BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 55, No. 2, 2007

GENERAL TOPOLOGY

On the Lifshits Constant for Hyperspaces

by

K. LEŚNIAK

Presented by Czesław BESSAGA

Summary. The Lifshits theorem states that any k-uniformly Lipschitz map with a bounded orbit on a complete metric space X has a fixed point provided $k < \varkappa(X)$ where $\varkappa(X)$ is the so-called Lifshits constant of X. For many spaces we have $\varkappa(X) > 1$. It is interesting whether we can use the Lifshits theorem in the theory of iterated function systems. Therefore we investigate the value of the Lifshits constant for several classes of hyperspaces.

1. Preliminaries. The standard method for showing that an iterated function system admits a unique invariant set makes use of the Banach contraction principle applied to the hyperspace (see [Hu]). But this method fails for noncontractive systems. Therefore one looks for other fixed point principles ([H], [Ha], [E], [LM], [L1], [O]). In [AG] the authors propose to use the Lifshits fixed point theorem for this purpose. This leads to the problem of calculation of the Lifshits constant for hyperspaces. In this article we investigate the Lifshits constant for several classes of hyperspaces, improving and extending our earlier results from [L2].

Let (X,d) be a metric space. We denote by $D(x,r)$ the closed r-ball with center at x, and by $\operatorname{diam} A$ the diameter of $A \subset X$. Let $c \geq 1$. We shall say that *balls are c-regular* in X if

$$(1)\qquad \forall k < c\, \exists \eta, \alpha \in (0,1)\, \forall x, y \in X\, \forall r > 0 \quad [d(x,y) \geq (1-\eta)r \Rightarrow \exists z \in X \\ D(x,(1+\eta)r) \cap D(y,k(1+\eta)r) \subset D(z,\alpha r)].$$

Set

$$\varkappa(X) = \sup\{c \geq 1 : \text{balls are } c\text{-regular}\}.$$

2000 *Mathematics Subject Classification*: 54B20, 51K99, 46B20.
Key words and phrases: Lifshits constant, hyperspace, Hausdorff metric, Pompeiu metric, ℓ^p-product, ψ-product.
Partially supported by the Nicolaus Copernicus University (grant no. 467-M).

Then $\varkappa(X)$ is called the *Lifshits constant* (or the *Lifshits characteristic*) of X. Below we recall the Lifshits theorem.

THEOREM 1. *Let (X,d) be a complete metric space. Let $f : X \to X$ be a k-uniformly Lipschitz map*:

$$\forall x, y \in X \ \forall n \in \mathbb{N} \quad d[\, f^n(x), f^n(y)\,] \le kd(x,y),$$

where $k < \varkappa(X)$ and f^n stands for the n-fold composition. If there exists x_0 such that the orbit $\{f^n(x_0) : n \in \mathbb{N}\}$ is bounded, then f has a fixed point.

This theorem is nontrivial because for many spaces we have $\varkappa(X) > 1$; e.g. if X is a Hilbert space then $\varkappa(X) = \sqrt{2}$. More details on the Lifshits constant can be found in [GK], [AT], [LF]. Some interesting modifications of this geometric constant are given in [WW].

For the rest of the paper the following technical lemma will play a crucial role.

LEMMA 1. *Let (X,d) be a metric space and let $p_1, p_2, q_1, q_2 \in X$ have the following distances*: $d(p_1,p_2) = d(q_i,p_j) = r$, $i,j = 1,2$, $d(q_1,q_2) = 2r$. *Then $\varkappa(X) = 1$.*

Proof. Put in (1) $\eta = 0$, $k = 1$, $x = p_1$, $y = p_2$. If $D(p_1,r) \cap D(p_2,r) \subset D(z,\alpha r)$ for some $z \in X$, $\alpha \in (0,1)$, then $\operatorname{diam}[D(p_1,r) \cap D(p_2,r)] \le \operatorname{diam} D(z,\alpha r) \le 2\alpha r < 2r$. On the other hand, since $q_1, q_2 \in D(p_1,r) \cap D(p_2,r)$, we get $\operatorname{diam}[D(p_1,r) \cap D(p_2,r)] \ge 2r$. This contradiction shows that balls are never c-regular for any choice of $c > k \ge 1$. ∎

