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FUNCTIONAL ANALYSIS

A Natural Class of Sequential Banach Spaces

by

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Summary. We introduce and study a natural class of variable exponent ℓ^p spaces, which generalizes the classical spaces ℓ^p and c_0. These spaces will typically not be rearrangement-invariant but instead they enjoy a good local control of some geometric properties. Some geometric examples are constructed by using these spaces.

1. Introduction. In this paper we will introduce a natural schema for producing geometrically complicated Banach spaces with a 1-unconditional basis. The idea is, however, very simple, almost to the extent of being naive. The resulting class of Banach spaces is still quite flexible, so that within it one can construct easily Banach spaces with various combinations of suitable geometric and isomorphic properties.

Consider a map $p\colon \mathbb{N} \to [1,\infty]$. We will study variable exponent sequential spaces $\ell^{p(\cdot)}$ given formally by

$$\ell^{p(\cdot)} = \cdots (\cdots (((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}) \oplus_{p(3)} \mathbb{R}) \oplus_{p(4)} \cdots$$

(This definition will be made rigorous shortly.) For example, for a constant function $p(\cdot) \equiv p \in [1,\infty]$ we have $\ell^{p(\cdot)} = \ell^p$ isometrically.

The variable exponent sequence spaces have been studied a great deal. For example, the class studied here appears to be related to modular spaces, whose norm is defined in a spirit similar to the definition of the Orlicz norm (see e.g. [5]). However, the class of $\ell^{p(\cdot)}$ spaces studied here does not coincide in the natural way, at least isometrically, with the modular spaces (see the discussion in Section 1.1). The author suspects that the class presented here has been previously overlooked because there exist attractive alternative

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definitions of variable exponent norms. The philosophy of Orlicz type norms is a bit different compared with the $\ell^{p(\cdot)}$ norm. One feature of Orlicz type norms is that they are often rearrangement-invariant in some sense, possibly weaker than usual, whereas the norm of $\ell^{p(\cdot)}$ spaces here is kind of *localized.* Another difference is that in Orlicz type spaces the norm is given by an infimum, whereas for finitely supported vectors in an $\ell^{p(\cdot)}$ space there is a simple formula for calculating the norm. Also, the norm of any vector can be approximated by using finitely supported vectors in the obvious way, so that the norm of $\ell^{p(\cdot)}$ becomes rather explicit.

By starting from an $\ell^{p(\cdot)}$ type space one can prove the following main result.

THEOREM 1.1. *There exists a Banach space with a* 1*-unconditional Schauder basis that contains all spaces* ℓ^p, $1 \leq p < \infty$, *isomorphically, in fact almost isometrically.*

The crux of this paper is that the sequential spaces studied are concrete. Recall the classical fact that ℓ^p and ℓ^q are not isomorphic when $p \neq q$. For this reason the above result is perhaps surprising, as the claimed space is separable but must contain continuum many (mutually non-isomorphic) ℓ^p spaces. Recall that $C([0,1])$ is a (concrete) separable space, which is universal for all separable spaces. However, it does not admit any unconditional basis (see [5, p. 24]). For a classical result on existence of a space universal for spaces with an unconditional basis, see Pełczyński's work [7].

We will also study the basic properties of $\ell^{p(\cdot)}$ spaces. These spaces have some nice properties analogous to the classical ℓ^p spaces. On the other hand, some unexpected open problems arise as well. For example, we do not know whether $[(e_n)] \subset \ell^{p(\cdot)}$, the closed linear span of the natural unit vectors, coincides with $c_0 \cap \ell^{p(\cdot)}$ in general. As regards the structural properties of $\ell^{p(\cdot)}$ spaces, it turns out for example that type and cotype become useful concepts and behave nicely in this setting.

1.1. Preliminaries. The references [1], [5] and [6] provide suitable background information on Banach spaces. We denote by X a real Banach space, by $\mathbf{B}_{\mathrm{X}}$ the closed unit ball and by $\mathbf{S}_{\mathrm{X}}$ the unit sphere. Given $t, s \in [0,\infty)$ and $p \in (0,\infty)$ we denote

$$t \boxplus_p s = (t^p + s^p)^{1/p} \quad \text{and} \quad t \boxplus_\infty s = \max(t,s).$$

Clearly $\boxplus_p$ gives a commutative semigroup for a fixed p.

FACT 1.2. *Let* $0 < p_0 \leq p_1 \leq \infty$ *and* $a, b, c \in [0,\infty)$. *Then*

$$(a \boxplus_{p_0} b) \boxplus_{p_1} c \leq a \boxplus_{p_0} (b \boxplus_{p_1} c).$$

Proof. The claim is clearly equivalent to

$$((a^{p_0} + b^{p_0})^{p_1/p_0} + c^{p_1})^{p_0/p_1} \leq a^{p_0} + (b^{p_1} + c^{p_1})^{p_0/p_1}, \tag{1.1}$$

where $p_0 \leq p_1 < \infty$, and which holds as an equality for $a = 0$. By substituting $u = a^{p_0}$, differentiating and using that $p_0/p_1 - 1 \leq 0$ we obtain

$$\begin{aligned}\frac{\partial}{\partial u}((u+b^{p_0})^{p_1/p_0}+c^{p_1})^{p_0/p_1} &= ((u+b^{p_0})^{p_1/p_0}+c_1^{p_1})^{p_0/p_1-1}(u+b^{p_0})^{p_1/p_0-1}\\ &\leq ((u+b^{p_0})^{p_1/p_0})^{p_0/p_1-1}(u+b^{p_0})^{p_1/p_0-1} = 1\\ &= \frac{\partial}{\partial u}(u+(b^{p_1}+c^{p_1})^{p_0/p_1})\end{aligned}$$

for $u \geq 0$, and this yields (1.1). ∎

We denote by $\mathrm{X} \oplus_p \mathrm{Y}$, $1 \leq p \leq \infty$, the direct sum of spaces X and Y with the norm $\|(x,y)\|_{\mathrm{X}\oplus_p\mathrm{Y}} = \|x\|_{\mathrm{X}} \boxplus_p \|y\|_{\mathrm{Y}}$ for $(x,y) \in \mathrm{X} \oplus \mathrm{Y}$. We will regard the real line $\mathbb{R}$ as a 1-dimensional Banach space and sometimes write $\mathbb{R} \oplus_p \mathbb{R}$ as ℓ_2^p.

