The Spaces of Closed Convex Sets in Euclidean Spaces with the Fell Topology

by

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Summary. Let \( \text{Conv}_F(\mathbb{R}^n) \) be the space of all non-empty closed convex sets in Euclidean space \( \mathbb{R}^n \) endowed with the Fell topology. We prove that \( \text{Conv}_F(\mathbb{R}^n) \approx \mathbb{R}^n \times Q \) for every \( n > 1 \) whereas \( \text{Conv}_F(\mathbb{R}) \approx \mathbb{R} \times I \).

Let \( \text{Conv}(X) \) be the set of all non-empty closed convex sets in a normed linear space \( X = (X, \| \cdot \|) \). We can consider various topologies on \( \text{Conv}(X) \). In [6], the AR-property of the spaces \( \text{Conv}(X) \) with the Hausdorff metric topology, the Attouch–Wets topology, and the Wijsman topology has been studied. In this paper, we shall consider the Fell topology on \( \text{Conv}(X) \), which is generated by the sets of the form

\[
U^- = \{ A \in \text{Conv}(X) \mid A \cap U \neq \emptyset \} \quad \text{and} \quad (X \setminus K)^+ = \{ A \in \text{Conv}(X) \mid A \subset X \setminus K \},
\]

where \( U \) is open and \( K \) is compact in \( X \). This topology is also defined on the set \( \text{Conv}^*(X) = \text{Conv}(X) \cup \{ \emptyset \} \). By \( \text{Conv}^*_F(X) \) and \( \text{Conv}_F(X) \), we denote the spaces \( \text{Conv}^*(X) \) and \( \text{Conv}(X) \) equipped with the Fell topology.

In case \( X \) is finite-dimensional (equivalently locally compact), \( \text{Conv}_F(X) \) is a locally compact metrizable space and \( \text{Conv}^*_F(X) \) is its Aleksandrov one-point compactification. It is easy to see that \( \text{Conv}_F((0, 1)) \) is homeomorphic

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to (≈) the triangle with two vertices removed, \( \Delta \setminus \{(0,0),(1,1)\} \), where \( \Delta = \{(x,y) \in \mathbb{I}^2 \mid x \leq y \} \subset \mathbb{I}^2 \). Since \( \text{Conv}_F(\mathbb{R}) \approx \text{Conv}_F((0,1)) \), we have

\[
\text{Conv}_F(\mathbb{R}) \approx \Delta \setminus \{(0,0),(1,1)\} \approx \mathbb{R} \times \mathbb{I},
\]

hence

\[
\text{Conv}_F^*(\mathbb{R}) \approx \Delta /\{(0,0),(1,1)\} \approx (S^1 \times \mathbb{I}) /\{(\text{pt}) \times \mathbb{I}\},
\]

where \( S^1 \) is the unit circle. For \( n > 1 \), the space \( \text{Conv}_F(\mathbb{R}^n) \) is infinite-dimensional. Let \( Q = [-1,1]^\mathbb{N} \) be the Hilbert cube. We prove the following result:

**Main Theorem.** For each \( n > 1 \), \( \text{Conv}_F(\mathbb{R}^n) \approx \mathbb{R}^n \times Q \) and

\[
\text{Conv}_F^*(\mathbb{R}^n) \approx (S^n \times Q) /\{(\text{pt}) \times Q\} \approx (B^n \times Q) /\{S^{n-1} \times Q\},
\]

where \( B^n \) and \( S^{n-1} \) are the closed unit ball and the unit sphere in \( \mathbb{R}^n \).

**Remark 1.** As studied in [6], \( \text{Conv}(X) \) has other metrizable topologies called the Attouch–Wets topology and the Wijsman topology. However, in case \( X \) is finite-dimensional, these are equal to the Fell topology. For the above topologies, we refer to the book [1].

**Remark 2.** The space \( \text{Conv}_H(X) \) with the Hausdorff metric topology is rather complicated. Concerning the subspace \( \text{Cld}_H(X) \subset \text{Conv}_H(X) \) consisting of non-empty compact convex sets, it is shown in [4] in case \( n > 1 \) that \( \text{CC}_H(\mathbb{R}^n) \approx Q \setminus \{0\} \). It should be remarked that \( \text{CC}_F(\mathbb{R}^n) = \text{CC}_H(\mathbb{R}^n) \), which can be obtained from [9, Theorem 3]. As is observed in [6, §2], \( \text{CC}_H(\mathbb{R}^n) \) is a component of \( \text{Conv}_H(\mathbb{R}^n) \) (1). However, as will be seen in Proposition 3, \( \text{CC}_F(\mathbb{R}^n) \) is homotopy dense in \( \text{Conv}_F(\mathbb{R}^n) \).

