BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 53, No. 4, 2005

GENERAL TOPOLOGY

Hyperspaces of Finite Sets in Universal Spaces for Absolute Borel Classes

by

Kotaro MINE, Katsuro SAKAI and Masato YAGUCHI

Presented by Czesław BESSAGA

Summary. By $\mathrm{Fin}(X)$ (resp. $\mathrm{Fin}^k(X)$), we denote the hyperspace of all non-empty finite subsets of X (resp. consisting of at most k points) with the Vietoris topology. Let $\ell_2(\tau)$ be the Hilbert space with weight τ and $\ell_2^{\mathrm{f}}(\tau)$ the linear span of the canonical orthonormal basis of $\ell_2(\tau)$. It is shown that if $E = \ell_2^{\mathrm{f}}(\tau)$ or E is an absorbing set in $\ell_2(\tau)$ for one of the absolute Borel classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$ of weight $\le \tau$ $(\alpha > 0)$ then $\mathrm{Fin}(E)$ and each $\mathrm{Fin}^k(E)$ are homeomorphic to E. More generally, if X is a connected E-manifold then $\mathrm{Fin}(X)$ is homeomorphic to E and each $\mathrm{Fin}^k(X)$ is a connected E-manifold.

1. Introduction. Throughout the paper, spaces are metrizable and maps are continuous. Let τ be an infinite cardinal. Let $\ell_2(\tau)$ be the Hilbert space of weight τ and $\ell_2^{\mathrm{f}}(\tau)$ the linear span of the canonical orthonormal basis of $\ell_2(\tau)$. We write $\ell_2(\aleph_0) = \ell_2$ and $\ell_2^{\mathrm{f}}(\aleph_0) = \ell_2^{\mathrm{f}}$. Given a space E, a paracompact Hausdorff space is called a *manifold modeled on* E or an *E-manifold* if it can be covered by open sets which are homeomorphic to $(\approx)$ open sets in E.

In [1], Bestvina and Mogilski constructed universal spaces for separable absolute Borel classes as absorbing sets in the Hilbert cube $Q = \mathbf{I}^{\mathbb{N}}$ or the pseudo-interior $s = (0,1)^{\mathbb{N}}$, and they also gave topological characterizations of those spaces and manifolds modeled on them. Recently, in [14] and [8], the present authors generalized the results of [1] to non-separable absolute Borel classes. For each countable ordinal $\alpha > 0$, let $\Lambda_\alpha(\tau)$ and $\Omega_\alpha(\tau)$ be absorbing sets in $\ell_2(\tau)$ for the classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$, respectively, where $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$ are respectively the additive and multiplicative absolute

---

2000 *Mathematics Subject Classification*: 54B20, 57N20, 54H05.

Key words and phrases: hyperspace of finite subsets, Vietoris topology, Hausdorff metric, universal spaces, absolute Borel classes, Hilbert space, absorbing set.

Borel classes α with weight $\leq \tau$. Then $\Lambda_\alpha(\tau)$ $(\alpha \geq 1)$ and $\Omega_\alpha(\tau)$ $(\alpha \geq 2)$ are universal spaces in the classes $\mathfrak{a}_\alpha(\tau)$ and $\mathfrak{M}_\alpha(\tau)$, respectively. The space $\Omega_1(\tau)$ is homeomorphic to $\ell_2(\tau) \times \ell_2^{\mathrm{f}}$, which is a universal space in the class of σ-completely metrizable spaces ([1]) with weight $\leq \tau$. Note that this class is a proper subclass of $\mathfrak{a}_2(\tau)$. Although $\mathfrak{a}_1(\aleph_0)$ is the class of σ-compact metrizable spaces, $\mathfrak{a}_1(\tau)$ is in general the class of σ-locally compact metrizable ([2]) spaces with weight $\leq \tau$ (cf. [15]). Moreover, $\Lambda_1(\tau) \approx \ell_2^{\mathrm{f}}(\tau) \times Q$ and $\Omega_2(\tau) \approx \Omega_1(\tau)^{\mathbb{N}} \approx \Lambda_1(\tau)^{\mathbb{N}}$ (see [8]).

On the other hand, in [3], Curtis and Nguyen To Nhu proved that

> *the hyperspace* $\mathrm{Fin}(X)$ *of non-empty finite subsets of* X *with the Vietoris topology is homeomorphic to* ℓ_2^{f} *if and only if* X *is non-degenerate, connected, locally path-connected* σ*-compact and strongly countable-dimensional.*

Moreover, Curtis [2] showed that

$$\mathrm{Fin}(Q) \approx \mathrm{Fin}(\ell_2^{\mathrm{f}} \times Q) \approx \ell_2^{\mathrm{f}} \times Q.$$

Recently, the last author [19] proved that

$$\mathrm{Fin}(\ell_2(\tau)) \approx \ell_2(\tau) \times \ell_2^{\mathrm{f}} \ (\approx \Omega_1(\tau)).$$

For each $k \in \mathbb{N}$, let $\mathrm{Fin}^k(X)$ be the subspace of $\mathrm{Fin}(X)$ consisting of subsets with cardinality $\leq k$. For this hyperspace, the following are known:

(1) $\mathrm{Fin}^k(Q) \approx Q$ (Fedorchuk [6]),
(2) $\mathrm{Fin}^k(\ell_2) \approx \ell_2$ and $\mathrm{Fin}^k(\ell_2^{\mathrm{f}} \times Q) \approx \ell_2^{\mathrm{f}} \times Q$ (Nguyen To Nhu [10]),
(3) $\mathrm{Fin}^k(\ell_2 \times \ell_2^{\mathrm{f}}) \approx \ell_2 \times \ell_2^{\mathrm{f}}$ and $\mathrm{Fin}^k((\ell_2^{\mathrm{f}})^{\mathbb{N}}) \approx (\ell_2^{\mathrm{f}})^{\mathbb{N}}$ (Nguyen To Nhu and the second author [11]),

where $\ell_2^{\mathrm{f}} \times Q \approx \Lambda_1(\aleph_0)$, $\ell_2 \times \ell_2^{\mathrm{f}} \approx \Omega_1(\aleph_0)$ and $(\ell_2^{\mathrm{f}})^{\mathbb{N}} \approx \Omega_2(\aleph_0)$. In this paper, we show the following theorem:

