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FUNCTIONAL ANALYSIS

Best Constants for the Inequalities between Equivalent Norms in Orlicz Spaces

by

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*Dedicated to academician S. M. Nikol'skii
on the occasion of his 105th birthday*

Summary. We investigate best constants for inequalities between the Orlicz norm and Luxemburg norm in Orlicz spaces.

1. Introduction. Let $G = \mathbb{R}$ or $\mathbb{R}^+$, and $\Phi : [0,\infty) \to [0,\infty)$ be an arbitrary Orlicz function (i.e., Φ is convex and vanishes only at zero). It is well-known that in Orlicz spaces $L_\Phi(G)$, the Orlicz norm $\|\cdot\|_{\Phi,G}$ and the Luxemburg norm $\|\cdot\|_{(\Phi,G)}$, to be defined below, are equivalent and satisfy

$$\|f\|_{(\Phi,G)} \le \|f\|_{\Phi,G} \le 2\|f\|_{(\Phi,G)} \quad \text{for all } f \in L_\Phi(G).$$

In this paper we investigate the best constants in these inequalities. Note that Lebesgue spaces and their extensions, Orlicz spaces, play an important role in analysis and have many applications (see [1–8]).

Denote by

$$\bar{\Phi}(t) = \sup_{s\ge 0}\{ts - \Phi(s)\}$$

the Young function conjugate to Φ, and $L_\Phi(G)$ be the Orlicz function space over the Lebesgue measure space (G, Σ, m), i.e., the space of all measurable functions u such that

$$|\langle u,v\rangle| = \Big|\int_G u(x)v(x)\,dx\Big| < \infty \quad \forall v : \rho(v,\bar{\Phi}) < \infty,$$

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where

$$\rho(v, \bar{\Phi}) = \int_G \bar{\Phi}(|v(x)|)\, dx.$$

Then $L_\Phi(G)$ is a Banach space with respect to the *Orlicz norm*

$$\|u\|_{\Phi,G} = \sup_{\rho(v,\bar{\Phi})\le 1} \Big| \int_G u(x)v(x)\, dx \Big|,$$

as well as the *Luxemburg norm*

$$\|f\|_{(\Phi,G)} = \inf\Big\{\lambda > 0 : \int_G \Phi(|f(x)|/\lambda)\, dx \le 1\Big\} < \infty.$$

Recall that $\|\cdot\|_{(\Phi,G)} = \|\cdot\|_{L_p(G)}$ where $\Phi(t) = t^p$ with $1 \le p < \infty$, and that an Orlicz function $\Phi : [0,\infty) \to [0,\infty)$ is called an *N-function* if $\lim_{t\to 0} \Phi(t)/t = 0$ and $\lim_{t\to\infty} \Phi(t)/t = \infty$.

We need the following results:

Theorem A ([7]). *Let Φ be an N-function. Then*

$$\|f\|_{\Phi,G} = \inf_{t>0} \frac{1}{t}\Big(1 + \int_G \Phi(t|f(x)|)\, dx\Big).$$

Young's inequality. *Let Φ be an N-function. Then*

$$xy \le \Phi(x) + \bar{\Phi}(y) \qquad \forall x, y \ge 0,$$

and equality holds iff $y \in [\psi(x), \eta(x)]$, where ψ, η are the left and right derivatives of Φ.