Next we will give a precise definition for the variable-exponent ℓ^p spaces. Let $p\colon \mathbb{N} \to [1,\infty]$ be a map and $x \in \ell^\infty$. We define seminorms $\|\cdot\|_k$ on ℓ^∞ recursively by putting $\|x\|_{(1)} = |x_1| \boxplus_{p(1)} |x_2|$ and $\|x\|_{(k)} = \|x\|_{(k-1)} \boxplus_{p(k)} |x_{k+1}|$ for $k \in \mathbb{N}$, $k \geq 2$. Observe that $(\|x\|_{(k)})$ is a non-decreasing sequence for each $x \in \ell^\infty$. Hence we may put $\Phi\colon \ell^\infty \to [0,\infty]$, $\Phi((x_n)) = \lim_{k\to\infty} \|x\|_{(k)}$ for $x \in \ell^\infty$. Consider the vector space

$$\ell^{p(\cdot)} = \{(x_n) \in \ell^\infty : \Phi((x_n)) < \infty\},$$

which is equipped with the usual pointwise linear structure. It is easy to see that the mapping $\|\cdot\|_{\ell^{p(\cdot)}} \doteq \Phi|_{\ell^{p(\cdot)}}$ is a norm on $\ell^{p(\cdot)}$.

Let $e_1 = (1,0,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, … be the canonical unit vectors of c_0. We denote $P_k\colon \ell^{p(\cdot)} \to [e_1,\ldots,e_k]$, $(x_n) \mapsto (x_1,\ldots,x_k,0,0,0,\ldots)$. Observe that by the construction of the norm $\|\cdot\|_{\ell^{p(\cdot)}}$ we have $\|P_i(x)\|_{\ell^{p(\cdot)}} \leq \|P_j(x)\|_{\ell^{p(\cdot)}}$ for $i,j \in \mathbb{N}$, $i \leq j$, $x \in \ell^{p(\cdot)}$. Note that $\|x\|_{\ell^{p(\cdot)}} = \sup_{k\in\mathbb{N}} \|P_k(x)\|_{\ell^{p(\cdot)}}$ for $x \in \ell^{p(\cdot)}$ and that P_n is a norm-1 projection for $n \in \mathbb{N}$. We will denote $Q_n \doteq \mathbf{I} - P_n$. Two sequences $x,y \in \ell^\infty$ have *disjoint supports* if $x_n y_n = 0$ for all $n \in \mathbb{N}$.

We will denote the Banach–Mazur distance of isomorphic Banach spaces X and Y by

$$d_{\mathrm{BM}}(\mathrm{X},\mathrm{Y}) = \inf\{\|T\|\cdot\|T^{-1}\| : T\colon \mathrm{X} \to \mathrm{Y} \text{ is an isomorphism}\}.$$

The spaces X and Y are *almost isometric* if $d_{\mathrm{BM}}(\mathrm{X},\mathrm{Y}) = 1$. Recall that a Banach space X is *contained almost isometrically* in a Banach space Z if for each $\epsilon > 0$ there is a subspace $\mathrm{Y} \subset \mathrm{Z}$ such that $d_{\mathrm{BM}}(\mathrm{X},\mathrm{Y}) < 1+\epsilon$. Here we will often encounter sequential Banach spaces X and Y such that X contains Y almost isometrically in such a way that

$$\lim_{n\to\infty} d_{\mathrm{BM}}(Q_n(\mathrm{X}),\mathrm{Y}) = 1. \tag{1.2}$$

In contrast, the modular norm is defined as follows:

$$\|(x_n)\|_{(M_n)} = \inf\Big\{\lambda > 0 : \sum_{n=1}^{n} M_n(|x_n|/\lambda) \le 1\Big\},$$

where $M_n\colon [0,\infty) \to [0,\infty)$ are suitable continuous, strictly increasing functions with $M_n(0) = 0$ (see [5] for discussion). For example, in our setting it would be natural to define $M_n(t) = t^{p(n)}$ for $n \in \mathbb{N}$. The resulting construction is justifiably a kind of variable exponent ℓ^p space. Observe that if $\pi\colon \mathbb{N} \to \mathbb{N}$ is a permutation, then $\|(x_n)\|_{(M_n)} = \|(x_{\pi(n)})\|_{(M_{\pi(n)})}$ for each sequence (x_n) such that one of the norms is defined. The $\ell^{p(\cdot)}$ space does not share the above property that the norm should be invariant under the equal permutation of the exponents and the coordinates of the space. For example, $\|(1,1,1)\|_{(\mathbb{R}\oplus_1\mathbb{R})\oplus_2\mathbb{R}} = \sqrt{5} \approx 2.236$ and $\|(1,1,1)\|_{(\mathbb{R}\oplus_2\mathbb{R})\oplus_1\mathbb{R}} = \sqrt{2} + 1 \approx 2.414$.

Recall that a Banach space X is an *Asplund space* if any separable subspace of X has a separable dual. Given a locally convex topology τ on X, the space X is said to be τ *locally uniformly rotund* (τ-LUR for short) if the following holds: For each sequence $(x_n) \subset \mathbf{S}_{\mathrm{X}}$ such that $\|x_1 + x_n\| \to 2$ as $n \to \infty$ we have $x_n \xrightarrow{\tau} x_1$ as $n \to \infty$. If τ is the norm topology then we write LUR instead of τ-LUR. The space X is *midpoint locally uniformly rotund* (MLUR) if for each point $x \in \mathbf{S}_{\mathrm{X}}$ and sequences $(y_n), (z_n) \subset \mathbf{S}_{\mathrm{X}}$ such that $\frac{1}{2}(y_n + z_n) \to x$ we have $\|y_n - z_n\| \to 0$ as $n \to \infty$.