Main Theorem. *Let* E *be one of the spaces* $\ell_2^{\mathrm{f}}(\tau)$, $\Lambda_\alpha(\tau)$, $\Omega_\alpha(\tau)$, *where* $\alpha > 0$. *Then* $\mathrm{Fin}(E) \approx E$ *and* $\mathrm{Fin}^k(E) \approx E$ *for each* $k \in \mathbb{N}$. *More generally, if* X *is a connected* E*-manifold then* $\mathrm{Fin}(X) \approx E$ *and each* $\mathrm{Fin}^k(X)$ *is a connected* E*-manifold.*

2. Preliminaries. For a metrizable space X, let $\mathfrak{a}_0(X)$ and $\mathfrak{M}_0(X)$ be the collections of all open sets and of all closed sets in X, respectively. For a countable ordinal $\alpha > 0$, by transfinite induction, we define $\mathfrak{a}_\alpha(X)$ (resp. $\mathfrak{M}_\alpha(X)$) as the collection of all countable unions (resp. intersections) of sets in $\bigcup_{\beta<\alpha} \mathfrak{a}_\beta(X) \cup \mathfrak{M}_\beta(X)$. Then $\mathfrak{M}_1(X)$ and $\mathfrak{a}_1(X)$ are the collections of all

([1]) A metrizable space is *σ-completely metrizable* if it is a countable union of completely metrizable <u>closed</u> subsets.

([2]) A metrizable space is *σ-locally compact metrizable* if it is a countable union of locally compact (closed) subsets. See footnote 1 in [8].

G_δ sets and of all F_σ sets in X, respectively. Sets in $\mathfrak{a}_2(X), \mathfrak{M}_2(X), \ldots$ are said to be $G_{\delta\sigma}$, $F_{\sigma\delta}, \ldots$ in X, respectively. For each subset $A \subset X$, we have $\mathrm{Fin}(A) \subset \mathrm{Fin}(X)$. It is easy to see that $\mathrm{Fin}(A)$ is open (resp. closed) in $\mathrm{Fin}(X)$ if A is open (resp. closed) in X. Then, by transfinite induction on α, we have

$$A \in \mathfrak{a}_\alpha(X) \;\Rightarrow\; \mathrm{Fin}(A) \in \mathfrak{a}_\alpha(\mathrm{Fin}(X)),$$
$$A \in \mathfrak{M}_\alpha(X) \;\Rightarrow\; \mathrm{Fin}(A) \in \mathfrak{M}_\alpha(\mathrm{Fin}(X)).$$

For an infinite cardinal τ, we denote by $\mathfrak{a}_\alpha(\tau)$ (resp. $\mathfrak{M}_\alpha(\tau)$) the class of all metrizable spaces X with $w(X) \le \tau$ such that $X \in \mathfrak{a}_\alpha(Y)$ (resp. $X \in \mathfrak{M}_\alpha(Y)$) whenever X is embedded in a metrizable space Y. Then $\mathfrak{M}_1(\tau)$, $\mathfrak{a}_1(\tau)$, $\mathfrak{M}_2(\tau)$, $\mathfrak{a}_2(\tau), \ldots$ are the classes of *absolutely* G_δ, *absolutely* F_σ, *absolutely* $G_{\delta\sigma}$, *absolutely* $F_{\sigma\delta}, \ldots$ spaces with weight $\le \tau$. Note that $\mathfrak{a}_0(\tau) = \emptyset$ and $\mathfrak{M}_0(\tau) = \mathfrak{M}_0(\aleph_0)$ is the class of compact metrizable spaces. As is well known, absolutely G_δ spaces are nothing else than completely metrizable spaces. A separable metrizable space is absolutely F_σ if and only if it is σ-compact. In the general case, a metrizable space is absolutely F_σ if and only if it is σ-locally compact (cf. [15]). We denote by $\mathfrak{a}_1^\omega(\tau)$ the class of all spaces with weight $\le \tau$ which are countable unions of locally compact, locally finite-dimensional closed sets.

Let $\mathcal{C}$ be a class of spaces. Then

- $\mathcal{C}$ is *topological* if $(X \in \mathcal{C},\, X \approx Y) \Rightarrow Y \in \mathcal{C}$,
- $\mathcal{C}$ is *closed* (resp. *open*) *hereditary* if $(X \in \mathcal{C},\, A \subset X$ is closed (resp. open) in $X) \Rightarrow A \in \mathcal{C}$,
- $\mathcal{C}$ is *additive* if $(X = X_1 \cup X_2$ and $X_1, X_2 \in \mathcal{C}$ are closed in $X) \Rightarrow X \in \mathcal{C}$.
- $\mathcal{C}$ is *productive* if $X_1, X_2 \in \mathcal{C} \Rightarrow X_1 \times X_2 \in \mathcal{C}$.

By $\mathcal{C}_\sigma$, we denote the class consisting of all metrizable spaces which can be expressed as countable unions of closed subspaces contained in $\mathcal{C}$. Then $\mathfrak{M}_1(\tau)_\sigma$ is the class of σ-completely metirzable spaces with weight $\le \tau$. Clearly, if $\mathcal{C}$ is closed hereditary then $\mathcal{C}_\sigma$ is closed and open hereditary.