2. Main results. Suppose that C_1 is the largest number and C_2 the smallest number such that

$$C_1\|f\|_{(\Phi,G)} \le \|f\|_{\Phi,G} \le C_2\|f\|_{(\Phi,G)} \quad \text{for all } f \in L_\Phi(G).$$

Let Φ be an N-function. It is well known that the Orlicz norm has the Fatou property, that is, if $0 \le f_n \le f \in L_\Phi(G)$ then $\|f_n\|_{\Phi,G} \to \|f\|_{\Phi,G}$ whenever $f_n \to f$ a.e. Hence,

$$C_1 = \inf\{\|f\|_{\Phi,G} : f \in A\}, \qquad C_2 = \sup\{\|f\|_{\Phi,G} : f \in A\}, \tag{1}$$

where A is the set of all simple functions $f \in L_\Phi(G)$ satisfying $\|f\|_{(\Phi,G)} = 1$. So, $1 \le C_1 \le C_2 \le 2$. For $t \ge 0$ we define

$$H(t) = \sup_{x>0} \frac{\Phi(tx)}{\Phi(x)}, \qquad D(t) = \inf_{x>0} \frac{\Phi(tx)}{\Phi(x)}. \tag{2}$$

Clearly, the functions $D(t), H(t)$ are increasing, $D(t) \le H(t) \le t$ for any $0 \le t \le 1$ and $t \le D(t) \le H(t)$ for any $t > 1$. In this paper, we denote by f^{-1} the inverse function of f.

We have the following theorem.

THEOREM 1. *Let Φ be an N-function. Then*

$$C_1 = \inf_{t>0} \frac{1}{t} \bar{\Phi}^{-1}(t) \Phi^{-1}(t) = \inf_{t>0} \frac{1 + D(t)}{t}. \tag{3}$$

Proof. Since Φ is an N-function, $\Phi(x)$ is strictly increasing and $\bar{\Phi}^{-1}(x)$ is well defined. From (2) we have

$$D(t) = \inf_{x>0} \frac{\Phi(t\Phi^{-1}(x))}{x}. \tag{4}$$

Then it follows from Young's inequality that

$$\begin{aligned} \frac{1}{t}(1 + D(t)) &= \frac{1}{t}\left(1 + \inf_{x>0} \frac{\Phi(t\Phi^{-1}(x))}{x}\right) = \inf_{x>0} \frac{1}{t} \frac{\bar{\Phi}(\bar{\Phi}^{-1}(x)) + \Phi(t\Phi^{-1}(x))}{x} \\ &\ge \inf_{x>0} \frac{\bar{\Phi}^{-1}(x) t\Phi^{-1}(x)}{tx} = \inf_{x>0} \frac{1}{x} \Phi^{-1}(x) \bar{\Phi}^{-1}(x). \end{aligned}$$

Therefore,

$$\inf_{t>0} \frac{1}{t}(1 + D(t)) \ge \inf_{x>0} \frac{1}{x} \Phi^{-1}(x) \bar{\Phi}^{-1}(x). \tag{5}$$

For each $x > 0$, we choose $t > 0$ satisfying $t\Phi^{-1}(x) = \varphi(\bar{\Phi}^{-1}(x))$, where φ is the left derivative of $\bar{\Phi}$. Then, from (4) and Young's equality, we obtain

$$1 + D(t) \le 1 + \frac{\Phi(t\Phi^{-1}(x))}{x} = \frac{\Phi(t\Phi^{-1}(x)) + \bar{\Phi}(\bar{\Phi}^{-1}(x))}{x} = \frac{t\Phi^{-1}(x)\bar{\Phi}^{-1}(x)}{x}.$$

Hence,

$$\inf_{t>0} \frac{1}{t}(1 + D(t)) \le \inf_{x>0} \frac{1}{x} \Phi^{-1}(x) \bar{\Phi}^{-1}(x). \tag{6}$$

From (5) and (6), we have

$$\inf_{t>0} \frac{1}{t}(1 + D(t)) = \inf_{x>0} \frac{1}{x} \Phi^{-1}(x) \bar{\Phi}^{-1}(x). \tag{7}$$