2. Hyperspaces with Hausdorff and Pompeiu metrics. We denote by $\mathcal{F}_2(X)$ the family of all nonempty subsets of X with at most two elements, $\mathcal{F}_b(X)$ the family of all nonempty closed bounded subsets of X, $\mathcal{F}_c(X)$ the family of all nonempty closed bounded connected subsets of X, $\mathcal{F}_k(X)$ the family of all nonempty closed bounded convex subsets of X, $\mathcal{K}(X)$ the family of all nonempty compact subsets of X, $\mathcal{K}_c(X)$ the family of all nonempty compact connected subsets of X, and $\mathcal{K}_k(X)$ the family of all nonempty compact convex subsets of X. The family $\mathcal{F}_2(X)$ is also referred to as the 2-fold symmetric product of X. A topologized family of sets is called a *hyperspace*.

Let $A, B \subset X$. The *excess* of the set A over B is

$$e(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b).$$

The *Hausdorff distance* between A and B is

$$d_{\mathrm{H}}(A,B) = \max\{e(A,B), e(B,A)\},$$

and the *Pompeiu distance* is

$$d_{\mathrm{P}}(A,B) = e(A,B) + e(B,A).$$

Recall that $\mathcal{F}_b(X)$ equipped with either d_{H} or d_{P} is a metric space. The geometric theory of hyperspaces is presented in [IN].

THEOREM 2. *Let X be a metric space containing at least three points x_0, x_1, x_2 which satisfy $0 < d(x_0,x_1) = d(x_1,x_2) = \frac{1}{2}d(x_0,x_2)$. Then $\varkappa(\mathcal{F}_2(X), d_{\mathrm{H}}) = 1 = \varkappa(\mathcal{F}_2(X), d_{\mathrm{P}})$.*

Proof. Set $r = d(x_0,x_1)$. In the case of the Hausdorff metric d_{H} take $P_1 = \{x_1\}$, $P_2 = \{x_1,x_2\}$, $Q_1 = \{x_0,x_2\}$, $Q_2 = \{x_2\}$. Then

$$\begin{aligned} &e(P_1,P_2) = 0,\ e(P_2,P_1) = r; && e(Q_1,Q_2) = 2r,\ e(Q_2,Q_1) = 0;\\ &e(Q_1,P_1) = r = e(P_1,Q_1); && e(P_2,Q_2) = r,\ e(Q_2,P_2) = 0;\\ &e(Q_1,P_2) = r = e(P_2,Q_1); && e(Q_2,P_1) = r = e(P_1,Q_2). \end{aligned}$$

Therefore we can use Lemma 1.

In the case of the Pompeiu metric d_{P} by taking $P_1 = \{x_1\}$, $P_2 = \{x_0,x_1,x_2\}$, $Q_1 = \{x_0,x_1\}$, $Q_2 = \{x_1,x_2\}$ we get

$$\begin{aligned} &e(P_1,P_2) = 0,\ e(P_2,P_1) = r; && e(Q_1,Q_2) = r = e(Q_2,Q_1);\\ &e(Q_1,P_1) = r,\ e(P_1,Q_1) = 0; && e(P_2,Q_2) = r,\ e(Q_2,P_2) = 0;\\ &e(Q_1,P_2) = 0,\ e(P_2,Q_1) = r; && e(Q_2,P_1) = r,\ e(P_1,Q_2) = 0. \end{aligned}$$

Again Lemma 1 applies. ∎

The above also holds true for $\mathcal{K}(X)$ which is larger than $\mathcal{F}_2(X)$.