Now, suppose that $\mathcal{C}$ is the topological class $\mathfrak{a}_\alpha(\tau)$ $(\alpha \ge 1)$, $\mathfrak{M}_\alpha(\tau)$ $(\alpha \ge 2)$ or $\mathfrak{a}_1^\omega(\tau)$. Then $\mathcal{C} = \mathcal{C}_\sigma$ is open and closed hereditary, additive, productive and contains $\mathbf{I}^n \times D(\tau)$ for all $n \in \mathbb{N}$, where $D(\tau)$ is the discrete space with $\operatorname{card} D(\tau) = \tau$.

For each space X, we denote by $\mathcal{E}(X)$ the class of all spaces which are homeomorphic to a closed subset of X. Note that

$$\mathcal{E}(\Lambda_\alpha(\tau)) = \mathfrak{a}_\alpha(\tau) \quad (\alpha \ge 1), \qquad \mathcal{E}(\Omega_\alpha(\tau)) = \mathfrak{M}_\alpha(\tau) \quad (\alpha \ge 2),$$
$$\mathcal{E}(\Omega_1(\tau)) = \mathfrak{M}_1(\tau)_\sigma \quad \text{and} \quad \mathcal{E}(\ell_2^{\mathrm{f}}(\tau)) = \mathfrak{a}_1^\omega(\tau).$$

For each open cover $\mathcal{U}$ of Y, two maps $f, g : X \to Y$ are *$\mathcal{U}$-close* (or f is *$\mathcal{U}$-close* to g) if each $\{f(x), g(x)\}$ is contained in some $U \in \mathcal{U}$. A closed set $A \subset X$ is called a *Z-set* (resp. a *strong Z-set*) in X provided, for each open cover $\mathcal{U}$ of X, there is a map $f : X \to X$ such that f is $\mathcal{U}$-close to id_X and $f(X) \cap A = \emptyset$ (resp. $\mathrm{cl}\, f(X) \cap A = \emptyset$). A subset $X \subset Y$ is *homotopy dense* in Y if there is a homotopy $h : Y \times \mathbf{I} \to Y$ such that $h_0 = \mathrm{id}_Y$ and $h_t(Y) \subset X$ for every $t > 0$. When X is an ANR, a closed set A is a Z-set in X if and only if $X \setminus A$ is homotopy dense in X (cf. Corollary 3.3 of [16]). A countable union of (strong) Z-sets in X is called a (*strong*) *Z_σ-set* in X. A (*strong*) *Z_σ-space* is a (*strong*) Z_σ-set in itself. A *Z-embedding* is an embedding whose image is a Z-set.

A space X is said to be *universal for a class $\mathcal{C}$* (simply, *$\mathcal{C}$-universal*) if every map $f : C \to X$ of $C \in \mathcal{C}$ can be approximated by Z-embeddings. We say that X is *strongly universal for $\mathcal{C}$* (simply, *strongly $\mathcal{C}$-universal*) when the following condition is satisfied:

($\mathrm{su}_{\mathcal{C}}$) for each $C \in \mathcal{C}$ and each closed set $D \subset C$, if $f : C \to X$ is a map such that $f|D$ is a Z-embedding, then, for each open cover $\mathcal{U}$ of X, there is a Z-embedding $h : C \to X$ such that $h|D = f|D$ and h is $\mathcal{U}$-close to f.

A *$\mathcal{C}$-absorbing set* in Y is a homotopy dense subset $X \subset Y$ such that $X \in \mathcal{C}_\sigma$ and X is a strongly $\mathcal{C}$-universal strong Z_σ-space. In [13, Theorem 3.8], Sakai and Yaguchi generalized a characterization of $\mathcal{C}$-absorbing sets by Bestvina and Mogilski [1, Theorem 5.3] to the following non-separable case:

THEOREM 2.1. *Let $\mathcal{C}$ be a closed hereditary additive topological class of spaces such that $\mathbf{I}^n \times D(\tau) \in \mathcal{C}$ for each $n \in \mathbb{N}$. Suppose that there exists a $\mathcal{C}$-absorbing set E in $\ell_2(\tau)$. Then an AR X with $w(X) \le \tau$ is homeomorphic to E if and only if $X \in \mathcal{C}_\sigma$, X is strongly $\mathcal{C}$-universal and X is a strong Z_σ-space.*

The open embedding theorem of E-manifolds [1, Corollary 5.7] can also be generalized to the non-separable case [13, Theorem 3.9] as follows:

THEOREM 2.2. *Under the assumption of Theorem* 2.1, *every connected E-manifold can be embedded in E as an open set.*

In this paper, $X^{\mathbb{N}}_{\mathrm{f}}$ denotes the weak product of X with a base point $* \in X$, that is,

$$X^{\mathbb{N}}_{\mathrm{f}} = \{(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \mid x_i = * \text{ except for finitely many } n \in \mathbb{N}\}.$$

In order to make the base point clear, we write $X^{\mathbb{N}}_{\mathrm{f}}(*)$. As is easily observed, Proposition 2.5 of [1] is valid for a non-separable AR X. Then we have the following proposition:

Proposition 2.3. *Under the assumptions of Theorem* 2.1, *if $\mathcal{C}$ is productive then $E \approx E_{\mathrm{f}}^{\mathbb{N}}$.*

Proof. Since $\mathcal{C} \subset \mathcal{E}(E) \subset \mathcal{E}(E_{\mathrm{f}}^{\mathbb{N}})$, it follows from Proposition 2.5 of [1] that $E_{\mathrm{f}}^{\mathbb{N}}$ is strongly $\mathcal{E}(E)$-universal. Moreover, $E_{\mathrm{f}}^{\mathbb{N}} \subset \ell_2(\tau)^{\mathbb{N}} \approx \ell_2(\tau)$ and E is homotopy dense in $\ell_2(\tau)$. Then $E_{\mathrm{f}}^{\mathbb{N}}$ can be embedded into $\ell_2(\tau)^{\mathbb{N}}$ as a homotopy dense subset. By Lemma 2.2 of [8], $E_{\mathrm{f}}^{\mathbb{N}}$ is a strong Z_σ-space. Since $E \in \mathcal{C}_\sigma$ and $\mathcal{C}$ is productive, we have $E_{\mathrm{f}}^{\mathbb{N}} \in \mathcal{C}_\sigma$. Hence, $E \approx E_{\mathrm{f}}^{\mathbb{N}}$ by Theorem 2.1. ∎

The following is due to Nguyen To Nhu [9, Theorem 2.1, Corollary 2.3]:

Theorem 2.4. *For every ANR* (*resp. AR*) *X and $k \in \mathbb{N}$, the hyperspaces* $\mathrm{Fin}(X)$ *and* $\mathrm{Fin}^k(X)$ *are also ANR's* (*resp. AR's*).

