On the Lifshits Constant for Hyperspaces

by

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Summary. The Lifshits theorem states that any $k$-uniformly Lipschitz map with a bounded orbit on a complete metric space $X$ has a fixed point provided $k < \kappa(X)$ where $\kappa(X)$ is the so-called Lifshits constant of $X$. For many spaces we have $\kappa(X) > 1$. It is interesting whether we can use the Lifshits theorem in the theory of iterated function systems. Therefore we investigate the value of the Lifshits constant for several classes of hyperspaces.

1. Preliminaries. The standard method for showing that an iterated function system admits a unique invariant set makes use of the Banach contraction principle applied to the hyperspace (see [Hu]). But this method fails for noncontractive systems. Therefore one looks for other fixed point principles ([H], [Ha], [E], [LM], [L1], [O]). In [AG] the authors propose to use the Lifshits fixed point theorem for this purpose. This leads to the problem of calculation of the Lifshits constant for hyperspaces. In this article we investigate the Lifshits constant for several classes of hyperspaces, improving and extending our earlier results from [L2].

Let $(X, d)$ be a metric space. We denote by $D(x, r)$ the closed $r$-ball with center at $x$, and by diam $A$ the diameter of $A \subset X$. Let $c \geq 1$. We shall say that balls are $c$-regular in $X$ if

$$\forall k < c \exists \eta, \alpha \in (0, 1) \forall x, y \in X \forall r > 0 \quad [d(x, y) \geq (1 - \eta)r \Rightarrow \exists z \in X \quad D(x, (1 + \eta)r) \cap D(y, k(1 + \eta)r) \subset D(z, \alpha r)].$$

Set

$$\kappa(X) = \sup\{c \geq 1 : \text{balls are } c\text{-regular}\}.$$
Then \( \kappa(X) \) is called the \textit{Lifshits constant} (or the \textit{Lifshits characteristic}) of \( X \). Below we recall the Lifshits theorem.

**Theorem 1.** Let \( (X, d) \) be a complete metric space. Let \( f : X \to X \) be a \( k \)-uniformly Lipschitz map:

\[
\forall x, y \in X \quad \forall n \in \mathbb{N} \quad d[ f^n(x), f^n(y) ] \leq kd(x, y),
\]

where \( k < \kappa(X) \) and \( f^n \) stands for the \( n \)-fold composition. If there exists \( x_0 \) such that the orbit \( \{ f^n(x_0) : n \in \mathbb{N} \} \) is bounded, then \( f \) has a fixed point.

This theorem is nontrivial because for many spaces we have \( \kappa(X) > 1 \); e.g. if \( X \) is a Hilbert space then \( \kappa(X) = \sqrt{2} \). More details on the Lifshits constant can be found in [GK], [AT], [LF]. Some interesting modifications of this geometric constant are given in [WW].

For the rest of the paper the following technical lemma will play a crucial role.

**Lemma 1.** Let \( (X, d) \) be a metric space and let \( p_1, p_2, q_1, q_2 \in X \) have the following distances: \( d(p_1, p_2) = d(q_1, p_j) = r, i, j = 1, 2, d(q_1, q_2) = 2r \). Then \( \kappa(X) = 1 \).

**Proof.** Put in (1) \( \eta = 0, k = 1, x = p_1, y = p_2 \). If \( D(p_1, r) \cap D(p_2, r) \subset D(z, \alpha r) \) for some \( z \in X, \alpha \in (0, 1) \), then \( \text{diam}[D(p_1, r) \cap D(p_2, r)] \leq \text{diam} D(z, \alpha r) \leq 2\alpha r < 2r \). On the other hand, since \( q_1, q_2 \in D(p_1, r) \cap D(p_2, r) \), we get \( \text{diam}[D(p_1, r) \cap D(p_2, r)] \geq 2r \). This contradiction shows that balls are never \( c \)-regular for any choice of \( c > k \geq 1 \). \( \blacksquare \)

2. **Hyperspaces with Hausdorff and Pompeiu metrics.** We denote by \( \mathcal{F}_2(X) \) the family of all nonempty subsets of \( X \) with at most two elements, \( \mathcal{F}_b(X) \) the family of all nonempty closed bounded subsets of \( X \), \( \mathcal{F}_c(X) \) the family of all nonempty closed bounded connected subsets of \( X \), \( \mathcal{F}_k(X) \) the family of all nonempty closed bounded convex subsets of \( X \), \( K(X) \) the family of all nonempty compact subsets of \( X \), \( K_c(X) \) the family of all nonempty compact connected subsets of \( X \), and \( K_k(X) \) the family of all nonempty compact convex subsets of \( X \). The family \( \mathcal{F}_2(X) \) is also referred to as the 2-fold symmetric product of \( X \). A topologized family of sets is called a \textit{hyperspace}.

