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Uniqueness of Cartesian Products of Compact Convex Sets

by

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Summary. Let X_i, $i \in I$, and Y_j, $j \in J$, be compact convex sets whose sets of extreme points are affinely independent and let φ be an affine homeomorphism of $\prod_{i\in I} X_i$ onto $\prod_{j\in J} Y_j$. We show that there exists a bijection $b\colon I \to J$ such that φ is the product of affine homeomorphisms of X_i onto $Y_{b(i)}$, $i \in I$.

1. Introduction. Throughout the paper, all topological spaces considered are assumed to be Hausdorff. By a compact convex set we mean a compact convex subset of a real locally convex space and we assume that all compact convex sets considered contain at least two points. If X is a compact convex set, we write $A(X)$ for the Banach space of all real-valued affine continuous functions on X endowed with the supremum norm and the pointwise order.

We recall that a compact convex set X is a *simplex* if the dual space $A(X)^*$ is a lattice (see [1, Chapter II, §3], [2, Section 2.7], [8, Chapter 6, §28], [10, Section 3], [15, Chapter 7, §20], [19, Chapter 10] or [20, Chapter 6, §23]). A *Bauer simplex* (also called a *regular simplex*) is a simplex whose set of extreme points is closed. A finite-dimensional compact convex set X is a simplex if and only if it is a usual n-simplex for some $n \in \mathbb{N}$; that is, $X = \operatorname{conv}\{e_0, \dots, e_n\}$ where the vectors $e_0, \dots, e_n$ are affinely independent.

Referring to the title of the paper, we note that Cartesian products of finitely many, respectively, countably many, finite-dimensional simplices appear in [13, Theorem II.5], [23, pp. 10 and 146] and [16, Theorem 1], re-

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spectively, [17, Lemma 1 and Theorem 1]. Moreover, Cartesian products of countably many more general Bauer simplices appear in [16, Theorem 5] and [18, Theorems 1 and 2].

Z. Lipecki posed in [17, p. 469] a problem which was solved by V. Losert (unpublished; see [17, p. 469, Postscript] and Remark 2.12 below) by proving the following result:

If X_i, $i \in I$, and Y_j, $j \in J$, are Bauer simplices and $\prod_{i\in I} X_i$ is affinely homeomorphic to $\prod_{j\in J} Y_j$, then there exists a bijection $b\colon I \to J$ such that X_i is affinely homeomorphic to $Y_{b(i)}$, $i \in I$.

The result, which is a consequence of Proposition 2.7 below and [20, Theorem 23.2.3], was then applied in [18, proof of Theorem 2].

The aim of this paper is to further improve this result by showing the following theorem due to Z. Lipecki and J. Spurný.

THEOREM 1.1. *Let X_i, $i \in I$, and Y_j, $j \in J$, be compact convex sets whose sets of extreme points are affinely independent and let φ be an affine homeomorphism of $\prod_{i\in I} X_i$ onto $\prod_{j\in J} Y_j$. Then there exist a bijection $b : I \to J$ and affine homeomorphisms $\varphi_i : X_i \to Y_{b(i)}$, $i \in I$, such that*

$$\varphi((x_i)_{i\in I}) = (\varphi_i(x_i))_{i\in I} \quad \textit{for } x_i \in X_i,\ i \in I.$$

The main ingredient of the proof of Theorem 1.1 is Proposition 2.7 which applies the key ideas of V. Losert's solution of Z. Lipecki's problem (see above).

Theorem 1.1 need not hold for general compact convex sets even if I is a singleton and J is a two-element set. This is seen by considering the unit square in the plane and the product of two unit intervals.

Theorem 1.1 together with this example suggest the following question.

QUESTION 1.2. *Can the independence assumption in Theorem* 1.1 *be relaxed by stipulating that, for all i, j, neither X_i nor Y_j be affinely homeomorphic to sets of the form $Z_1 \times Z_2$, where Z_1 and Z_2 are compact convex sets?*

For another question related to Theorem 1.1 see [18, Remark 2].