Note that every map $f : X \to Y$ induces a map $\widetilde{f} : \mathrm{Fin}(X) \to \mathrm{Fin}(Y)$ defined by $\widetilde{f}(A) = f(A) = \{f(x) \mid x \in A\}$. Moreover, for a homotopy $h : X \times \mathbf{I} \to Y$, we define $\widetilde{h} : \mathrm{Fin}(X) \times \mathbf{I} \to \mathrm{Fin}(Y)$ and $\widetilde{h}^k : \mathrm{Fin}^k(X) \times \mathbf{I} \to \mathrm{Fin}^k(Y)$ for each $k \in \mathbb{N}$ by $\widetilde{h}_t(A) = \widetilde{h}_t^k(A) = h_t(A) = \{h_t(x) \mid x \in A\}$. Then it is easy to see that $\widetilde{h}$ and $\widetilde{h}^k$ are continuous, so they are also homotopies.

3. Universality. Given an admissible metric d for X, we use the admissible metric for $X^{\mathbb{N}}$ defined as follows:

$$\varrho\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big) = \sup_{i\in\mathbb{N}} \min\{d(x_i, y_i), 2^{-i}\}.$$

Then ϱ_{H} is the Hausdorff metric induced by the metric ϱ.

Proposition 3.1. *Let X be a non-degenerate AR and W an open set in* $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$ *or* $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$ *for some $k \in \mathbb{N}$. Then W is universal for $\mathcal{E}(X)$.*

Proof. Because of similarity, we shall prove only the case of $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$. Let $Y = X_{\mathrm{f}}^{\mathbb{N}} \subset X^{\mathbb{N}}$. Since the weak product $Y_{\mathrm{f}}^{\mathbb{N}}(*)$ of Y with a base point $* \in Y$ is homeomorphic to $X_{\mathrm{f}}^{\mathbb{N}}$, we may show the universality of every open set W in $\mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*))$. Let $f : A \to W$ be a map of $A \in \mathcal{E}(X)$. For each open cover $\mathcal{U}$ of W, take a collection $\widetilde{\mathcal{U}}$ of open sets of $\mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$ such that $\mathcal{U} = \{U \cap \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*)) \mid U \in \widetilde{\mathcal{U}}\}$. Then $\widetilde{W} = \bigcup \widetilde{\mathcal{U}}$ is an open subset of $\mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$. Suppose that $\alpha : \widetilde{W} \to (0,1)$ is a map such that if a map $g : Y \to \mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$ is α-close to f then $g(Y) \subset \widetilde{W}$ and g is $\widetilde{\mathcal{U}}$-close to f. Since Y is an AR, we have a map $\lambda : Y \times Y \times \mathbf{I} \to Y$ such that $\lambda(x, x, t) = x$ for every $t \in \mathbf{I}$, $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ (such a map is called an *equi-connecting map*). Using this map, we can define a homotopy $\varphi : Y^{\mathbb{N}} \times Y \times \mathbf{I} \to Y^{\mathbb{N}}$ as follows:

$$\varphi(x,z,1) = (x_1, z, z, z, *, *, \dots),$$
$$\varphi(x,z,2^{-1}) = (x_1, x_2, z, z, z, *, *, \dots),$$
$$\dots$$
$$\varphi(x,z,2^{-n}) = (x_1, \dots, x_n, z, z, z, *, *, \dots),$$
$$\dots$$
$$\varphi(x,z,0) = (x_1, x_2, x_3, \dots) = x$$

and for $2^{-n} < t < 2^{-n+1}$,

$$\varphi(x,z,t) = (x_1, \dots, x_n, \lambda(x_{n+1}, z, 2^n t - 1), z, z, \lambda(z, *, 2^n t - 1), *, *, \dots).$$

Observe that $\varrho(x, \varphi(x,z,t)) < t$ for any $t > 0$.

Since $A \in \mathcal{E}(X)$, we can take a closed embedding $h : A \hookrightarrow Y = X_{\mathrm{f}}^{\mathbb{N}}$ such that $* \notin h(A)$ and $h(A)$ is closed in $X^{\mathbb{N}}$. Define $g : A \to \mathrm{Fin}(Y^{\mathbb{N}})$ by

$$g(y) = \{\varphi(x, h(y), \alpha(f(y))) \mid x \in f(y)\}.$$

It is clear that g is continuous. Since $\varrho(x, \varphi(x, h(y), \alpha(f(y))) < \alpha(f(y))$, it follows that $\varrho_{\mathrm{H}}(f(y), g(y)) < \alpha(f(y))$, that is, g is α-close to f. Note that $\varphi(Y^{\mathbb{N}} \times Y \times (0,1]) \subset Y_{\mathrm{f}}^{\mathbb{N}}(*)$, which means $g(A) \subset W$. Thus, it remains to prove that $g : A \to W$ is a Z-embedding.

To see that g is injective, let $g(y) = g(y')$ and fix a point

$$x = (x_1, \dots, x_n, *, *, *, \dots) \in g(y) = g(y')$$

with $x_n \neq *$. Then $x_{n-1} = h(y) = h(y')$ by the definition of φ. Since h is an embedding, we have $y = y'$.