THEOREM 3. *Each of the following hyperspaces of a normed space X has Lifshits constant* 1: $\mathcal{F}_b(X)$, $\mathcal{F}_c(X)$, $\mathcal{K}(X)$, $\mathcal{K}_c(X)$, $\mathcal{K}_k(X)$ *when endowed with d_{H} or d_{P}.*

Proof. Fix $0 < r < 1/3$ and some vector v with $\|v\| = 1$. Put

$$\begin{aligned} P_1 &= \{tv : t \in [r,1]\}, & P_2 &= \{tv : t \in [r,1-r]\},\\ Q_1 &= \{tv : t \in [0,1]\}, & Q_2 &= \{tv : t \in [2r,1-r]\}. \end{aligned}$$

Then we have

$$\left\{\begin{array}{l} e(P_1,P_2) = r, \quad e(Q_1,Q_2) = 2r,\\ e(P_1,Q_2) = e(P_2,Q_2) = e(Q_1,P_1) = e(Q_1,P_2) = r, \end{array}\right. \tag{2}$$

and 0 for the reverse excesses. Thus (regardless of the metric in the hyperspace) the assumptions of Lemma 1 are fulfilled and the assertion follows. ∎

We remark that the proof of the above result given in [L2] for $\mathcal{K}(X)$, $\mathcal{K}_c(X)$ and $\mathcal{K}_k(X)$ relied on the compactness of the unit ball in the normed space.

3. Other metrizations. Instead of the Hausdorff or Pompeiu metric we now consider some of their variations. First, by analogy with the ℓ^p-product, we define the following distance between $A, B \subset X$:

$$d_{M^p}(A,B) = (e(A,B)^p + e(B,A)^p)^{1/p},$$

where $p \in [1,\infty)$. Of course $d_{M^1} = d_{\mathrm{P}}$.

THEOREM 4. *Each of the hyperspaces $\mathcal{F}_b(X)$, $\mathcal{F}_c(X)$, $\mathcal{K}(X)$, $\mathcal{K}_c(X)$, $\mathcal{K}_k(X)$ of a normed space X endowed with d_{M^p} has Lifshits constant* 1.

Proof. The argument for Theorem 3 can be repeated here. Indeed,

$$Q_2 \subset P_2 \subset P_1 \subset Q_1 \tag{3}$$

for the sets defined there, so one only has to observe that

$$d_{M^p}(A,B) = d_{\mathrm{H}}(A,B) \tag{4}$$

whenever $A \subset B \subset X$. ∎

THEOREM 5. *Let (X,d) be a metric space containing four points x_0, x_1, x_2, x_3 which satisfy $d(x_i, x_{i+1}) = r$, $i = 0,1,2$, $d(x_i, x_{i+2}) = 2r$, $i = 0,1$, $d(x_0, x_3) = 3r$ for some $r > 0$. Then $\varkappa(\mathcal{F}_2(X), d_{M^p}) = 1 = \varkappa(\mathcal{K}(X), d_{M^p})$.*

Proof. Put $Q_1 = \{x_0, x_1, x_2, x_3\}$, $Q_2 = \{x_2, x_3\}$, $P_1 = \{x_1, x_2, x_3\}$, $P_2 = \{x_1, x_2\}$. Then (2) and (3) hold for our sets. Hence by (4) we obtain the same conclusion as in Theorem 3. ∎

The above argument is also valid for d_{H}, but involves slightly stronger geometric assumptions on X than those in Theorem 2.

Now, as suggested by [KST], we introduce the ψ-distance between $A, B \subset X$ as

$$d_\psi(A,B) = \psi(e(A,B), e(B,A)),$$

where $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is any function making d_ψ a metric in $\mathcal{F}_b(X)$. In particular $\psi(\alpha,\beta) = (\alpha^p + \beta^p)^{1/p}$ gives $d_\psi = d_{M^p}$.

If $\psi(\alpha, 0) = \alpha = \psi(0,\alpha)$, then the reasoning from Theorem 4 is also valid for d_ψ. Therefore hyperspaces metrized with d_ψ turn out to have Lifshits constant 1.