The open ball and the closed ball in \( \mathbb{R}^n \) centered at the point \( x \in \mathbb{R}^n \) with radius \( r > 0 \) are respectively denoted as follows:

\[
B(x,r) = \text{int}(x + rB^n) \quad \text{and} \quad \overline{B}(x,r) = x + rB^n.
\]

**Proposition 1.** For every \( n \in \mathbb{N} \), \( \text{Conv}_F^*(\mathbb{R}^n) \) is compact, hence it is the Aleksandrov one-point compactification of \( \text{Conv}_F(\mathbb{R}^n) \).

**Proof.** Since the hyperspace \( \text{Cld}_F(\mathbb{R}^n) \) of all closed sets in \( \mathbb{R}^n \) with the Fell topology is compact [1, Theorem 5.1.3], it suffices to show that \( \text{Conv}_F(\mathbb{R}^n) \) is closed in \( \text{Cld}_F(\mathbb{R}^n) \). For \( A \in \text{Cld}^*(\mathbb{R}^n) \setminus \text{Conv}^*(\mathbb{R}^n) \), we have \( a, b \in A \) and \( c \in \langle a, b \rangle \setminus A \), where \( \langle a, b \rangle \) is the convex hull of \( \{a, b\} \). Choose \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( \overline{B}(c,\varepsilon) \cap A = \emptyset \) and \( \langle x, y \rangle \cap B(c,\varepsilon) \neq \emptyset \) if \( \|x - a\| < \delta \) and \( \|y - b\| < \delta \). Then

\[
(\mathbb{R}^n \setminus \overline{B}(c,\varepsilon))^+ \cap B(a,\delta)^- \cap B(b,\delta)^-
\]

is a neighborhood of \( A \) which misses \( \text{Conv}^*(\mathbb{R}^n) \). \( \blacksquare \)

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(1) The subspace \( \text{Conv}_H^*(\mathbb{R}^n) \subset \text{Conv}_H(\mathbb{R}^n) \) consisting of all bounded closed convex sets coincides with \( \text{CC}_H(\mathbb{R}^n) \).

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Every locally compact Hausdorff space $X$ has the Aleksandrov one-point compactification, which is denoted by $\alpha X = X \cup \{\infty\}$. Let $f : X \to Y$ be a map between locally compact Hausdorff spaces. If $f$ is proper, that is, $f^{-1}(C)$ is compact for each compact set $C \subseteq Y$, then $f$ extends to a map $\bar{f} : \alpha X \to \alpha Y$ such that $\bar{f}(\infty) = \infty$. By identifying $X$ with the subset of $\text{Cld}_F(X)$ consisting of singletons and $\infty$ with $\emptyset$, we can regard $\alpha X \subseteq \text{Cld}_F^*(X)$.

For $A \in \text{Conv}(\mathbb{R}^n)$, let $p(A)$ be the nearest point of $A$ from the origin $0 \in \mathbb{R}^n$ with respect to the Euclidean metric (cf. the proof of [5, Lemma 1.6]).

**Lemma 2.** The function $p : \text{Conv}_F(\mathbb{R}^n) \to \mathbb{R}^n$ is continuous and proper, hence it extends to a map $p^* : \text{Conv}_F^*(\mathbb{R}^n) \to \alpha \mathbb{R}^n$ with $p^*(\emptyset) = \infty$.

**Proof.** For each $\varepsilon > 0$, $A \in \text{Conv}(\mathbb{R}^n)$ has the following neighborhood:

\[
\mathcal{U} = B(p(A), \varepsilon) \cap (\mathbb{R}^n \setminus (\|p(A)\| - \varepsilon)B^n)^+ \cap \text{Conv}(\mathbb{R}^n),
\]

where $(\|p(A)\| - \varepsilon)B^n = \emptyset$ if $\|p(A)\| - \varepsilon < 0$. Then, for every $B \in \mathcal{U}$, $\|p(A)\| - \varepsilon < \|p(B)\| < \|p(A)\| + \varepsilon$, which implies $\|p(A) - p(B)\| < \varepsilon$. Hence, $p$ is continuous at $A$.