To see that g is closed, let $a_i \in A$ $(i \in \mathbb{N})$ and $G \in \widetilde{W}$ with $g(a_i) \to G$. We show that $(a_i)_{i \in \mathbb{N}}$ has a convergent subsequence. By taking a subsequence, we may assume that $\alpha(f(a_i)) \to t \in \mathbf{I}$. Then $t > 0$. Otherwise, $f(a_i) \to G$ because $\varrho_{\mathrm{H}}(f(a_i), g(a_i)) < \alpha(f(a_i)) \to 0$. Hence, $\alpha(f(a_i)) \to \alpha(G) > 0$, which is a contradiction. Thus, we can choose $n \in \mathbb{N}$ so that $2^{-n} < t < 2^{-n+2}$. Take $z = (z_n)_{n \in \mathbb{N}} \in G \subset (X^{\mathbb{N}})^{\mathbb{N}}$. Note that $\varrho(z, g(a_i)) \to 0$. For each $i \in \mathbb{N}$, we can choose $x_i \in f(a_i)$ so that $\varphi(x_i, h(a_i), \alpha(f(a_i))) \to z$. For sufficiently large $i \in \mathbb{N}$, $2^{-n} < \alpha(f(y_i)) < 2^{-n+2}$, in which case

$$\mathrm{pr}_{n+2} \circ \varphi(x_i, h(a_i), \alpha(f(a_i))) = h(a_i),$$

where $\mathrm{pr}_n : Y^{\mathbb{N}} \to Y$ is the projection onto the nth factor. Therefore, $h(a_i) \to z_{n+2} \in X^{\mathbb{N}}$. Since h is a closed embedding of A not only into Y but also into X^N, it follows that $(a_i)_{i \in \mathbb{N}}$ is convergent in A. Thus, $g : A \to W$ is closed. Moreover, $g(A)$ is a closed subset of $\widetilde{W}$.

Now, we shall show that $g(A)$ is a Z-set in W. Let $Y_{\mathrm{f}}^{\mathbb{N}}(*')$ be the weak product of Y with a base point $*'$ different from $*$. Suppose $\mathcal{V}$ is an open cover of W. Choose a collection $\widetilde{\mathcal{V}}_1$ of open sets of $\mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$ so that $\mathcal{V} = \{U \cap \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*)) \mid U \in \widetilde{\mathcal{V}}_1\}$ and $\widetilde{V} = \bigcup \widetilde{\mathcal{V}}_1 \subset \widetilde{W}$. Let $\widetilde{\mathcal{V}}_2$ be an open cover of V which is a star-refinement of $\widetilde{\mathcal{V}}_1$. Since $\mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*'))$ is homotopy dense in $\mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$, there exists a map $i_1 : \widetilde{V} \to \widetilde{V} \cap \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*'))$ such that i_1 is $\widetilde{\mathcal{V}}_2$-close to $\mathrm{id}_{\widetilde{V}}$. Recall that $g(A)$ is closed in $\widetilde{V}$. Hence, $\widetilde{V} \setminus g(A)$ is open in $\mathrm{Fin}((X^{\mathbb{N}})^{\mathbb{N}})$. Thus, we can find a map

$$i_2 : \widetilde{V} \setminus g(A) \to (\widetilde{V} \setminus g(A)) \cap \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*))$$

such that i_2 is $\widetilde{\mathcal{V}}_2$-close to $\mathrm{id}_{\widetilde{V}}$. Observe that

$$g(A) \subset \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*)) \quad \text{and} \quad \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*)) \cap \mathrm{Fin}(Y_{\mathrm{f}}^{\mathbb{N}}(*')) = \emptyset.$$

Then we have $i_1(\widetilde{V}) \subset \widetilde{V} \setminus g(A)$. The map $i = i_2 \circ i_1 : W \to W$ is $\mathcal{V}$-close to id_W and $i(W) \cap g(A) = \emptyset$. Therefore, $g(A)$ is a Z-set in W. ∎

By replacing $Y = X_{\mathrm{f}}^{\mathbb{N}}$ with $Y = X^{\mathbb{N}}$ in the proof above, we can also show the following proposition:

PROPOSITION 3.2. *Let X be a non-degenerate AR and W an open set in* $\mathrm{Fin}(X^{\mathbb{N}})$ *or* $\mathrm{Fin}^k(X^{\mathbb{N}})$ *for some $k \in \mathbb{N}$. Then W is universal for $\mathcal{E}(X)$.* ∎

By the same proof as for Proposition 2.2 of [1], we can obtain the following non-separable version:

PROPOSITION 3.3. *Let $\mathcal{C}$ be an open and closed hereditary topological class. If each open subset of an ANR X is $\mathcal{C}$-universal and every Z-set in X is a strong Z-set, then X is strongly $\mathcal{C}$-universal.*

Due to Proposition 2.4 in [14], when $\mathcal{C} = \mathfrak{M}_1(\tau)$ in the above, it is not necessary to assume that every Z-set in X is a strong Z-set. Thus, we have the following generalization of Proposition 7.3 of [19]:

COROLLARY 3.4. *Let X be a non-degenerate AR such that $\mathfrak{M}_1(\tau) \subset \mathcal{E}(X)$ and let $k \in \mathbb{N}$. Then* $\mathrm{Fin}(X^{\mathbb{N}})$, $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$, $\mathrm{Fin}^k(X^{\mathbb{N}})$ *and* $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$ *are strongly $\mathfrak{M}_1(\tau)$-universal.* ∎

By Toruńczyk's characterization of Hilbert spaces [17] (cf. [18]), Theorem 2.4 and Corollary 3.4 imply the following non-separable version of Corollary 2.4 of [9]:

THEOREM 3.5. *For each $k \in \mathbb{N}$, the hyperspace* $\mathrm{Fin}^k(\ell_2(\tau))$ *of the Hilbert space $\ell_2(\tau)$ with weight τ is homeomorphic to $\ell_2(\tau)$.* ∎