An analogous uniqueness problem concerning Cartesian products of metric spaces and homeomorphisms is first discussed in K. Borsuk [5]. For a result similar to our Theorem 1.1 in that case see R. Cauty [7].

2. Proof of Theorem 1.1. If E and E_i, $i \in I$, are sets and $j \in I$, we say that a function $f : \prod_{i\in I} E_i \to E$ *depends on the j-th axis* if there is no function $h : \prod_{i\in I\setminus\{j\}} E_i \to E$ such that $f((x_i)_{i\in I}) = h((x_i)_{i\in I\setminus\{j\}})$ for $(x_i)_{i\in I} \in \prod_{i\in I} E_i$.

Lemma 2.1. *Let E_1, E_2 be sets and let $g : E_1 \times E_2 \to \mathbb{R}$. Then the following assertions are equivalent:*

(i) *g depends on at most one axis,*
(ii) *$(g \vee \lambda)(x_1, x_2) + (g \vee \lambda)(y_1, y_2) = (g \vee \lambda)(x_1, y_2) + (g \vee \lambda)(y_1, x_2)$ for all $x_1, y_1 \in E_1$, $x_2, y_2 \in E_2$ and $\lambda \in \mathbb{R}$.*

We recall that the symbol $g \vee \lambda$ denotes the pointwise maximum of g and the constant function λ.

Proof. If g does not depend on the first or second axis, then it clearly satisfies the condition in (ii).

Conversely, assuming that (ii) holds, for any $x_1, y_1 \in E_1$ and $x_2, y_2 \in E_2$ we have

$$g(x_1, x_2) + g(y_1, y_2) = g(x_1, y_2) + g(y_1, x_2). \tag{1}$$

Using the assumption again with $\lambda = g(x_1, y_2) \vee g(y_1, x_2)$ we obtain

$$g(x_1, x_2) \vee g(y_1, y_2) \le g(x_1, y_2) \vee g(y_1, x_2).$$

By symmetry,

$$g(x_1, y_2) \vee g(y_1, x_2) \le g(x_1, x_2) \vee g(y_1, y_2).$$

Therefore, in view of (1), we have

$$\{g(x_1, x_2), g(y_1, y_2)\} = \{g(x_1, y_2), g(y_1, x_2)\}, \qquad x_1, y_1 \in E_1,\ x_2, y_2 \in E_2. \tag{2}$$

Suppose that g depends on the first axis. Then there exists $x_2 \in E_2$ such that $x \mapsto g(x, x_2)$ is not constant on E_1. We fix an arbitrary element $x_1 \in E_1$ and take $y_1 \in E_1$ with $g(x_1, x_2) \neq g(y_1, x_2)$. It follows from (2) that $g(x_1, x_2) = g(x_1, y_2)$ for all $y_2 \in E_2$. Thus g does not depend on the second axis. ∎

We shall need the following straightforward observation.

Lemma 2.2. *Let F be an affinely independent set in a linear space E over $\mathbb{R}$ and $g : F \to \mathbb{R}$. Then there exists an affine function $h : E \to \mathbb{R}$ such that $h = g$ on F.*

Proof. We fix $x_0 \in F$ and take a linear functional f on E such that $f(x - x_0) = g(x) - g(x_0)$ for each $x \in F$. By setting $h(x) = f(x - x_0) + g(x_0)$, $x \in E$, we get the desired extension. ∎

In connection with Remark 2.12, we note that if the set F in Lemma 2.2 is finite and E is a locally convex space, then, by the Hahn–Banach theorem, the function h can be chosen continuous.

Definition 2.3. For a compact convex set X we denote by $A_0(X)$ the set of all $f \in A(X)$ with the following property: given $\lambda \in \mathbb{R}$ and a finite subset $F \subset \operatorname{ext} X$, there exists an affine function $h : X \to \mathbb{R}$ such that $h = f \vee \lambda$ on F.