2. Results. The variable-exponent ℓ^p spaces can be used in constructing Banach spaces which admit in some sense pathological, yet 1-unconditional bases. Next we list examples of such results, the proofs of which are given subsequently.

THEOREM 2.1. *The class of Banach spaces of the type $\ell^{p(\cdot)}$ contains a universal space up to almost isometric containment. The analogous statement holds for spaces of the type $[(e_n)] \subset \ell^{p(\cdot)}$.*

THEOREM 2.2. *Let $1 \le p < \infty$. Then there is a Banach space* X *with a* 1*-unconditional basis (f_n) such that*

(i) *ℓ^p does not contain an isomorphic copy of* X.
(ii) *For each strictly increasing subsequence $(i) \subset \mathbb{N}$ and $\epsilon > 0$ there is a further subsequence (i_j) such that $[f_{i_j} : j \in \mathbb{N}]$ is asymptotically isometric to ℓ^p in the sense of* (1.2) *via equivalence of bases.*

More specifically, here X *is of the type $\ell^{p(\cdot)}$.*

THEOREM 2.3. *Each space $\ell^{q(\cdot)}$ contains ℓ^p almost isometrically in the sense of* (1.2) *for some $p \in [1,\infty]$.*

2.1. Basic properties. It turns out that spaces of the type $\ell^{p(\cdot)}$ enjoy some basic properties similar to those of classical ℓ^p spaces.

PROPOSITION 2.4. *Let $p\colon \mathbb{N} \to [1,\infty]$. Then $\ell^{p(\cdot)}$ is a Banach space. Moreover, (e_n) is a 1-unconditional basis of the space $[(e_n)]$.*

Proof. Clearly $[e_1,\dots,e_n] \subset \ell^{p(\cdot)}$ is a Banach space for $n \in \mathbb{N}$. Let $(x^{(j)}) \subset \ell^{p(\cdot)}$ be a Cauchy sequence. Note that $(x^{(j)})$ is bounded in ℓ^∞. Since P_k is a contractive projection for $k \in \mathbb{N}$, it follows that $(P_n(x^{(j)}))$ is a Cauchy sequence for a fixed n and hence converges in $[e_1,\dots,e_n]$. We conclude that $x^{(j)} \to x$ pointwise as $j \to \infty$, for some $x \in \ell^\infty$. Let $\epsilon > 0$. Since $(x^{(j)}) \subset \ell^{p(\cdot)}$ is Cauchy, there is $i_0 \in \mathbb{N}$ such that $\|x^{(j)} - x^{(i_0)}\| < \epsilon$ for $j \geq i_0$. In particular

$$\|P_k(x^{(j)} - x^{(i_0)})\|_{\ell^{p(\cdot)}} < \epsilon \quad \text{for } n \in \mathbb{N}. \tag{2.1}$$

By the definition of $\varPhi$ and (2.1) we get

$$\varPhi(x - x^{(i_0)}) = \sup_k \|P_k(x - x^{(i_0)})\|_{\ell^{p(\cdot)}} < \epsilon.$$

This yields $\varPhi(x) \leq \varPhi(x^{(i_0)}) + \epsilon < \infty$. Moreover, since ϵ was arbitrary, $x^{(j)} \to x$ in $\ell^{p(\cdot)}$ as $j \to \infty$. This completes the proof that $\ell^{p(\cdot)}$ is complete.

To check the last claim, let $x = (x_n) \in [(e_n)]$. We apply an auxiliary sequence $(y^{(j)}) \subset \operatorname{span}((e_n))$ such that $y^{(j)} \to x$ in $\ell^{p(\cdot)}$ as $j \to \infty$. Set $k_i \in \mathbb{N}$ for $i \in \mathbb{N}$ such that $P_{k_i}(y^{(j)}) = y^{(j)}$ for $j \leq i$. Observe that $\|x - P_{k_i}(x)\|_{\ell^{p(\cdot)}} \leq \|x - P_{k_i}(y^{(j)})\|_{\ell^{p(\cdot)}}$ for $j \leq i$. This yields

$$\sup_{i \geq j} \|x - P_{k_i}(x)\|_{\ell^{p(\cdot)}} \leq \sup_{i \geq j} \|x - P_{k_i}(y^{(j)})\|_{\ell^{p(\cdot)}} = \|x - y^{(j)}\|_{\ell^{p(\cdot)}} \to 0 \text{ as } j \to \infty.$$

We conclude that (e_n) is a Schauder basis. By the construction of $\varPhi$, we have $\|(x_n)\|_{\ell^{p(\cdot)}} = \|(\theta(n)x_n)\|_{\ell^{p(\cdot)}}$ for any sequence of signs $\theta \in \{-1,1\}^{\mathbb{N}}$. Thus (e_n) is a 1-unconditional basis of $[(e_n)]$. ■

Clearly $\ell^{p(\cdot)} \cap c_0$ is a closed subspace of $\ell^{p(\cdot)}$ for any p, and hence it contains $[(e_n)]$. We do not know if the space $\ell^{p(\cdot)} \cap c_0$ coincides with $[(e_n)] \subset \ell^{p(\cdot)}$. If $\limsup_{n\to\infty} p(n) < \infty$, then it is easy to check that $\ell^{p(\cdot)} \subset c_0$.