Let \( A, B \subseteq X \). The \textit{excess} of the set \( A \) over \( B \) is

\[
e(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).
\]

The \textit{Hausdorff distance} between \( A \) and \( B \) is

\[
d_H(A, B) = \max \{ e(A, B), e(B, A) \},
\]

and the \textit{Pompeiu distance} is

\[
d_p(A, B) = e(A, B) + e(B, A).
\]
Recall that $\mathcal{F}_b(X)$ equipped with either $d_H$ or $d_P$ is a metric space. The geometric theory of hyperspaces is presented in [IN].

**Theorem 2.** Let $X$ be a metric space containing at least three points $x_0, x_1, x_2$ which satisfy $0 < d(x_0, x_1) = d(x_1, x_2) = \frac{1}{2} d(x_0, x_2)$. Then $\kappa(\mathcal{F}_2(X), d_H) = 1 = \kappa(\mathcal{F}_2(X), d_P)$.

**Proof.** Set $r = d(x_0, x_1)$. In the case of the Hausdorff metric $d_H$ take $P_1 = \{x_1\}, P_2 = \{x_1, x_2\}, Q_1 = \{x_0, x_2\}, Q_2 = \{x_2\}$. Then

$$
e(P_1, P_2) = 0, e(P_2, P_1) = r; \quad e(Q_1, Q_2) = 2r, e(Q_2, Q_1) = 0;$$

$$e(Q_1, P_1) = r = e(P_1, Q_1); \quad e(P_2, Q_2) = r, e(Q_2, P_2) = 0;$$

$$e(Q_1, P_2) = r = e(P_2, Q_1); \quad e(Q_2, P_1) = r = e(P_1, Q_2).$$

Therefore we can use Lemma 1.

In the case of the Pompeiu metric $d_P$ by taking $P_1 = \{x_1\}, P_2 = \{x_0, x_1, x_2\}, Q_1 = \{x_0, x_1\}, Q_2 = \{x_1, x_2\}$ we get

$$e(P_1, P_2) = 0, e(P_2, P_1) = r; \quad e(Q_1, Q_2) = r = e(Q_2, Q_1);$$

$$e(Q_1, P_1) = r, e(P_1, Q_1) = 0; \quad e(P_2, Q_2) = r, e(Q_2, P_2) = 0;$$

$$e(Q_1, P_2) = 0, e(P_2, Q_1) = r; \quad e(Q_2, P_1) = r, e(P_1, Q_2) = 0.$$

Again Lemma 1 applies.

The above also holds true for $\mathcal{K}(X)$ which is larger than $\mathcal{F}_2(X)$.

**Theorem 3.** Each of the following hyperspaces of a normed space $X$ has Lifshits constant 1: $\mathcal{F}_b(X), \mathcal{F}_c(X), \mathcal{K}(X), \mathcal{K}_c(X), \mathcal{K}_k(X)$ when endowed with $d_H$ or $d_P$.

**Proof.** Fix $0 < r < 1/3$ and some vector $v$ with $\|v\| = 1$. Put

$$P_1 = \{tv : t \in [r, 1]\}, \quad P_2 = \{tv : t \in [r, 1 - r]\},$$

$$Q_1 = \{tv : t \in [0, 1]\}, \quad Q_2 = \{tv : t \in [2r, 1 - r]\}.$$

Then we have

$$e(P_1, P_2) = r, \quad e(Q_1, Q_2) = 2r,$$

$$e(P_1, Q_1) = e(P_2, Q_2) = e(Q_1, P_1) = e(Q_1, P_2) = r,$$

and 0 for the reverse excesses. Thus (regardless of the metric in the hyperspace) the assumptions of Lemma 1 are fulfilled and the assertion follows.

We remark that the proof of the above result given in [L2] for $\mathcal{K}(X)$, $\mathcal{K}_c(X)$ and $\mathcal{K}_k(X)$ relied on the compactness of the unit ball in the normed space.

**3. Other metrizations.** Instead of the Hausdorff or Pompeiu metric we now consider some of their variations. First, by analogy with the $l^p$-product,
we define the following distance between \( A, B \subset X \):
\[
d_{M^p}(A, B) = (e(A, B)^p + e(B, A)^p)^{1/p},
\]
where \( p \in [1, \infty) \). Of course \( d_{M^1} = d_p \).

**Theorem 4.** Each of the hyperspaces \( \mathcal{F}_b(X), \mathcal{F}_c(X), \mathcal{K}(X), \mathcal{K}_c(X), \mathcal{K}_k(X) \) of a normed space \( X \) endowed with \( d_{M^p} \) has Lifshits constant 1.

**Proof.** The argument for Theorem 3 can be repeated here. Indeed,
\[
(3) \quad Q_2 \subset P_2 \subset P_1 \subset Q_1
\]
for the sets defined there, so one only has to observe that
\[
(4) \quad d_{M^p}(A, B) = d_H(A, B)
\]
whenever \( A \subset B \subset X \).