We denote by $\mathcal{V}(X)$ the family of all linear subspaces A of $A(X)$ with $A \subset A_0(X)$. We further denote by $\mathcal{V}_0(X)$ the family of those elements in $\mathcal{V}(X)$ which are maximal with respect to inclusion.

If E_1, E_2 are sets and $f_i : E_i \to \mathbb{R}$ are functions, $i = 1, 2$, we denote by $f_1 \otimes f_2 : E_1 \times E_2 \to \mathbb{R}$ the function

$$(x_1, x_2) \mapsto f_1(x_1) f_2(x_2), \qquad (x_1, x_2) \in E_1 \times E_2.$$

We write χ_E for the characteristic function of a subset E of a set F. If X_i, $i \in I$, are compact convex sets and $X = \prod_{i\in I} X_i$ is their Cartesian product, we understand each $A(X_j)$ as a subspace of $A(X)$; that is, we identify $A(X_j)$ with

$$\{f \otimes \chi_{\prod_{i\in I\setminus\{j\}} X_i} : f \in A(X_j)\}.$$

Proposition 2.4. *If X_1, X_2 are compact convex sets, then*

$$A_0(X_1 \times X_2) \subset A(X_1) \cup A(X_2).$$

Proof. Let $E_i := \operatorname{ext} X_i$, $i = 1, 2$. We claim that, for each $f \in A_0(X_1 \times X_2)$, the function $g := f|_{E_1 \times E_2}$ satisfies (ii) of Lemma 2.1.

Indeed, for fixed $x_i, y_i \in E_i$, $i = 1, 2$, and $\lambda \in \mathbb{R}$ we take an affine function $h : X_1 \times X_2 \to \mathbb{R}$ with

$$h(z) = g(z) \vee \lambda, \quad z \in \{(x_1, x_2), (y_1, y_2), (x_1, y_2), (y_1, x_2)\}.$$

Then

$$\frac{1}{2}(h(x_1, x_2) + h(y_1, y_2)) = \frac{1}{2}(h(x_1, y_2) + h(y_1, x_2)),$$

proving the claim.

It follows from Lemma 2.1 that g depends on at most one axis. Since f is affine and continuous, an application of the Krein–Milman theorem yields the assertion. ∎

Lemma 2.5. *Let E_i, $i \in I$, and F be topological spaces and let $f : \prod_{i\in I} E_i \to F$ be a continuous nonconstant function. Then there exists $j \in I$ such that f depends on the j-th axis.*

Proof. Assume that f does not depend on any axis $j \in I$. Given $x, x' \in \prod_{i\in I} E_i$ and a neighbourhood U of x', it is easy to deduce from the assumption that there exists $x'' \in U$ such that $f(x) = f(x'')$. Since U is arbitrary, $f(x) = f(x')$ and f is a constant function. ∎

Remark 2.6. We note that the continuity of f in Lemma 2.5 is essential. Indeed, it is enough to consider $E_n := \{0, 1\}$, $n \in \mathbb{N}$, and $f : \prod_{n\in\mathbb{N}} E_n \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 1, & x \text{ is eventually constant,} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.7. *If X_i, $i \in I$, are compact convex sets with* $\operatorname{ext} X_i$ *affinely independent, then*

$$\mathcal{V}_0\Big(\prod_{i\in I} X_i\Big) = \{A(X_i) : i \in I\}.$$

Proof. We set $X = \prod_{i\in I} X_i$. In view of Lemma 2.2, $A(X_i)$, $i \in I$, are all in $\mathcal{V}(X)$. Clearly, they are mutually incomparable with respect to inclusion. Thus it is enough to show that, given $A \in \mathcal{V}(X)$, we have $A \subset A(X_j)$ for some $j \in I$.

We may asssume that A contains a nonconstant function f. Lemma 2.5 provides an axis $j \in I$ such that f depends on j. We set $Y = \prod_{i\in I\setminus\{j\}} X_i$ and apply Proposition 2.4 to X_j and Y to get $f \in A(X_j)$. It follows that, in fact, every function $g \in A$ is in $A(X_j)$. Indeed, if $g \in A \setminus A(X_j)$, then $g \in A(Y)$ by Proposition 2.4, but

$$f + g \notin A(X_j) \cup A(Y),$$

which contradicts Proposition 2.4.