REMARK 1. Due to Proposition 6.1 of [19], $\mathrm{Fin}(X)$ is a strong Z-space for every normed linear space X with $\dim X \geq 1$. As a combination of Theorems 2.1, 2.4 and Corollary 3.4, we have the main result of [19], that is,

THEOREM 3.6. *The hyperspace* $\mathrm{Fin}(\ell_2(\tau))$ *of the Hilbert space $\ell_2(\tau)$ with weight τ is homeomorphic to $\ell_2(\tau) \times \ell_2^{\mathrm{f}}$.* ∎

4. Z-sets in $\mathrm{Fin}(X)$

Lemma 4.1. *Let X be an ANR and A a Z-set in X. Then* $\mathrm{Fin}(A)$ *is a Z-set in* $\mathrm{Fin}(X)$, *and* $\mathrm{Fin}^k(A)$ *is a Z-set in* $\mathrm{Fin}^k(X)$ *for any $k \in \mathbb{N}$. Thus, if X is a Z_σ-space then* $\mathrm{Fin}(A)$ *and* $\mathrm{Fin}^k(A)$ *are also Z_σ-spaces.*

Proof. We deal with the case of $\mathrm{Fin}(A)$. Since X is an ANR, $X \setminus A$ is homotopy dense in X, hence $\mathrm{Fin}(X \setminus A)$ is homotopy dense in $\mathrm{Fin}(X)$. Since $\mathrm{Fin}(A) \subset \mathrm{Fin}(X) \setminus \mathrm{Fin}(X \setminus A)$, it follows that $\mathrm{Fin}(A)$ is a Z-set in $\mathrm{Fin}(X)$. It can be similarly shown that $\mathrm{Fin}^k(A)$ is a Z-set in $\mathrm{Fin}^k(X)$ for any $k \in \mathbb{N}$. ∎

Note that every Z-set in $\ell_2(\tau)$ is a strong Z-set [7]. Since $\ell_2(\tau) \times \ell_2^{\mathrm{f}}$ is homotopy dense in $\ell_2(\tau) \times \ell_2 \approx \ell_2(\tau)$, every Z-set in $\ell_2(\tau) \times \ell_2^{\mathrm{f}}$ is a strong Z-set by Lemma 2.2 of [8].

Proposition 4.2. *Let X be a non-degenerate AR. In the spaces* $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$ *and* $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$, $k \in \mathbb{N}$, *every Z-set is a strong Z-set. Thus*, $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$ *and* $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$ *are strong Z_σ-spaces.*

Proof. We may assume that $X_{\mathrm{f}}^{\mathbb{N}}$ can be embedded into Hilbert space as a homotopy dense subset. Indeed, X can be embedded into a completely metrizable AR $\widetilde{X}$ as a homotopy dense subset [12]. Hence, $X_{\mathrm{f}}^{\mathbb{N}}$ is homotopy dense in $\widetilde{X}^{\mathbb{N}}$ which is homeomorphic to $\ell_2(\tau)$ [17]. Thus, $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$ and $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$ are homotopy dense subsets of $\mathrm{Fin}(\ell_2(\tau))$ and $\mathrm{Fin}^k(\ell_2(\tau))$, respectively. Since $\mathrm{Fin}(\ell_2(\tau)) \approx \ell_2(\tau) \times \ell_2^{\mathrm{f}}$ and $\mathrm{Fin}^k(\ell_2(\tau)) \approx \ell_2(\tau)$ (Theorems 3.6 and 3.5), it follows from Lemma 2.2 of [8] that every Z-set in $\mathrm{Fin}(X)$ is a strong Z-set. Since $X_{\mathrm{f}}^{\mathbb{N}}$ is a Z_σ-space, $\mathrm{Fin}(X_{\mathrm{f}}^{\mathbb{N}})$ and $\mathrm{Fin}^k(X_{\mathrm{f}}^{\mathbb{N}})$ are Z_σ-spaces. Thus, they are strong Z_σ-spaces. ∎

Remark 2. It can also be shown that $\mathrm{Fin}(X^{\mathbb{N}})$ is a strong Z_σ-space if X is a non-degenerate AR. Indeed, $\mathrm{Fin}(X^{\mathbb{N}})$ can also be embedded into the strong Z_σ-space $\mathrm{Fin}(\ell_2(\tau)) \approx \ell_2(\tau) \times \ell_2^{\mathrm{f}}$ (Theorem 3.6). Since every AR which is a homotopy dense subset of a Z_σ-space is also a Z_σ-space, we see that $\mathrm{Fin}(X^{\mathbb{N}})$ is a Z_σ-space.

5. Absolute Borel classes. Let d be an admissible metric for X. Then the Vietoris topology on $\mathrm{Fin}(X)$ is induced by the Hausdorff metric

$$d_{\mathrm{H}}(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\Big\},$$

where $d(a, A) = \inf_{y \in A} d(x, y)$. For each $k \in \mathbb{N}$, let ϱ be the metric for X^k defined as follows:

$$\varrho(x, y) = \max_{i \le k} d(x_i, y_i).$$

Let $q_k : X^k \to \mathrm{Fin}^k(X)$ be the natural surjection defined by

$$q_k((x_1, \dots, x_k)) = \{x_1, \dots, x_k\}.$$

Then it is clear that

$$d_{\mathrm{H}}(q_k(x), q_k(y)) \leq \varrho(x, y) \quad \text{for any } x, y \in X^k.$$

This means that q_k is uniformly continuous. Note that

$$\operatorname{card} q_k^{-1}(A) \leq k! \quad \text{for each } A \in \operatorname{Fin}^k(X).$$