For each $r > 0$, $p^{-1}(rB^n)$ is a closed subset of

\[
\text{Conv}_F(\mathbb{R}^n) \setminus (\mathbb{R}^n \setminus rB^n)^+ = \text{Conv}_F^*(\mathbb{R}^n) \setminus (\mathbb{R}^n \setminus rB^n)^+,
\]

which is compact by Proposition 1. Then $p^{-1}(rB^n)$ is also compact. It follows that $p$ is proper. $\blacksquare$

**Proposition 3.** There is a homotopy $h : \text{Conv}_F^*(\mathbb{R}^n) \times I \to \text{Conv}_F^*(\mathbb{R}^n)$ such that $h_0 = \text{id}$, $h_1 = p^*$, $h_t|\alpha \mathbb{R}^n = \text{id}$ and $p^*h_t = p^*$ for every $t \in I$,

\[
h(t \{\emptyset\} \times I) = \{\emptyset\} \quad \text{and} \quad h(\text{Conv}(\mathbb{R}^n) \times (0, 1]) \subset \text{CC}(\mathbb{R}^n).
\]

Thus, $\alpha \mathbb{R}^n$ (resp. $\mathbb{R}^n$) is a strong deformation retract of $\text{Conv}_F^*(\mathbb{R}^n)$ (resp. $\text{Conv}_F(\mathbb{R}^n)$), $\text{CC}^*(\mathbb{R}^n)$ (resp. $\text{CC}(\mathbb{R}^n)$) is homotopy dense in $\text{Conv}_F^*(\mathbb{R}^n)$ (resp. $\text{Conv}_F(\mathbb{R}^n)$) and each fiber of $p^*$ is contractible (hence $p^*$ is a $CE$-map).

**Proof.** The desired homotopy $h$ is defined as follows:

\[
h_0 = \text{id}, \quad h(t \{\emptyset\} \times I) = \{\emptyset\}, \quad h_t(A) = A \cap \left(p(A) + \frac{1-t}{t}B^n\right)
\]

for $A \in \text{Conv}(\mathbb{R}^n)$ and $t > 0$.

Obviously, $h$ satisfies the desired conditions. It remains to verify the continuity of $h$. Since $p(h_t(A)) = p(A)$ for all $A \in \text{Conv}(\mathbb{R}^n)$ and $t \in I$,

\[
h^{-1}((\mathbb{R}^n \setminus rB^n)^+) = (\mathbb{R}^n \setminus rB^n)^+ \times I \quad \text{for} \, r > 0,
\]

hence $h$ is continuous at $(\emptyset, t)$.

Let $A \in \text{Conv}(\mathbb{R}^n)$ and $t \in I$. Assume that $K \subseteq \mathbb{R}^n$ is compact and $h_t(A) \cap K = \emptyset$. When $t = 0$, $V = (\mathbb{R}^n \setminus K)^+ \cap \text{Conv}(\mathbb{R}^n)$ is a neighborhood
of $A$ in $\text{Conv}_F(\mathbb{R}^n)$ and $h_s(B) \cap K = \emptyset$ for all $B \in \mathcal{V}$ and $s \in I$. In case $t > 0$, choose $0 < \varepsilon < t/2$ so that

$$K \cap A \cap \left( p(A) + \frac{1 - t + 2 \varepsilon}{t - 2 \varepsilon} B^n \right) = \emptyset.$$  

Since $p$ is continuous, $A$ has a neighborhood $\mathcal{U}$ in $\text{Conv}(\mathbb{R}^n)$ such that $B \in \mathcal{U}$ implies

$$\|p(A) - p(B)\| < \frac{1 - t + 2 \varepsilon}{t - 2 \varepsilon} - \frac{1 - t + \varepsilon}{t - \varepsilon},$$  

and then for $s > t - \varepsilon$,

$$p(B) + \frac{1 - s}{s} B^n \subset p(B) + \frac{1 - t + \varepsilon}{t - \varepsilon} B^n \subset p(A) + \frac{1 - t + 2 \varepsilon}{t - 2 \varepsilon} B^n.$$

Thus, $A$ has the following neighborhood in $\text{Conv}_F(\mathbb{R}^n)$:

$$\mathcal{V} = \mathcal{U} \cap \left( \mathbb{R}^n \setminus \left( K \cap \left( p(A) + \frac{1 - t + 2 \varepsilon}{t - 2 \varepsilon} B^{n-1} \right) \right) \right)^+. $$

Then $h_s(B) \cap K = \emptyset$ for every $B \in \mathcal{V}$ and $s > t - \varepsilon$.