It is known that if $f \in L_\Phi(G)$ is a simple function and $\|f\|_{(\Phi,G)} = 1$ then

$$\int_G \Phi(|f(x)|)\,dx = 1.$$

Therefore, it follows from Theorem A that

$$\begin{aligned} \|f\|_{\Phi,G} &= \inf\left\{\frac{1}{t}\left(1 + \int_G \Phi(t|f(x)|)\,dx\right) : t > 0\right\} \\ &\ge \inf\left\{\frac{1}{t}\left(1 + D(t)\int_G \Phi(|f(x)|)\,dx\right) : t > 0\right\} = \inf_{t>0} \frac{1 + D(t)}{t}, \end{aligned}$$

which together with (1) implies

$$C_1 \ge \inf_{t>0} \frac{1 + D(t)}{t}. \tag{8}$$

For each $t > 0$, we define $h(x) = \chi_{(0,1/t)}(x)$. Clearly, $\|h\|_{(\Phi,G)} = 1/\Phi^{-1}(t)$ and it follows from Young's equality and Theorem A that

$$\|h\|_{\Phi,G} = \inf_{k>0} \frac{1}{k}\left(1 + \frac{1}{t}\Phi(k)\right) \le \frac{1}{\varphi(\bar{\Phi}^{-1}(t))}\left(1 + \frac{1}{t}\Phi(\varphi(\bar{\Phi}^{-1}(t)))\right)$$
$$= \frac{1}{t}\,\frac{t + \Phi(\varphi(\bar{\Phi}^{-1}(t)))}{\varphi(\bar{\Phi}^{-1}(t))} = \frac{1}{t}\,\frac{\bar{\Phi}(\bar{\Phi}^{-1}(t)) + \Phi(\varphi(\bar{\Phi}^{-1}(t)))}{\varphi(\bar{\Phi}^{-1}(t))} = \frac{1}{t}\bar{\Phi}^{-1}(t).$$

Hence,

$$C_1 \le \frac{1}{t}\bar{\Phi}^{-1}(t)\Phi^{-1}(t) \quad \forall t > 0,$$

which implies

$$C_1 \le \inf_{t>0} \frac{1}{t}\Phi^{-1}(t)\bar{\Phi}^{-1}(t). \tag{9}$$

Combining (7)–(9), we obtain (3). The proof is complete. ∎

THEOREM 2. *Let Φ be an N-function. Then*

$$C_2 \le \inf_{t>0} \frac{1 + H(t)}{t} \tag{10}$$

and

$$\sup_{t>0} \frac{1}{t}\bar{\Phi}^{-1}(t)\Phi^{-1}(t) \le C_2. \tag{11}$$

Proof. Let $f \in L_\Phi(G)$ be a simple function satisfying $\|f\|_{(\Phi,G)} = 1$. Then

$$\int_G \Phi(|f(x))|)\,dx = 1.$$

Therefore, it follows from Theorem A that

$$\|f\|_{\Phi,G} \le \frac{1}{t}\Big(1 + \int_G \Phi(t|f(x)|)\,dx\Big)$$
$$\le \frac{1}{t}\Big(1 + H(t)\int_G \Phi(|f(x)|)\,dx\Big) = \frac{1 + H(t)}{t} \quad \forall t > 0.$$

So, by (1) we obtain

$$C_2 \le \inf_{t>0} \frac{1 + H(t)}{t}.$$

For each $t > 0$, we put $h(x) = \chi_{(0,1/t)}(x)$. Then, clearly,

$$\|h\|_{(\Phi,G)} = \frac{1}{\Phi^{-1}(t)} \quad \text{and} \quad \|h\|_{\Phi,G} = \inf_{k>0} \frac{1}{k}\left(1 + \frac{1}{t}\Phi(k)\right) = \frac{1}{t}\bar{\Phi}^{-1}(t).$$

Hence,

$$C_2 \ge \frac{\|h\|_{\Phi,G}}{\|h\|_{(\Phi,G)}} = \frac{1}{t}\bar{\Phi}^{-1}(t)\Phi^{-1}(t) \quad \forall t > 0,$$

which gives

$$C_2 \geq \sup_{t>0} \frac{1}{t} \Phi^{-1}(t) \bar{\Phi}^{-1}(t).$$

The proof is complete. ■

Recall that an Orlicz function Φ satisfies the Δ_2-*condition* (we write $\Phi \in \Delta_2$) if there exists $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$ for all $t > 0$, and Φ satisfies the ∇_2-*condition* (we write $\Phi \in \nabla_2$) if there exists a number $l > 1$ such that $\Phi(x) \leq \frac{1}{2l}\Phi(lx)$ for all $x \geq 0$.