LEMMA 5.1. *The map q_k is perfect.*

Proof. It suffices to show that a sequence $(x_n)_{n\in\mathbb{N}}$ in X^k has a convergent subsequence if $(q_k(x_n))_{n\in\mathbb{N}}$ is convergent in $\operatorname{Fin}^k(X)$. Let $q_k(x_n)$ be convergent to $A \in \operatorname{Fin}^k(X)$. For each $j \leq k$,

$$d(\mathrm{pr}_j(x_n), A) \leq d_{\mathrm{H}}(q_k(x_n), A) \to 0 \quad (n \to \infty),$$

where $\mathrm{pr}_j : X^k \to X$ is the projection onto the jth factor. Since A is finite, any subsequence of $(\mathrm{pr}_j(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence. Then it is easy to find a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that $(\mathrm{pr}_j(x_{n_i}))_{i\in\mathbb{N}}$ is convergent in X for every $j \leq k$, which means that $(x_{n_i})_{i\in\mathbb{N}}$ is convergent in X^k. ∎

PROPOSITION 5.2. *For a σ-locally compact metric space X*, $\operatorname{Fin}(X)$ *is also σ-locally compact*, *i.e.*, $X \in \mathfrak{a}_1(\tau) \Rightarrow \operatorname{Fin}(X) \in \mathfrak{a}_1(\tau)$.

Proof. Let $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n is a locally compact subset of X with $X_n \subset X_{n+1}$. Since the perfect image of a locally compact space is also locally compact ([4, Theorem 3.7.12]), it follows from Lemma 5.1 that $\operatorname{Fin}^k(X_k)$ is locally compact. Then $\operatorname{Fin}(X) = \bigcup_{k\in\mathbb{N}} \operatorname{Fin}^k(X_k)$ is σ-locally compact. ∎

Note that if $f : X \to Y$ is a closed map and $0 < \operatorname{card} f^{-1}(y) \leq k\ (< \infty)$ for every $y \in Y$ then $\dim Y \leq \dim X + k - 1$ [5, Theorem 3.3.7]. Then, by the same proof as for Proposition 5.2 above, we have the following:

PROPOSITION 5.3. $X \in \mathfrak{a}_1^{\omega}(\tau) \Rightarrow \operatorname{Fin}(X) \in \mathfrak{a}_1^{\omega}(\tau)$. ∎

By Proposition 5.1 of [19], if X is completely metrizable then $\operatorname{Fin}(X)$ is σ-completely metrizable, that is,

$$X \in \mathfrak{M}_1(\tau) \;\Rightarrow\; \operatorname{Fin}(X) \in \mathfrak{M}_1(\tau)_\sigma \subset \mathfrak{a}_2(\tau).$$

We also have the following:

$$X \in \mathfrak{M}_1(\tau)_\sigma \;\Rightarrow\; \operatorname{Fin}(X) \in \mathfrak{M}_1(\tau)_\sigma.$$

PROPOSITION 5.4. *For each countable ordinal $\alpha \geq 2$*,

$$X \in \mathfrak{a}_\alpha(\tau) \;\Rightarrow\; \operatorname{Fin}(X) \in \mathfrak{a}_\alpha(\tau),$$
$$X \in \mathfrak{M}_\alpha(\tau) \;\Rightarrow\; \operatorname{Fin}(X) \in \mathfrak{M}_\alpha(\tau).$$

Proof. We handle the cases of $X \in \mathfrak{a}_2(\tau)$ and $X \in \mathfrak{M}_2(\tau)$. Then the result can be obtained by transfinite induction.

If $X \in \mathfrak{a}_2(\tau)$, let $X = \bigcup_{i\in\mathbb{N}} X_i$, where each X_i is completely metrizable. Without loss of generality, we may assume $X_i \subset X_{i+1}$ for each $i \in \mathbb{N}$; then $\mathrm{Fin}(X) = \bigcup_{i\in\mathbb{N}} \mathrm{Fin}(X_i)$. Since each $\mathrm{Fin}(X_i)$ is σ-completely metrizable, $\mathrm{Fin}(X)$ is also σ-completely metrizable. This means $\mathrm{Fin}(X) \in \mathfrak{a}_2(\tau)$.

If $X \in \mathfrak{M}_2(\tau)$, let $\widetilde{X}$ be the completion of X. Since X is $F_{\sigma\delta}$ in $\widetilde{X}$, $\mathrm{Fin}(X)$ is also $F_{\sigma\delta}$ in $\mathrm{Fin}(\widetilde{X})$. By Proposition 5.1 of [19], $\mathrm{Fin}(\widetilde{X})$ is F_σ in the completely metrizable space $\mathrm{Cld}_{\mathrm{H}}(\widetilde{X})$. This means $\mathrm{Fin}(X)$ is $F_{\sigma\delta}$ in $\mathrm{Cld}_{\mathrm{H}}(\widetilde{X})$. Thus, $\mathrm{Fin}(X) \in \mathfrak{M}_2(\tau)$. ∎

Remark 3. For each $k \in \mathbb{N}$, $\mathrm{Fin}^k(X)$ is closed in $\mathrm{Fin}(X)$. For the spaces $\mathrm{Fin}^k(X)$, $k \in \mathbb{N}$, we have the same results as for $\mathrm{Fin}(X)$ above.