PROPOSITION 2.5 (Hölder inequality). *For a given sequence $p\colon \mathbb{N} \to [1,\infty]$ define $p^*\colon \mathbb{N} \to [1,\infty]$ by $1/p(n) + 1/p^*(n) = 1$ for each $n \in \mathbb{N}$. If (x_n) and (y_n) are real-valued sequences then*

$$\sum_{n\in\mathbb{N}} |x_n y_n| \leq \|(x_n)\|_{\ell^{p(\cdot)}} \|(y_n)\|_{\ell^{p^*(\cdot)}}.$$

Proof. By induction. ■

PROPOSITION 2.6. *Let $p\colon \mathbb{N} \to [1,\infty]$ and let $\mathrm{X} = [(e_n)] \subset \ell^{p(\cdot)}$. Then $\mathrm{X}^* = \ell^{p^*(\cdot)}$, where the duality is given by $x^*(x) = \sum_n x_n x_n^*(e_n)$.*

Proof. Recall that (e_n) is the 1-unconditional basis of $[(e_n)]$. We claim that $P_m^*(\mathrm{X}^*) = \{x^* \circ P_m : x^* \in \mathrm{X}^*\}$ is isometric to $P_m(\ell^{p^*(\cdot)})$ for each $m \in \mathbb{N}$.

Indeed, $(\mathbb{R} \oplus_p \mathbb{R})^* = \mathbb{R} \oplus_{p^*} \mathbb{R}$, so that the claim holds for $m = 2$. More generally, $(\mathrm{X} \oplus_p \mathbb{R})^* = \mathrm{X}^* \oplus_{p^*} \mathbb{R}$ for any Banach space X and $1 \leq p \leq \infty$. Thus, by induction on m we obtain $P_m^*(\mathrm{X}^*) = (P_m \ell^{p(\cdot)})^* = P_m \ell^{p^*(\cdot)}$ isometrically. For each m we identify these spaces and the corresponding projection is denoted by $P^*_{[e_1^*,\ldots,e_m^*]}\colon \mathrm{X}^* \to [e_1^*, \ldots, e_m^*]$. The projections of the type $P^*_{[e_1^*,\ldots,e_m^*]}$ clearly commute. Since (e_n) is a Schauder basis of $[(e_n)]$, we obtain

$$x^*(x) = \lim_{m\to\infty} P^*_{[e_1^*,\ldots,e_m^*]}(x^*)(x) = \lim_{m\to\infty} x^*(P_m(x)) \tag{2.2}$$

for each $x \in \mathrm{X}$.

There exists for each $x^* \in \mathrm{X}^*$ a unique vector $f_{x^*} \in \ell^\infty$ such that $f_{x^*} = \omega^*\text{-}\lim_{m\to\infty} P^*_{[e_1^*,\ldots,e_m^*]}(x^*)$, where we consider $[e_1^*, \ldots, e_m^*]$ in the canonical way as a subspace of ℓ^∞. Note that the continuity of given functionals $f, g \in \mathrm{X}^*$ implies that if $f(e_n) = g(e_n)$ for $n \in \mathbb{N}$, then $f - g = 0$. Thus, if $x^* \neq y^*$, then $f_{x^*} \neq f_{y^*}$.

In fact $f_{x^*} \in \ell^{p^*(\cdot)}$ and moreover $\|f_{x^*}\|_{\ell^{p^*(\cdot)}} = \|x^*\|_{\mathrm{X}^*}$. Indeed, by using (2.2) and the basic properties of (e_n) we get

$$\begin{aligned}\|x^*\|_{\mathrm{X}^*} &= \sup_{x\in \mathbf{S}_\mathrm{X}} |x^*(x)| = \sup_{x\in \mathbf{S}_\mathrm{X}} \lim_{m\to\infty} |P^*_{[e_1^*,\ldots,e_m^*]}(x^*)(x)| \\ &= \lim_{m\to\infty} \|P^*_{[e_1^*,\ldots,e_m^*]}(x^*)\|_{\ell^{p^*(\cdot)}} = \|f_{x^*}\|_{\ell^{p^*(\cdot)}}.\end{aligned}$$

Hence X^* can be regarded isometrically as a subspace of $\ell^{p^*(\cdot)}$ respecting the given duality. Finally, the Hölder inequality gives $\mathrm{X}^* = \ell^{p^*(\cdot)}$. ∎

2.2. Almost isometric containment. Next we will prove the results formulated previously. The arguments share some common auxiliary observations and we proceed by proving these facts.

FACT 2.7. *If $(n_k) \subset \mathbb{N}$ is a sequence, then*

$$\{(x_j) \in \ell^{p(\cdot)} : x_j \neq 0 \Rightarrow j \in \{n_1, n_1 + 1, n_2 + 1, n_3 + 1, \ldots\}\} \subset \ell^{p(\cdot)}$$

is isometric to $\ell^{q(\cdot)}$, where $q(k) = p(n_k)$ for $k \in \mathbb{N}$.

This justifies the notation $\ell^{q(\cdot)} \hookrightarrow \ell^{p(\cdot)}$. It is clear that if mappings $p_1, p_2\colon \mathbb{N} \to [1, \infty]$ satisfy $p_1 \leq p_2$ pointwise, then $\ell^{p_2(\cdot)} \subset \ell^{p_1(\cdot)}$.

Recall that we are denoting the 2-dimensional space $\mathbb{R} \oplus_p \mathbb{R}$ by ℓ_2^p (where p may possibly depend on some given parameters). For $p_1, p_2\colon \mathbb{N} \to [1, \infty]$ and $k \in \mathbb{N}$ we denote by $p_1|^{(k)}p_2\colon \mathbb{N} \to [1, \infty]$ the map given by $p_1|^{(k)}p_2(n) = p_2(n)$ for $n \geq k$ and $p_1|^{(k)}p_2(n) = p_1(n)$ for $n < k$. Consider a finite sequence of mappings

$$\ell^{p_1|^{(k)}p_2} \xrightarrow{\psi_{k-1}} \ell^{p_1|^{(k-1)}p_2} \xrightarrow{\psi_{k-2}} \cdots \xrightarrow{\psi_{k-i}} \ell^{p_1|^{(k-i)}p_2} \xrightarrow{\psi_{k-i-1}} \cdots \xrightarrow{\psi_1} \ell^{p_2(\cdot)},$$

where $\psi_j = \mathbf{I}\colon \ell^{p_1|^{(j+1)}p_2} \to \ell^{p_1|^{(j)}p_2}$ for $1 \leq j \leq k-1$. Then $\|\psi_1 \circ \cdots \circ \psi_{k-1}\| \leq \prod_{j=1}^{k-1} \|\psi_j\|$.