THEOREM B ([7]). *Let Φ be an N-function. Then the following conditions are equivalent:*

(i) $\Phi \in \nabla_2$.
(ii) *There exists $\beta > 1$ such that $x\psi(x) > \beta\Phi(x)$ for all $x > 0$, where $\psi(x)$ is the left derivative of Φ.*
(iii) *There exist $l > 1$ and $\delta_l > 0$ such that $\Phi(lx) \geq (l + \delta_l)\Phi(x)$ for all $x > 0$.*

Now we find conditions so that $C_1 = 1$ or $C_2 = 2$:

THEOREM 3. *Let Φ be an N-function. Then $C_1 > 1$ if and only if $\Phi \in \Delta_2 \cap \nabla_2$.*

Proof. Necessity. Assume $C_1 > 1$. We have to prove that $\Phi \in \Delta_2 \cap \nabla_2$. Indeed, assume the contrary, that is, $\Phi \notin \Delta_2 \cap \nabla_2$. Then $\Phi \notin \Delta_2$ or $\Phi \notin \nabla_2$. From Theorem 1, we have

$$C_1 = \inf_{t>0} \frac{1 + D(t)}{t}.$$

If $\Phi \notin \Delta_2$, there exists a sequence $\{x_n\}$ of positive numbers such that $\Phi(x_n) \geq n\Phi(x_n/2)$ for all $n \in \mathbb{N}$. Fix $t \in (0, 1)$ and choose $n_0 \in \mathbb{N}$ such that $1/2 \geq t^{n_0}$. Then for all $n > n_0$ we have $\Phi(x_n) \geq n\Phi(x_n/2) \geq n\Phi(t^{n_0}x_n)$. Then it follows from $\Phi(t^{n_0}x_n) \geq (D(t))^{n_0}\Phi(x_n)$ that $1 \geq n(D(t))^{n_0}$ for all $n > n_0$, and so $D(t) = 0$ for all $t \in (0, 1)$. Hence,

$$C_1 \leq \inf_{t\in(0,1)} \frac{1 + D(t)}{t} = \inf_{t\in(0,1)} \frac{1}{t} = 1.$$

Therefore, it follows from $C_1 \geq 1$ that $C_1 = 1$.

If $\Phi \notin \nabla_2$, it follows from Theorem B that for any $t > 1$ and $\delta > 0$ there exists $x > 0$ such that

$$\Phi(tx) < (t + \delta)\Phi(x).$$

Therefore,

$$D(t) = \inf_{x>0} \frac{\Phi(tx)}{\Phi(x)} \leq t + \delta.$$

Letting $\delta \to 0$, we obtain $D(t) = t$ for all $t > 1$. So we have

$$C_1 \le \inf_{t>1} \frac{1 + D(t)}{t} = \inf_{t>1} \frac{1 + t}{t} = 1.$$

From this inequality and since $C_1 \ge 1$, we get $C_1 = 1$, which contradicts $C_1 > 1$. So, $\Phi \in \Delta_2 \cap \nabla_2$ has been proved.