Next, assume $U \subset \mathbb{R}^n$ is open and $h_t(A) \cap U \neq \emptyset$. When $t = 1$, $p(A) \in U$. By continuity of $p$, $\mathcal{V} = p^{-1}(U)$ is a neighborhood of $A$ in $\text{Conv}_F(\mathbb{R}^n)$, and $p(B) \in h_s(B) \cap U$ for all $B \in \mathcal{V}$. In case $t < 1$, choose $0 < \varepsilon < (1 - t)/2$ so that

$$U \cap A \cap \left( p(A) + \frac{1 - t - 2 \varepsilon}{t + 2 \varepsilon} B^n \right) \neq \emptyset.$$  

We have a neighborhood $\mathcal{U}$ of $A$ in $\text{Conv}_F(\mathbb{R}^n)$ such that $B \in \mathcal{U}$ implies

$$\|p(A) - p(B)\| < \frac{1 - t - \varepsilon}{t + \varepsilon} - \frac{1 - t - 2 \varepsilon}{t + 2 \varepsilon},$$  

and then for $s < t + \varepsilon$,

$$p(A) + \frac{1 - t - 2 \varepsilon}{t + 2 \varepsilon} B^n \subset p(B) + \frac{1 - t - \varepsilon}{t + \varepsilon} B^n \subset p(B) + \frac{1 - s}{s} B^n.$$  

Thus, $\mathcal{V} = \mathcal{U} \cap U^-$ is a neighborhood of $A$ in $\text{Conv}_F(\mathbb{R}^n)$ and $h_s(B) \cap U \neq \emptyset$ for every $B \in \mathcal{V}$ and $s < t + \varepsilon$. ■

A separable metrizable space $M$ is called a **Hilbert cube manifold** or a **$Q$-manifold** if each point of $M$ has an open neighborhood which is homeomorphic to an open set in $Q$.

**Corollary 4.** For every $n > 1$, $\text{Conv}_F(\mathbb{R}^n)$ is a $Q$-manifold.

*Proof.* As observed in Remark 2, $\text{CC}_F(\mathbb{R}^n) = \text{CC}_V(\mathbb{R}^n) \approx Q \setminus \{0\}$ for every $n > 1$. Since $\text{CC}_F(\mathbb{R}^n)$ is homotopy dense in $\text{Conv}_F(\mathbb{R}^n)$ by Proposition 3, we can apply the Toruńczyk characterization of $Q$-manifolds [8] to show that $\text{Conv}_F(\mathbb{R}^n)$ is a $Q$-manifold. ■

Now, we prove the Main Theorem.
Proof of Main Theorem. First, note that \( \mathbb{R}^n \times Q \) is a \( Q \)-manifold. Since \( p \) is a CE-map by Proposition 3, \( p \times \text{id} : \text{Conv}_F(\mathbb{R}^n) \times Q \to \mathbb{R}^n \times Q \) is a near homeomorphism by the CE Approximation Theorem \([2, 43.1]\). By the Stability Theorem \([2, 15.1]\), \( \text{Conv}_F(\mathbb{R}^n) \times Q \cong \text{Conv}_F(\mathbb{R}^n) \) \((^2)\). Then, it follows that \( \text{Conv}_F(\mathbb{R}^n) \cong \mathbb{R}^n \times Q \). Moreover, by Proposition 1, we have

\[
\text{Conv}_F^*(\mathbb{R}^n) \approx \alpha(\mathbb{R}^n \times Q) \approx (S^n \times Q)/\{\text{pt}\} \times Q).
\]

The proof is complete. \( \blacksquare \)

The following is a direct consequence of the above proof:

Corollary 5. For each \( n \in \mathbb{N} \), \( \text{Conv}_F^*(\mathbb{R}^n) \) has the unique singular point \( \emptyset \) and \( \text{Conv}_F^*(\mathbb{R}^n) \) has the homotopy type of \( S^n \). If \( m \neq n \) then neither \( \text{Conv}_F^*(\mathbb{R}^n) \cong \text{Conv}_F^*(\mathbb{R}^m) \) nor \( \text{Conv}_F(\mathbb{R}^n) \cong \text{Conv}_F(\mathbb{R}^m) \). \( \blacksquare \)

References


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\(^2\) For non-compact \( Q \)-manifolds, the book \([3]\) is not sufficient—one should refer to Chapman’s lecture notes \([2]\).