LEMMA 2.8. *Under the above notation,*

$$\|\psi_j\| = \|\mathbf{I}\colon \ell_2^{p_1(j)} \to \ell_2^{p_2(j)}\|. \tag{2.3}$$

Proof. Write $C = \|\mathbf{I}\colon \ell_2^{p_1(j)} \to \ell_2^{p_2(j)}\|$; then clearly $C \geq 1$. By inspecting the subspace $[e_j, e_{j+1}]$ we note that $\|\psi_j\| \geq C$. Next we will check that the converse inequality holds. Fix $x = \sum_{n=1}^{k} a_n e_n$. Then $\|\sum_{n=1}^{j} a_n e_n\|_{\ell^{p_1|^{(j+1)}p_2}} = \|\sum_{n=1}^{j} a_n e_n\|_{\ell^{p_1|^{(j)}p_2}} = \|\sum_{n=1}^{j} a_n e_n\|_{\ell^{p_1}}$. Thus

$$\begin{aligned}\Big\|\sum_{n=1}^{j+1} a_n e_n\Big\|_{\ell^{p_1|^{(j)}p_2}} &= \Big\|\sum_{n=1}^{j} a_n e_n\Big\|_{\ell^{p_1}} \boxplus_{p_2(j)} |a_{j+1}| \\ &\leq C\Big(\Big\|\sum_{n=1}^{j} a_n e_n\Big\|_{\ell^{p_1}} \boxplus_{p_1(j)} |a_{j+1}|\Big) = C\Big\|\sum_{n=1}^{j+1} a_n e_n\Big\|_{\ell^{p_1|^{(j+1)}p_2}}.\end{aligned}$$

Then

$$\begin{aligned}&\Big\|\sum_{n=1}^{j+1} a_n e_n\Big\|_{\ell^{p_1|^{(j)}p_2}} \boxplus_{p_2(j+1)} |Ca_{j+2}| \\ &\qquad\qquad \leq C\Big(\Big\|\sum_{n=1}^{j+1} a_n e_n\Big\|_{\ell^{p_1|^{(j+1)}p_2}} \boxplus_{p_2(j+1)} |a_{j+2}|\Big).\end{aligned}$$

More generally, we obtain

$$\Big\|\sum_{n=1}^{j+1} a_n e_n + \sum_{n=j+2}^{k} Ca_n e_n\Big\|_{\ell^{p_1|^{(j)}p_2}} \leq C\Big\|\sum_{n=1}^{k} a_n e_n\Big\|_{\ell^{p_1|^{(j+1)}p_2}}$$

and in particular $\|x\|_{\ell^{p_1|^{(j)}p_2}} \leq C\|x\|_{\ell^{p_1|^{(j+1)}p_2}}$. ∎

LEMMA 2.9. *Let $p, q\colon \mathbb{N} \to [1,\infty]$ and $\epsilon > 0$. If $\liminf_{n\to\infty} |q(k) - p(n)| = 0$ for $k \in \mathbb{N}$, then there is a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\phi\colon y_k \mapsto x_{n_k}$ defines an embedding $\ell^{q(\cdot)} \hookrightarrow \ell^{p(\cdot)}$ such that $(1+\epsilon)^{-1}\|y\| \leq \|\phi(y)\| \leq (1+\epsilon)\|y\|$ for $y \in \ell^{q(\cdot)}$. Moreover, ϕ is an embedding satisfying* (1.2).

Proof. Extract a subsequence $(n_k) \subset \mathbb{N}$ such that

$$\prod_{k\in\mathbb{N}} \|\mathbf{I}\colon \ell_2^{p(n_k)} \to \ell_2^{q(k)}\| \leq 1+\epsilon \quad \text{and} \quad \prod_{k\in\mathbb{N}} \|\mathbf{I}\colon \ell_2^{q(k)} \to \ell_2^{p(n_k)}\| \leq 1+\epsilon.$$

Define a linear map $\phi\colon \ell^{q(\cdot)} \to \ell^{\infty}$ by $\phi\colon a_{k+1}e_{k+1} \mapsto a_{k+1}e_{n_k+1}$. Fix $y \in \ell^{q(\cdot)}$ and $m \in \mathbb{N}$. Next we will apply the preceding observations in (2.3). We obtain

$$(1+\epsilon)^{-1}\|P_m(y)\|_{\ell^{q(\cdot)}} \leq \|\phi(P_m(y))\|_{\ell^{p(\cdot)}} \leq (1+\epsilon)\|P_m(y)\|_{\ell^{q(\cdot)}}.$$

Thus by recalling the definition of the norms $\|\cdot\|_{\ell^{q(\cdot)}}$ and $\|\cdot\|_{\ell^{p(\cdot)}}$ we see that $\phi\colon \ell^{q(\cdot)} \to \ell^{p(\cdot)}$ is defined, and this is the claimed isomorphism. By inspecting the construction of ϕ it is clear that also the last part of the statement holds. ∎

Proof of Theorem 2.1. Enumerate $\mathbb{Q}\cap[1,\infty)=(q(n))$, where $q\colon \mathbb{N}\to\mathbb{Q}$. Let $\ell^{q(\cdot)}$ be the corresponding space. It is easy to see that, given $p\colon \mathbb{N}\to[1,\infty]$ and $k\in\mathbb{N}$, we have $\liminf_{n\to\infty}|q(n)-p(k)|=0$. Lemma 2.9 shows that $\ell^{q(\cdot)}$ contains $\ell^{p(\cdot)}$ almost isometrically, so that the first claim holds.