Sufficiency. Assume $\Phi \in \Delta_2 \cap \nabla_2$; we have to show $C_1 > 1$. Indeed, since $\Phi \in \Delta_2$, $D(1/2) > 0$. Since $\Phi \in \nabla_2$, there exists $\beta > 1$ such that

$$\frac{x\psi(x)}{\Phi(x)} > \beta \quad \forall x > 0,$$

where ψ is the left derivative of Φ (see (ii) in Theorem B). Therefore, for all $t > 1$ we have

$$\ln \frac{\Phi(tx)}{\Phi(x)} = \int_x^{tx} \frac{\psi(y)}{\Phi(y)}\, dy \ge \int_x^{tx} \frac{\beta}{y}\, dy = \beta \ln t \quad \forall x > 0.$$

This implies $D(t) \ge t^\beta$. Hence,

$$\inf_{t\ge 1} \frac{1 + D(t)}{t} \ge \inf_{t>1} \frac{1 + t^\beta}{t} > 1.$$

Then it follows from

$$\inf_{1>t\ge 1/2} \frac{1 + D(t)}{t} \ge \inf_{1>t\ge 1/2} (1 + D(t)) \ge 1 + D(1/2) > 1$$

and

$$\inf_{1/2\ge t>0} \frac{1 + D(t)}{t} \ge 2,$$

that

$$C_1 = \inf_{t>0} \frac{1 + D(t)}{t} > 1.$$

The proof is complete. ∎

THEOREM 4. *Let Φ be an N-function and suppose its left derivative ψ is continuous. Then $C_2 = 2$ if and only if*

$$\inf_{x>0} \frac{x\psi(x)}{\Phi(x)} \le 2 \le \sup_{x>0} \frac{x\psi(x)}{\Phi(x)}. \tag{12}$$

To prove Theorem 4, we need the following result:

LEMMA 5. *Let Φ be an N-function with continuous left derivative ψ, and*

$$H(t) := \sup_{x>0} \frac{\Phi(tx)}{\Phi(x)}, \quad a := \sup_{x>0} \frac{x\psi(x)}{\Phi(x)}, \quad b := \inf_{x>0} \frac{x\psi(x)}{\Phi(x)}.$$

Then H has the left derivative and the right derivative at 1 and $H'_+(1) = a$, $H'_-(1) = b$.

Proof. For $t > 1$ and $x > 0$ we have

$$\ln \frac{\Phi(tx)}{\Phi(x)} = \int_x^{tx} \frac{\psi(y)}{\Phi(y)}\, dy \le \int_x^{tx} \frac{a}{y}\, dy = a \ln t.$$

Thus $H(t) \le t^a$ and from $H(1) = 1$, we have

$$\limsup_{t\to 1^+} \frac{H(t) - H(1)}{t-1} \le \lim_{t \to 1^+} \frac{t^a - 1}{t-1} = a. \tag{13}$$

For each $c \in (0, a)$, there exist $x_0 > 0$ and $\delta > 0$ such that

$$\frac{x\psi(x)}{\Phi(x)} > c \quad \forall x \in (x_0, x_0 + \delta).$$

It is obvious that for any $t \in (1, 1 + \delta/x_0)$, we have $(x_0, tx_0) \subset (x_0, x_0 + \delta)$, and the last inequality gives

$$\ln \frac{\Phi(tx_0)}{\Phi(x_0)} = \int_{x_0}^{tx_0} \frac{\psi(y)}{\Phi(y)}\, dy \ge \int_{x_0}^{tx_0} \frac{c}{y}\, dy = c \ln t.$$

This implies

$$H(t) \ge \frac{\Phi(tx_0)}{\Phi(x_0)} \ge t^c.$$

Hence $H(1) = 1$ yields

$$\liminf_{t \to 1^+} \frac{H(t) - 1}{t-1} \ge \lim_{t\to 1^+} \frac{t^c - 1}{t-1} = c.$$

