.1, and moreover it is the unique observable measure. This proves that physical measures are not necessarily ergodic.

Example 5.6. Consider a partially hyperbolic C^2 diffeomorphism f as defined in Section 11.2 of [BDV05]. In this family of examples, we will assume that for all $x \in M$ there exists a df-invariant dominated splitting $TM =$

$E^u \oplus E^{cs}$, where the subbundle E^u is uniformly expanding, has positive constant dimension, and the expanding exponential rate of $df|_{E^u}$ dominates that of $df|_{E^{cs}}$. Through every $x \in M$ there exists a unique C^2 injectively immersed unstable manifold $F^u(x)$ tangent to E^u.

We provide below a concrete example for which SRB measures according to Definition 1.1 do not exist. Nevertheless, in Subsection 11.2.3 of [BDV05] the authors prove that f does have probability measures μ that are Gibbs u-states. Namely, such a μ has conditional measures μ_x with respect to the unstable foliation $\mathcal{F}^u$ that are absolutely continuous with respect to the internal Lebesgue measures m^u_x along the leaves $\mathcal{F}^u_x$. Precisely, Theorem 11.16 of [BDV05] states that for all x in a set $E \subset M$ of initial states such that $m^u_y(\mathcal{F}^u_y \setminus E) = 0$ for all $y \in M$, convergent subsequences of the empirical probabilities (1.1) converge to Gibbs u-states (depending, a priori, on the point $x \in E$).

We provide below an example for which the set E has full Lebesgue measure in the ambient manifold M. Therefore, in this example Theorem 11.16 of [BDV05] implies that for Lebesgue almost all $x \in M$, the limit set $p\omega(x)$ of the sequence (1.1) is contained in the set of Gibbs u-states. Combining this result with Theorem 1.5 of this paper, we deduce that all the observable or SRB-like measures are Gibbs u-states in this example. Nevertheless, not all Gibbs u-states are necessarily observable, since the Gibbs u-states form a convex set but $\mathcal{O}$ is not necessarily convex. Moreover, by Theorems 1.6 and 1.7, and since in the example below there does not exist any SRB measure, the set $\mathcal{O}$ (and thus also the set of Gibbs u-states) is uncountable. Moreover, in the example below this fact holds simultaneously with the property that the sequence (1.1) of empirical probabilities converges for Lebesgue almost all initial states. This property, and the statement that the observable measures are Gibbs u-states, are two remarkable differences between Example 5.6 and Example 5.5. For both, no SRB measure exists and the set $\mathcal{O}$ is uncountable.

Let us consider the following (trivial but illustrative) example of partially hyperbolic system: Denote by $f : \mathbb{T}^3 \to \mathbb{T}^3$ the C^2 map on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{S}^1)^3$, defined by $f(x, y, z) = (x, g(x, y))$, where $g : \mathbb{T}^2 \to \mathbb{T}^2$ is a transitive C^2 Anosov diffeomorphism. By the Sinai Theorem there exists a g-ergodic SRB measure μ_1 on the two-torus, which is a Gibbs u-state for g. Thus, for Lebesgue almost all initial states $(x, y, z) \in \mathbb{T}^3$, the sequence (1.1) of the empirical probabilities converges to a measure $\mu_x = \delta_x \times \mu_1$, which is supported on a 1-dimensional unstable manifold injectively immersed in the two-torus $\{x\} \times \mathbb{T}^2$. For different values of $x \in \mathbb{S}^1$ the measures μ_x are mutually singular, since they are supported on disjoint compact two-tori embedded on $\mathbb{T}^3$. For each measure μ_x in $\mathbb{T}^3$, the basin of attraction $A(\mu_x)$ (as defined in 1.1) has zero Lebesgue measure in the ambient manifold $\mathbb{T}^3$. So, none of the probabilities μ_x is SRB for f. Nevertheless, by Theorem 1.5, the set of all those measures μ_x (which is easily checked to be weak* compact) coincides with the set $\mathcal{O}$ of observable SRB-like measures for f. By the construction of this concrete example, any $\mu \in \mathcal{O}$ is a Gibbs u-state. Moreover, any $\mu \in \mathcal{O}$ is ergodic. Since there exist many observable probabilities and since every convex combination of Gibbs u-states is also a Gibbs u-state, we conclude that there exist Gibbs u-states that are not observable.

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