Hence $A \subset A(X_j)$, and the assertion follows. ∎

We recall the following well-known fact.

Lemma 2.8 (cf. [9, p. 119]). *Let X, Y be compact convex sets and let $T : A(X) \to A(Y)$ be a positive surjective isometry. Then there exists an affine homeomorphism $\varphi\colon Y \to X$ such that*

$$Tf = f \circ \varphi, \quad f \in A(X).$$

Proof. A compact convex set X and its state space

$$S_X = \{s \in A(X)^*\colon s(\chi_X) = \|s\| = 1\},$$

considered with the w^*-topology, can be identified via the evaluation mapping $\phi_X\colon X \to S_X$ given by $\phi_X(x)(h) = h(x)$, $h \in A(X)$ (see [20, Theorem 23.2.3]). Given compact convex sets X, Y and a positive surjective isometry $T\colon A(X) \to A(Y)$, the dual operator $T^*\colon A(Y)^* \to A(X)^*$ restricts to an affine homeomorphism of S_Y onto S_X. By the identification above, the mapping

$$\varphi = (\phi_X)^{-1} \circ T^* \circ \phi_Y$$

is the required affine homeomorphism. ∎

Remark 2.9. It is worth noting that, without the assumption of positivity, X and Y in Lemma 2.8 need not even be affinely homeomorphic; see Example in [9] attributed to J. T. Chan.

Proof of Theorem 1.1. Define $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$. Let $T : A(Y) \to A(X)$ be the positive surjective isometry given by the mapping φ, that is, $Tg = g \circ \varphi$, $g \in A(Y)$. Then T provides a bijection between

$\mathcal{V}_0(Y)$ and $\mathcal{V}_0(X)$. By Proposition 2.7, each element of $\mathcal{V}_0(X)$ or $\mathcal{V}_0(Y)$ can be uniquely identified with some $A(X_i)$ or $A(Y_j)$, respectively.

For $i \in I$, let $b(i) \in J$ be the unique index such that $T(A(Y_{b(i)})) = A(X_i)$. By the reasoning above, $b : I \to J$ is indeed a bijection and $T : A(Y_{b(i)}) \to A(X_i)$ is a positive surjective isometry. In view of Lemma 2.8, there exists an affine homeomorphism $\varphi_i : X_i \to Y_{b(i)}$ such that

$$(g \otimes \chi_{\prod_{j\in J\setminus\{b(i)\}} Y_j}) \circ \varphi = (g \circ \varphi_i) \otimes \chi_{\prod_{k\in I\setminus\{i\}} X_k}, \qquad g \in A(Y_{b(i)}).$$

Then, for all $g \in A(Y_{b(i)})$ and $x \in X$, we have

$$\begin{aligned} g(\varphi(x)_{b(i)}) &= (g \otimes \chi_{\prod_{j\in J\setminus\{b(i)\}} Y_j})(\varphi(x)) \\ &= ((g \circ \varphi_i) \otimes \chi_{\prod_{k\in I\setminus\{i\}} X_k})(x) = g(\varphi_i(x_i)). \end{aligned}$$

Hence

$$\varphi(x)_{b(i)} = \varphi_i(x_i), \qquad i \in I,\ x \in X,$$

and thus

$$\varphi((x_i)_{i\in I}) = (\varphi_i(x_i))_{i\in I}, \quad (x_i)_{i\in I} \in X.$$

This concludes the proof. ■

Remark 2.10. We note that the proof of Theorem 1.1 used only the fact that each four-point subset of $\operatorname{ext} X_i$, $i \in I$, is affinely independent.