Letting $c \to a$ and using (13), we see that H has the right derivative at 1 and $H'_+(1) = a$. Next, we will prove that $H'_-(1) = b$. Indeed, for $t < 1$ we have

$$\ln \frac{\Phi(x)}{\Phi(tx)} = \int_{tx}^{x} \frac{\psi(y)}{\Phi(y)}\, dy \ge \int_{tx}^{x} \frac{b}{y}\, dy = -b \ln t = -\ln t^b \qquad \forall x > 0,$$

which gives $H(t) \le t^b$. Therefore,

$$\liminf_{t \to 1^-} \frac{1 - H(t)}{1-t} \ge \lim_{t \to 1^-} \frac{1 - t^b}{1-t} = b. \tag{14}$$

On the other hand, for each $d > b$, there exists $x_0 > 0$ satisfying

$$\frac{x_0 \psi(x_0)}{\Phi(x_0)} < d.$$

So, there exists $\delta > 0$ such that

$$\frac{x\psi(x)}{\Phi(x)} < d \quad \forall x \in (x_0 - \delta, x_0).$$

Since for $1-\delta/x_0<t<1$ we have $(tx_0,x_0)\subset(x_0-\delta,x_0)$, it follows that

$$\ln\frac{\Phi(x_0)}{\Phi(tx_0)}=\int_{tx_0}^{x_0}\frac{\psi(y)}{\Phi(y)}\,dy\le\int_{tx_0}^{x_0}\frac{d}{y}\,dy=-\ln t^d.$$

Consequently,

$$H(t)\ge\frac{\Phi(tx_0)}{\Phi(x_0)}\ge t^d\qquad\forall t\in(1-\delta/x_0,1).$$

Therefore,

$$\limsup_{t\to1^-}\frac{1-H(t)}{1-t}\le\lim_{t\to1^+}\frac{1-t^d}{1-t}=d.$$

Letting $d\to b$, we get

$$\limsup_{t\to1^-}\frac{1-H(t)}{1-t}\le b. \tag{15}$$

Combining (14) and (15) shows that H has the left derivative at 1, and $H'_-(1)=b$. The proof is complete. ■

Now we will prove Theorem 4:

Proof of Theorem 4. Necessity. Assume $C_2=2$; we have to prove (12). Indeed, put $g(t)=(1+H(t))/t$. Then $g(1)=2$ and using Theorem 2, we get $C_2\le\inf\{g(t):t>0\}$. So, $g(1)=\min\{g(t):t>0\}$. Since H has the left derivative and the right derivative at 1, so does g. Moreover, it follows from $g(t)\ge g(1)$ for all $t>0$ that $g'_+(1)\ge0\ge g'_-(1)$. Thus

$$H'_+(1)\ge2\ge H'_-(1).$$

From this, by using Lemma 5, we obtain (12).

Sufficiency. Assuming that (12) is true, we have to show that $C_2=2$. Indeed, for all $\epsilon\in(0,1)$, by (12) and the continuity of ψ and Φ, there exists $x_0>0$ such that

$$\frac{x_0\psi(x_0)}{\Phi(x_0)}\in(2-\epsilon,2+\epsilon).$$

We define

$$f(x)=x_0\chi_{(0,t)}(x),\qquad g(x)=\psi(x_0)\chi_{(0,t)}(x),$$

where t is chosen such that $t\Phi(x_0)=1-\epsilon$. Hence,

$$\int_G\Phi(|f(x)|)\,dx=1-\epsilon$$

and

$$\Big|\int_G f(x)g(x)\,dx\Big| = \int_0^t x_0\psi(x_0)\,dx$$
$$= \frac{x_0\psi(x_0)}{\Phi(x_0)}(t\Phi(x_0)) \in ((1-\epsilon)(2-\epsilon),(1-\epsilon)(2+\epsilon)).$$

Thus

$$2-3\epsilon \le \Big|\int_G f(x)g(x)\,dx\Big| = \int_0^t x_0\psi(x_0)\,dx = \frac{x_0\psi(x_0)}{\Phi(x_0)}(t\Phi(x_0)) \le 2-\epsilon.$$