For the other claim, $[(e_n)]\subset\ell^{q(\cdot)}$ is a suitable universal space. Indeed, according to Proposition 2.4, (e_n) is an unconditional basis and the $1+\epsilon$-isomorphism ϕ appearing in the proof of Lemma 2.9 takes $(e_n)\subset\ell^{p(\cdot)}$ to $(f_{n_1},f_{n_1+1},f_{n_2+1},\ldots)\subset\ell^{q(\cdot)}$. ∎

In the above proof the space $\ell^{q(\cdot)}$ contains ℓ^∞ almost isometrically in the sense of (1.2). Hence it is easy to see that $\ell^{q(\cdot)}/c_0$ contains ℓ^∞/c_0 isometrically. It is a classical fact that ℓ^∞ contains all separable Banach spaces isometrically, and using the same argument, so does ℓ^∞/c_0. We conclude that $\ell^{q(\cdot)}/c_0$ contains all separable spaces isometrically.

Proof of Theorem 2.2. Let $1<p<\infty$. Let $(r_i)\subset(1,\infty)$ be a sequence such that $r_i\to p$ as $i\to\infty$, $r_n<p$ for $n\in\mathbb{N}$ if $p\le 2$, and $r_n>p$ for $n\in\mathbb{N}$ if $p>2$. The space ℓ^p has type p if $p\le 2$ and cotype p if $p\ge 2$, and in both cases, given $i\in\mathbb{N}$, ℓ^p does not contain $\ell^{r_i}_n$s uniformly (see e.g. [8]). Hence we may pick for each $i\in\mathbb{N}$ a number $j_i\in\mathbb{N}$ such that

$$\inf_E d_{\mathrm{BM}}(E,\ell^{r_i}_{j_i})>i,$$

where the infimum is taken over all j_i-dimensional subspaces E of ℓ^p. Define $q\colon\mathbb{N}\to[1,\infty]$ by putting $q(n)=r_1$ for $1\le n\le r_1$ and $q(n)=r_l$ for $\sum_{i<l}r_i<n\le\sum_{i\le l}r_i$ and $l>1$. Then it follows from the selection of $q\colon\mathbb{N}\to[1,\infty]$ that $\ell^{q(\cdot)}$ does not embed linearly into ℓ^p.

Consider the canonical unit vectors (f_n) of $\ell^{q(\cdot)}$. According to Proposition 2.4, (f_n) is an unconditional basis of $\mathrm{X}=[(f_n)]$. Since $q(n)\to p$ as $n\to\infty$, an application of Lemma 2.9 yields the claim. ∎

Proof of Theorem 2.3. Let $q\colon\mathbb{N}\to[1,\infty]$. Since $[1,\infty]$ is a compact metrizable space, therefore sequentially compact, there exists a subsequence $(q(n_k))\subset(q(n))$ convergent in $[1,\infty]$. Let $p=\lim_{k\to\infty}q(n_k)\in[1,\infty]$. Thus we may apply Lemma 2.9 to conclude that $\ell^{q(\cdot)}$ contains ℓ^p almost isometrically. ∎

THEOREM 2.10. *Let $p\colon\mathbb{N}\to[1,\infty]$. Then the following conditions are equivalent:*

(1) *$\ell^{p(\cdot)}$ is reflexive.*
(2) *$\ell^{p(\cdot)}$ is superreflexive.*
(3) $\liminf_{n\to\infty}p(n)>1$ *and* $\limsup_{n\to\infty}p(n)<\infty$.
(4) *$\ell^{p(\cdot)}$ is an Asplund space.*

Let us observe before passing to the proof that ℓ^1 has the RNP but is not a reflexive space and thus one cannot replace Asplund by RNP in (4).

Proof. Recall the well-known characterization of superreflexive spaces, due to Enflo: a space is superreflexive if and only if it is isomorphic to a space both uniformly convex and uniformly smooth. Clearly (2)⇒(1). By using Lemma 2.9 we deduce that if $\liminf_{n\to\infty} p(n) = 1$ (resp. $\limsup_{n\to\infty} p(n) = \infty$), then $\ell^{p(\cdot)}$ contains ℓ^1 (resp. ℓ^∞) almost isometrically, and thus fails to be reflexive. Indeed, the bidual of ℓ^1 is the non-separable space ℓ^∞, and, on the other hand, if a Banach space contains an isomorphic copy of a non-reflexive space, then it is itself non-reflexive (see the first chapters of [1]).

Hence (1)⇒(3). Similarly (4)⇒(3) and it is well-known that reflexive spaces are Asplund.

It suffices to verify (3)⇒(2), so let us assume that (3) holds. Then there exist $p_0 \in (1, \liminf_{n\to\infty} p(n)]$, $q_0 \in [\limsup_{n\to\infty} p(n), \infty)$ and $k_0 \in \mathbb{N}$ such that $p(n) \in [p_0, q_0]$ for $n \geq k_0$. Define $\widetilde{p}\colon \mathbb{N} \to [p_0, q_0]$ by $\widetilde{p}(n) = \min(q_0, \max(p(n), p_0))$ for $n \in \mathbb{N}$. Note that $\mathrm{span}(e_1, \ldots, e_k)$ is a bicontractively complemented subspace regardless of whether it is considered as being contained in $\ell^{p(\cdot)}$ or $\ell^{\widetilde{p}(\cdot)}$. It follows that the identity mapping $I\colon \ell^{p(\cdot)} \to \ell^{\widetilde{p}(\cdot)}$ is an isomorphism. Thus our task is reduced to proving that $\ell^{\widetilde{p}(\cdot)}$ is superreflexive.