6. Proof of the Main Theorem. First, we prove the following:

Theorem 6.1. *Let $\mathcal{C}$ be an open and closed hereditary, additive, productive and topological class of spaces such that $\mathbf{I}^n \times D(\tau) \in \mathcal{C}$ for each $n \in \mathbb{N}$. Suppose that there exists a $\mathcal{C}$-absorbing set E in $\ell_2(\tau)$. Then* $\mathrm{Fin}(E)$ *and* $\mathrm{Fin}^k(E)$, $k \in \mathbb{N}$, *are strongly $\mathcal{C}$-universal.*

Proof. By the $\mathcal{C}$-universality of E, we have $\mathcal{C} \subset \mathcal{E}(E)$. Since $\mathrm{Fin}(E_{\mathrm{f}}^{\mathbb{N}})$ and $\mathrm{Fin}^k(E_{\mathrm{f}}^{\mathbb{N}})$ are AR's by Theorem 2.4 and every Z-set is a strong Z-set in these spaces, it follows from Propositions 3.1 and 3.3 that $\mathrm{Fin}(E_{\mathrm{f}}^{\mathbb{N}})$ and $\mathrm{Fin}^k(E_{\mathrm{f}}^{\mathbb{N}})$ are strongly $\mathcal{C}$-universal. On the other hand, $E_{\mathrm{f}}^{\mathbb{N}} \approx E$ by Proposition 2.3, hence $\mathrm{Fin}(E) \approx \mathrm{Fin}(E_{\mathrm{f}}^{\mathbb{N}})$. Thus, we have the result. ∎

Now, we shall prove the main theorem.

Theorem 6.2. *Suppose that E is homeomorphic to $\ell_2^{\mathrm{f}}(\tau)$, $\Lambda_\alpha(\tau)$ or $\Omega_\alpha(\tau)$, where $\alpha \ge 1$ is a countable ordinal. Then the hyperspaces* $\mathrm{Fin}(E)$ *and* $\mathrm{Fin}^k(E)$, $k \in \mathbb{N}$, *are homeomorphic to E.*

Proof. First, note that E is strongly universal for the class $\mathcal{C} = \mathcal{E}(E)$ and $E \in \mathcal{C}_\sigma = \mathcal{C}$. In §5, we have shown that $\mathrm{Fin}(E), \mathrm{Fin}^k(E) \in \mathcal{C}$. These spaces are strong Z_σ-spaces by Proposition 4.2 and are strongly $\mathcal{C}$-universal by Theorem 6.1. Thus, $\mathrm{Fin}(E) \approx \mathrm{Fin}^k(E) \approx E$ by Theorem 2.1. ∎

Since every connected E-manifold X can be embedded into E as an open set by Theorem 2.2, $\mathrm{Fin}(X)$ and $\mathrm{Fin}^k(X)$ can also be embedded into $\mathrm{Fin}(E)$ and $\mathrm{Fin}^k(E)$ as open sets, respectively. Since X is connected, $\mathrm{Fin}(X)$ is an AR (cf. Proposition 3.1 of [19]) and each $\mathrm{Fin}^k(X)$ is connected. Hence, we have the following theorem.

Theorem 6.3. *Suppose that E is homeomorphic to $\ell_2^{\mathrm{f}}(\tau)$, $\Lambda_\alpha(\tau)$ or $\Omega_\alpha(\tau)$, where $\alpha \ge 1$ is a countable ordinal. Let X be a connected E-manifold. Then* $\mathrm{Fin}(X)$ *is homeomorphic to E and each* $\mathrm{Fin}^k(X)$ *is a connected E-manifold.* ∎

References

[1] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J. 33 (1986), 291–313.

[2] D. Curtis, *Hyperspaces of finite subsets as boundary sets*, Topology Appl. 22 (1986), 97–107.

[3] D. Curtis and Nguyen To Nhu, *Hyperspaces of finite subsets which are homeomorphic to $\aleph_0$-dimensional linear metric spaces*, ibid. 19 (1985), 251–260.

[4] R. Engelking, *General Topology*, rev. ed., Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1989.

[5] —, *Theory of Dimensions, Finite and Infinite*, Sigma Ser. Pure Math. 10, Heldermann, Berlin, 1995.

[6] V. V. Fedorchuk, *Covariant functors in the category of compacta, absolute retracts, and Q-manifolds*, Uspekhi Mat. Nauk 36 (1981), no. 3, 177–195 (in Russian); English transl.: Russian Math. Surveys 36 (1981), no. 3, 211–233.

[7] D.W. Henderson, *Z-sets in ANR's*, Trans. Amer. Math. Soc. 213 (1975), 205–216.

[8] K. Mine, *Universal spaces of non-separable absolute Borel classes*, Tsukuba J. Math., to appear.

[9] Nguyen To Nhu, *Investigating the ANR-property of metric spaces*, Fund. Math. 124 (1984), 243–254.

[10] —, *Hyperspaces of compact sets in metric linear spaces*, Topology Appl. 22 (1986), 109–122.

[11] Nguyen To Nhu and K. Sakai, *Probability measure functions preserving infinite-dimensional spaces*, Colloq. Math. 70 (1996), 291–304.

[12] K. Sakai, *The completion of metric ANR's and homotopy dense subsets*, J. Math. Soc. Japan 52 (2000), 835–846.

[13] K. Sakai and M. Yaguchi, *Characterizing manifolds modeled on certain dense subspaces of non-separable Hilbert spaces*, Tsukuba J. Math. 27 (2003), 143–159.

[14] —, —, *Hyperspaces of Banach spaces with the Attouch–Wets topology*, Set-Valued Anal. 12 (2004), 329–344.

[15] A. H. Stone, *Absolute F_σ spaces*, Proc. Amer. Math. Soc. 13 (1962), 495–499.

[16] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of ℓ_2-manifolds*, Fund. Math. 101 (1978), 93–110.

[17] —, *Characterizing Hilbert space topology*, ibid. 111 (1981), 247–262.

[18] —, *A correction of two papers concerning Hilbert manifolds*, ibid. 125 (1985), 89–93.

[19] M. Yaguchi, *Hyperspaces of finite subsets of non-separable Hilbert spaces*, Tsukuba J. Math., to appear.

K. Mine, K. Sakai and M. Yaguchi
Institute of Mathematics
University of Tsukuba
Tsukuba, 305-8571 Japan
E-mail: pen@math.tsukuba.ac.jp
sakaiktr@sakura.cc.tsukuba.ac.jp
masato@math.tsukuba.ac.jp

Received May 30, 2005;
received in final form February 20, 2006 (7461)