Using Young's equality, we get

$$\int_G \Phi(|f(x)|)\,dx + \int_G \bar{\Phi}(|g(x)|)\,dx = \Big|\int_G f(x)g(x)\,dx\Big|,$$

which together with $\int_G \Phi(|f(x)|)\,dx = 1-\epsilon$ implies that

$$\int_G \bar{\Phi}(|g(x)|)\,dx \le 1.$$

So, we obtain

$$\|g\|_{\bar{\Phi},G} \le 1, \quad \|f\|_{(\Phi,G)} \le 1, \quad \text{and} \quad \Big|\int_G f(x)g(x)\,dx\Big| \ge 2-3\epsilon.$$

Hence,

$$C_2 \ge \frac{\|f\|_{\Phi,G}}{\|f\|_{(\Phi,G)}} \ge \|f\|_{\Phi,G} \ge \Big|\int_G f(x)g(x)\,dx\Big| \ge 2-3\epsilon.$$

Letting $\epsilon \to 0$, we get $C_2 \ge 2$ and so $C_2 = 2$. The proof is complete. ∎

REMARK 1. Theorems 1–4 still hold if G is an arbitrary measurable set in $\mathbb{R}^n$ satisfying $m(G)=\infty$, where m is the Lebesgue measure.

Indeed, let g be an arbitrary measurable function on G. Denote by g^* the non-increasing rearrangement of g:

$$g^*(x) = \inf\{\lambda > 0 : \mu_g(\lambda) \le x\},$$

with $x > 0$, where μ_g denotes the distribution function of g defined by $\mu_g(t) = \mu(\{x \in G : |g(x)| > t\})$ for $t \ge 0$. Then $\int_G |g(x)|\,dx = \int_{\mathbb{R}^+} g^*(x)\,dx$. So, if $f \in L_\Phi(G)$ then $f^* \in L_\Phi(\mathbb{R}^+)$ and $\|f\|_{\Phi,G} = \|f^*\|_{\Phi,\mathbb{R}^+}$, $\|f\|_{(\Phi,G)} = \|f^*\|_{(\Phi,\mathbb{R}^+)}$. Therefore,

$$C_1 \ge C_1', \quad C_2 \le C_2', \tag{16}$$

where C_1', C_2' are the best constants for the inequalities between the Orlicz norm and Luxemburg norm in $L_\Phi(\mathbb{R}^+)$. Moreover, for each $\epsilon > 0$, by (1), there exists a simple function $f = \sum_{i=1}^k x_i\chi_{A_i} \in L_\Phi(\mathbb{R}^+)$ with $A_i \cap A_j = \emptyset$

$(i \neq j)$ satisfying

$$\|f\|_{\Phi,\mathbb{R}^+} \leq (C_1' + \epsilon)\|f\|_{(\Phi,\mathbb{R}^+)}.$$

For $i = 1, \dots, k$ we choose $B_i \subset G$ satisfying $m(B_i) = m(A_i)$ and $B_i \cap B_j = \emptyset$ $(i \neq j)$, and put $g = \sum_{i=1}^{k} x_i \chi_{B_i}$. Then $g \in L_\Phi(G)$, $g^* = f^*$ and

$$\|g\|_{\Phi,G} = \|g^*\|_{\Phi,G} = \|f^*\|_{\Phi,\mathbb{R}^+} = \|f\|_{\Phi,\mathbb{R}^+}$$

and

$$\|g\|_{(\Phi,G)} = \|g^*\|_{(\Phi,G)} = \|f^*\|_{(\Phi,\mathbb{R}^+)} = \|f\|_{(\Phi,\mathbb{R}^+)}.$$

Therefore,

$$\|g\|_{\Phi,G} \leq (C_1' + \epsilon)\|g\|_{(\Phi,G)},$$

which gives $C_1 \leq C_1' + \epsilon$. Letting $\epsilon \to 0$ and using (16), we get $C_1 = C_1'$. Similarly, $C_2 = C_2'$. The proof is complete.

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