We will require the notions of upper p-estimate and lower q-estimate of Banach lattices. If X is a Banach lattice and $1 \leq p \leq q < \infty$ then the upper p-estimate and the lower q-estimate, respectively, are defined (for the relevant multiplicative constants being 1) as follows:

$$\Big\| \sum_{1\leq i\leq n} x_i \Big\| \leq \Big(\sum_{1\leq i\leq n} \|x_i\|^p \Big)^{1/p}, \qquad \Big\| \sum_{1\leq i\leq n} x_i \Big\| \geq \Big(\sum_{1\leq i\leq n} \|x_i\|^q \Big)^{1/q},$$

respectively, for any vectors $x_1, \ldots, x_n \in \mathrm{X}$ with pairwise disjoint supports. We will apply the fact that a Banach lattice which satisfies an upper p-estimate and a lower q-estimate for some $1 < p < q < \infty$ is isomorphic to a Banach space both uniformly convex and uniformly smooth (see [6, 1.f.1, 1.f.7]). Note that in that case X is superreflexive and the space $\ell^{\widetilde{p}(\cdot)}$, having a 1-unconditional basis, carries a natural Banach lattice structure. Thus, it suffices to show that $\ell^{\widetilde{p}(\cdot)}$ satisfies upper and lower estimates for p_0 and q_0, respectively.

Denote by $P_m\colon \ell^{\widetilde{p}(\cdot)} \to \mathrm{span}(e_1, \ldots, e_m)$ the natural projection preserving the first m coordinates. To check that $\ell^{\widetilde{p}(\cdot)}$ satisfies the upper p_0-estimate, let $x_1, \ldots, x_n \in \ell^{\widetilde{p}(\cdot)}$ be disjointly supported vectors. We claim that

$$\Big\| \sum_{1\leq i\leq n} x_i \Big\|_{\ell^{\widetilde{p}(\cdot)}} \leq \Big(\sum_{1\leq i\leq n} \|x_i\|_{\ell^{\widetilde{p}(\cdot)}}^{p_0} \Big)^{1/p_0}.$$

Indeed, assume to the contrary that this does not hold and let $m \in \mathbb{N}$ be the least natural number such that

$$\Big\|\sum_{1\le i\le n} P_m x_i\Big\|_{\ell^{\widetilde{p}(\cdot)}} > \Big(\sum_{1\le i\le n} \|P_m x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0}.$$

Clearly $m > 1$. We may assume without loss of generality that $(P_m - P_{m-1})x_n \doteq ae_m \neq 0$. It follows from the disjointness of the supports of $x_1,\dots,x_n$ that $(P_m - P_{m-1})x_i = 0$ for $1 \le i < n$. Observe that

$$\Big\|\sum_{1\le i\le n} P_{m-1} x_i\Big\|_{\ell^{\widetilde{p}(\cdot)}} \le \Big(\sum_{1\le i\le n} \|P_{m-1} x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0} \tag{2.4}$$

by the selection of m. Then

$$\begin{aligned}
\Big\|\sum_{1\le i\le n} P_m x_i\Big\|_{\ell^{\widetilde{p}(\cdot)}} &= \Big\|\sum_{1\le i\le n} P_{m-1} x_i\Big\|_{\ell^{\widetilde{p}(\cdot)}} \boxplus_{\widetilde{p}(m-1)} |a| \\
&\le \Big(\sum_{1\le i\le n} \|P_{m-1} x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0} \boxplus_{\widetilde{p}(m-1)} |a| \\
&= \bigg(\bigg(\Big(\sum_{1\le i< n} \|P_m x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0} \boxplus_{p_0} \|P_{m-1} x_n\|_{\ell^{\widetilde{p}(\cdot)}}\bigg)^{p_0}\bigg)^{1/p_0} \boxplus_{\widetilde{p}(m-1)} |a| \\
&\le \Big(\sum_{1\le i< n} \|P_m x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0} \boxplus_{p_0} (\|P_{m-1} x_n\|_{\ell^{\widetilde{p}(\cdot)}} \boxplus_{\widetilde{p}(m-1)} |a|) \\
&= \Big(\sum_{1\le i< n} \|P_m x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0} \boxplus_{p_0} (\|P_m x_n\|_{\ell^{\widetilde{p}(\cdot)}} \\
&= \Big(\sum_{1\le i\le n} \|P_m x_i\|^{p_0}_{\ell^{\widetilde{p}(\cdot)}}\Big)^{1/p_0}.
\end{aligned}$$

Above we applied the selection of m, (2.4), the disjointness of the supports and Fact 1.2. Thus we arrive at a contradiction, which means that $\ell^{\widetilde{p}(\cdot)}$ satisfies the upper p_0-estimate. The lower q_0-estimate is checked in a similar manner. ∎

3. Final remarks. A question was raised in [4, p. 174] whether each MLUR Banach space X is LUR. It has been established by now that this is not the case (see e.g. [2]). Next we will give a rather simple and natural example, which is related to these convexity conditions.

PROPOSITION 3.1. *Let* $p\colon \mathbb{N} \to (1,\infty)$ *be such that*

$$\prod_{k\in\mathbb{N}} \|\mathbf{I}\colon \ell_2^{p(k)} \to \ell_2^1\| < 2.$$

Then $\ell^{p(\cdot)}$ *satisfies the following condition: Given* $x \in \mathbf{S}_{\ell^{p(\cdot)}}$ *and sequences* $(y_n), (z_n) \subset \mathbf{B}_{\ell^{p(\cdot)}}$ *such that* $\frac{1}{2}(y_n + z_n) \to x$ *and* $\|P_k y_n\|, \|P_k z_n\| \to \|P_k x\|$

as $n \to \infty$ for each $k \in \mathbb{N}$, then $\|y_n - z_n\| \to 0$ as $n \to \infty$. However, $\ell^{p(\cdot)}$ is not ω-LUR.

Recall that X is ω-LUR if for each $x \in \mathbf{S}_{\mathrm{X}}$ and each sequence $(x_n) \subset \mathbf{B}_{\mathrm{X}}$ such that $\lim_{n\to\infty} \|x + x_n\| = 2$ the sequence x_n converges weakly to x as n tends to infinity.