Remark 2.11. By [1, Proposition II.3.19], there exists a compact convex set X with $\operatorname{ext} X$ affinely independent that is not a simplex. We note that such sets, with the set of extreme points closed and countable, can be constructed in the Hilbert space ℓ_2 (and even in an arbitrary infinite-dimensional Banach space by [11, Theorem]).

To construct such a set, let $\{e_n : n \in \mathbb{N}\}$ be the standard basis of ℓ_2 and let

$$X_0 = \left\{ \sum_{n=1}^{\infty} \frac{\lambda_n}{n} e_n \colon \lambda_n \geq 0,\ \sum_{n=1}^{\infty} \lambda_n \leq 1 \right\}.$$

We have $\operatorname{ext} X_0 = \{0\} \cup \{n^{-1}e_n \colon n \in \mathbb{N}\}$. Denote by N_1 and N_2 the sets of odd and even natural numbers, respectively, and let

$$x_i = \sum_{n\in N_i} \frac{2^{-n}}{n} e_n, \qquad i = 1, 2, \qquad x_0 = 2x_1 - x_2.$$

Then

$$X := \operatorname{conv}(X_0 \cup \{x_0\})$$

is a compact convex set with $\operatorname{ext} X = \operatorname{ext} X_0 \cup \{x_0\}$. Moreover, $x_1, x_2 \in X_0$, $x_0 \notin \operatorname{span}\{e_n \colon n \in \mathbb{N}\}$ and

$$\sum_{n\in N_1} \frac{2^{-n}}{n} e_n = x_1 = \frac{1}{2}(x_0 + x_2) = \frac{1}{2}\left(x_0 + \sum_{n\in N_2} \frac{2^{-n}}{n} e_n \right).$$

It follows that ext X is an affinely independent set but X is not a simplex. Indeed, the point x_1 can be expressed as a σ-convex combination of extreme points of X in two different ways (see [1, Theorem II.3.6]). Hence X has all the desired properties.

Remark 2.12. Recall that a compact convex set X is a Bauer simplex if and only if $A(X)$ is a lattice under pointwise order of functions ([20, Theorem 23.7.1]). By similar arguments to those above (and this was the original idea of the proof for the case of Bauer simplices), one can show the following: Assume that X_i, $i \in I$, are Bauer simplices. If A is a maximal linear sublattice of $A(\prod_{i\in I} X_i)$ containing the constant functions, then there exists $j \in J$ such that $A = A(X_j)$ (embedded as above). For the more general class of sets X_i appearing in Theorem 1.1, one can formulate a corresponding statement as follows: If F is a finite set of extreme points of X, consider its convex hull X_F and the corresponding projection $p_F : A(X) \to A(X_F)$ obtained by restriction of functions. Then, if X_i, $i \in I$, are compact convex sets whose sets of extreme points are affinely independent and $X = \prod_{i\in I} X_i$, the following holds. If $A \subseteq A(X)$ contains the constant functions and A is maximal for the property that $p_F(A)$ is a linear sublattice of $A(X_F)$ for every finite set F of extreme points of X, then there exists $j \in J$ such that $A = A(X_j)$ (in fact, the sets F with four elements are sufficient for this conclusion).

3. Three more questions. In the following, for compact convex sets X and Y we write $X \approx Y$ whenever X and Y are affinely homeomorphic.

Question 3.1. *Let S be a simplex and let X and Y be compact convex sets such that $S \times X \approx S \times Y$. Does it follow that $X \approx Y$?*

Question 3.2. *Which compact convex sets X are prime in the sense that there are no compact convex sets X_1 and X_2 (both containing at least two points) such that $X \approx X_1 \times X_2$?*

Question 3.3. *Do there exist compact convex sets X and Y such that $X \not\approx Y$ but $X \times X \approx Y \times Y$?*

Similar questions are classical in some other categories. For a few examples in connection with Question 3.1 see [3], [4], [6], [14] or [21].

As for Question 3.2, see [5, p. 137]. Finally, Question 3.3 is the problem of the "extraction of the square root", according to [22, Section III.3]; see also [21] and some of its reference items, and [12].

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