**Theorem 5.** Let \( (X, d) \) be a metric space containing four points \( x_0, x_1, x_2, x_3 \) which satisfy \( d(x_i, x_{i+1}) = r, i = 0, 1, 2, d(x_i, x_{i+2}) = 2r, i = 0, 1, d(x_0, x_3) = 3r \) for some \( r > 0 \). Then \( \kappa(\mathcal{F}_2(X), d_{M^p}) = 1 = \kappa(\mathcal{K}(X), d_{M^p}) \).

**Proof.** Put \( Q_1 = \{x_0, x_1, x_2, x_3\}, Q_2 = \{x_2, x_3\}, P_1 = \{x_1, x_2, x_3\}, P_2 = \{x_1, x_2\} \). Then (2) and (3) hold for our sets. Hence by (4) we obtain the same conclusion as in Theorem 3.

The above argument is also valid for \( d_H \), but involves slightly stronger geometric assumptions on \( X \) than those in Theorem 2.

Now, as suggested by [KST], we introduce the \( \psi \)-distance between \( A, B \subset X \) as
\[
d_{\psi}(A, B) = \psi(e(A, B), e(B, A)),
\]
where \( \psi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) is any function making \( d_\psi \) a metric in \( \mathcal{F}_b(X) \). In particular \( \psi(\alpha, \beta) = (\alpha^p + \beta^p)^{1/p} \) gives \( d_\psi = d_{M^p} \).

If \( \psi(\alpha, 0) = \alpha = \psi(0, \alpha) \), then the reasoning from Theorem 4 is also valid for \( d_\psi \). Therefore hyperspaces metrized with \( d_\psi \) turn out to have Lifshits constant 1.

Let \( A, B \subset X \). The *spread* of the set \( A \) over \( B \) is
\[
s(A, B) = \inf \{\sup_{a \in A} d(a, f(a)) : f : A \rightarrow B \text{ is continuous} \}.
\]
The Borsuk distance of continuity between \( A \) and \( B \) is
\[
d_B(A, B) = \max \{s(A, B), s(B, A)\}.
\]
One can describe the excess functional \( e \) in an analogous way to \( s \), namely by omitting the requirement of continuity. In particular \( s(A, B) \geq e(B, A) \) and \( d_B \geq d_H \). More details on the Borsuk metric can be found e.g. in [B1], [B2], [G] and [M] (the last two works are devoted to the applications of the Borsuk metric in the fixed point theory of multivalued mappings).
Since $d_B$ coincides with $d_H$ on the family of finite sets, we immediately see that under the hypotheses of either Theorem 2 or 5, $\varkappa(F_2(X), d_B) = 1 = \varkappa(K(X), d_B)$.

Let $X$ be a normed space and $v \in X$ a vector with $\|v\| = 1$. We put
\[ I = \{tv : t \in [a_1, a_2]\}, \quad J = \{tv : t \in [b_1, b_2]\}, \]
where $a_1 < a_2$ and $b_1 < b_2$ are reals. Then, as is well known, $d_H(I, J) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$. Moreover,
\[ d_B(I, J) = d_H(I, J). \tag{5} \]
To see this, simply take the affine map $f : I \to J$ given by
\[ f(tv) = (\alpha t + \beta) \cdot v, \quad \alpha = \frac{b_2 - b_1}{a_2 - a_1}, \quad \beta = b_1 - \alpha a_1 \]
and calculate
\[ s(I, J) \leq \sup_{a \in I} \|a - f(a)\| = \max\{|a_1 - b_1|, |a_2 - b_2|\}. \]
Finally, by (5) and the proof of Theorem 3, we find that $\varkappa(K_c(X), d_B) = 1 = \varkappa(K_k(X), d_B)$.

4. Final remarks. The following question (asked by L. Górniewicz and J. Andres) arises: Are there any “natural” hyperspaces with the Lifshits constant strictly greater than 1? As we argued at the beginning, such spaces could be interesting for fixed point theory, especially the theory of iterated function systems. We do not know a satisfactory answer to this question. Of course the family of singletons $F_1(X) = \{\{x\} : x \in X\}$ under any of the metrics above considered here is isometric to $X$, so $\varkappa(F_1(X)) = \varkappa(X)$. We provide an example of a hyperspace different from $F_1(X)$.

Example. Let $\varepsilon > 0$ and $\mathcal{H} = \{[a, a + \varepsilon] \subset [0, 1] : a \in [0, 1 - \varepsilon]\}$. Then $\varkappa(\mathcal{H}, d_H) = 2$. Indeed, $\mathcal{H}$ is isometric to $[0, 1 - \varepsilon]$ and $\varkappa([0, 1 - \varepsilon]) = 2$.

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Received December 19, 2006 (7572)