Proof of Proposition 3.1. Clearly the sequence of canonical unit basis vectors $(e_n) \subset \ell^1 \subset \ell^{p(\cdot)}$ is a Schauder basis for $\ell^{p(\cdot)}$, since $\ell^{p(\cdot)}$ and ℓ^1 are isomorphic by the construction of $p(\cdot)$. We denote by $P_k\colon \ell^{p(\cdot)} \to [e_1, \ldots, e_k]$ the projection given by $\sum_{i\in\mathbb{N}} a_i e_i \mapsto \sum_{i=1}^{k} a_i e_i$ for $k \in \mathbb{N}$.

Fix $x \in \mathbf{S}_{\ell^{p(\cdot)}}$ and $(y_n), (z_n) \subset \mathbf{B}_{\ell^{p(\cdot)}}$ as in the assumptions; we wish to show that $y_n - z_n \to 0$ as $n \to \infty$. Indeed, if this is not the case, that is, $\limsup_{n\to\infty} \|y_n - z_n\| = c > 0$, then there exists a subsequence $(n_j) \subset \mathbb{N}$ such that $\|y_{n_j} - z_{n_j}\| \to c$ as $j \to \infty$. However, this possibility is excluded, as it turns out that there is a further subsequence $(n_{j_k}) \subset (n_j)$ such that $\lim_{k\to\infty} \|y_{n_{j_k}} - z_{n_{j_k}}\| = 0$. Let us write $u_j = y_{n_j}$ and $v_j = z_{n_j}$ for $j \in \mathbb{N}$.

By the continuity of P_k we find that $\frac{1}{2}P_k(u_j + v_j) \to P_k(x)$ for $k \in \mathbb{N}$. Observe that $[e_1, \ldots, e_k] \subset \ell^{p(\cdot)}$ is a uniformly convex subspace for each $k \in \mathbb{N}$. Thus $P_k(u_j), P_k(v_j) \to P_k(x)$ as $j \to \infty$ for all $k \in \mathbb{N}$ by the assumptions and the uniform convexity of $[e_1, \ldots, e_k]$. Hence one can pick a sequence $(j_k) \subset \mathbb{N}$ such that $P_k(u_{j_k}) - P_k(v_{j_k}) \to 0$ as $k \to \infty$.

As earlier in (2.3), one can consider $\mathbf{I}\colon \ell^{p(\cdot)} \to \ell^1$ formally as $\psi_1 \circ \psi_2 \circ \cdots\colon \ell^{p(\cdot)} \to \ell^1$. By applying the fact that $\lim_{i\to\infty} \lim_{k\to\infty} \|\psi_{k-i} \circ \cdots \circ \psi_k\| = 1$ we conclude that the sequence of mappings $R_k\colon \ell^{p(\cdot)} \to \mathbb{R}$ given by $R_k(x) = \|P_k(x)\| + \|(\mathbf{I} - P_k)(x)\|$ for $k \in \mathbb{N}$ satisfies $\|R_k\| \to 1$ as $k \to \infty$.

This is applied as follows. Since $\|P_k(u_{j_k})\|, \|P_k(v_{j_k})\| \to 1$ as $k \to \infty$, we see that $\|(\mathbf{I} - P_k)(u_{j_k} - v_{j_k})\| \to 0$ as $k \to \infty$. Hence

$$\|u_{j_k} - v_{j_k}\|_{\ell^{p(\cdot)}} \leq (\|P_k(u_{j_k} - v_{j_k})\|_{\ell^{p(\cdot)}} + \|(\mathbf{I} - P_k)(u_{j_k} - v_{j_k})\|_{\ell^{p(\cdot)}}) \xrightarrow{k\to\infty} 0.$$

Consequently, $\ell^{p(\cdot)}$ satisfies the first part of the claim.

One can pick a strictly increasing sequence $(n_i) \subset \mathbb{N}$ such that

$$\|(1 - 2^{-i}, 1 - 2^{-i})\|_{\ell_2^{p^*(n_i)}} \leq 1 - 2^{-i-1} \quad \text{for all } i \in \mathbb{N}.$$

Define a sequence (x_n) by $x_{n_i+1} = 1 - 2^{-i}$ for all n_i and $x_n = 0$ for all $n \in \mathbb{N} \setminus (n_i + 1)$. Observe that $(x_n) \in \mathbf{B}_{\ell^{p^*(\cdot)}}$ and that $x_1 = 0$. According to the Hölder inequality, $f\colon \ell^{p(\cdot)} \to \mathbb{R}$, $(y_n) \mapsto \sum_{n\in\mathbb{N}} x_n y_n$, is defined and $f \in \mathbf{B}_{(\ell^{p(\cdot)})^*}$.

Finally, observe that $\|e_1 + e_{n_i+1}\|_{\ell^{p(\cdot)}} \to 2$ as $i \to \infty$. However, $\lim_{i\to\infty} f(e_{n_i+1}) = 1 \neq 0 = f(e_1)$. This means that $\ell^{p(\cdot)}$ is not ω-LUR. ■

Observe that for each $\epsilon > 0$ the space $\ell^{p(\cdot)}$ above can be additionally defined so that its Banach–Mazur distance to ℓ^1 is less than $1+\epsilon$. In fact,

$$\lim_{n\to\infty} d_{\mathrm{BM}}(Q_n(\ell^{p(\cdot)}), \ell^1) = 1.$$

Finally we reiterate the open problems that have arised in this note:

- We do not know if the space in Proposition 3.1 is MLUR.
- Does $\ell^{p(\cdot)} \cap c_0$ always coincide with $[(e_n)] \subset \ell^{p(\cdot)}$?
- Given $p\colon \mathbb{N} \to (1,\infty)$, is $\ell^{p(\cdot)}$ necessarily strictly convex or smooth?

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