Let $A, B \subset X$. The *spread* of the set A over B is

$$s(A,B) = \inf\{\sup_{a\in A} d(a, f(a)) : f : A \to B \text{ is continuous}\}.$$

The *Borsuk distance of continuity* between A and B is

$$d_{\mathrm{B}}(A,B) = \max\{s(A,B), s(B,A)\}.$$

One can describe the excess functional e in an analogous way to s, namely by omitting the requirement of continuity. In particular $s(A,B) \geq e(B,A)$ and $d_{\mathrm{B}} \geq d_{\mathrm{H}}$. More details on the Borsuk metric can be found e.g. in [B1], [B2], [G] and [M] (the last two works are devoted to the applications of the Borsuk metric in the fixed point theory of multivalued mappings).

Since d_{B} coincides with d_{H} on the family of finite sets, we immediately see that under the hypotheses of either Theorem 2 or 5, $\varkappa(\mathcal{F}_2(X), d_{\mathrm{B}}) = 1 = \varkappa(\mathcal{K}(X), d_{\mathrm{B}})$.

Let X be a normed space and $v \in X$ a vector with $\|v\| = 1$. We put

$$I = \{tv : t \in [a_1, a_2]\}, \qquad J = \{tv : t \in [b_1, b_2]\},$$

where $a_1 < a_2$ and $b_1 < b_2$ are reals. Then, as is well known, $d_{\mathrm{H}}(I, J) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$. Moreover,

$$d_{\mathrm{B}}(I, J) = d_{\mathrm{H}}(I, J). \tag{5}$$

To see this, simply take the affine map $f : I \to J$ given by

$$f(tv) = (\alpha t + \beta) \cdot v, \qquad \alpha = \frac{b_2 - b_1}{a_2 - a_1},\ \beta = b_1 - \alpha\, a_1$$

and calculate

$$s(I, J) \le \sup_{a \in I} \|a - f(a)\| = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Finally, by (5) and the proof of Theorem 3, we find that $\varkappa(\mathcal{K}_c(X), d_{\mathrm{B}}) = 1 = \varkappa(\mathcal{K}_k(X), d_{\mathrm{B}})$.

4. Final remarks. The following question (asked by L. Górniewicz and J. Andres) arises: Are there any "natural" hyperspaces with the Lifshits constant strictly greater than 1? As we argued at the beginning, such spaces could be interesting for fixed point theory, especially the theory of iterated function systems. We do not know a satisfactory answer to this question. Of course the family of singletons $\mathcal{F}_1(X) = \{\{x\} : x \in X\}$ under any of the metrics above considered here is isometric to X, so $\varkappa(\mathcal{F}_1(X)) = \varkappa(X)$. We provide an example of a hyperspace different from $\mathcal{F}_1(X)$.

Example. Let $\varepsilon > 0$ and $\mathcal{H} = \{[a, a + \varepsilon] \subset [0, 1] : a \in [0, 1 - \varepsilon]\}$. Then $\varkappa(\mathcal{H}, d_{\mathrm{H}}) = 2$. Indeed, $\mathcal{H}$ is isometric to $[0, 1 - \varepsilon]$ and $\varkappa([0, 1 - \varepsilon]) = 2$.

References

[AG] J. Andres and L. Górniewicz, *On the Banach contraction principle for multivalued mappings*, in: Approximation, Optimization and Mathematical Economics, M. Lassonde (ed.), Physica-Verlag, 2001, 1–23.

[AT] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel, 1997.

[B1] K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fund. Math. 41 (1955), 168–202.

[B2] —, *On a metrization of the hyperspace of a metric space*, ibid. 94 (1977), 191–207.

[E] A. Edalat, *Dynamical systems, measures and fractals via domain theory*, Inform. Comput. 120 (1995), 32–48.

[GK] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, 1990.

[G] L. Górniewicz, *Present state of the Brouwer fixed point theorem for multivalued mappings*, Ann. Sci. Math. Québec 22 (1998), 169–179.

[H] M. Hata, *On some properties of set-dynamical systems*, Proc. Japan Acad. Ser. A 61 (1985), 99–102.

[Ha] S. Hayashi, *Self-similar sets as Tarski's fixed points*, Publ. RIMS Kyoto Univ. 21 (1985), 1059–1066.

[Hu] J. E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. 30 (1981), 713–747.

[IN] A. Illanes and S. B. Nadler Jr., *Hyperspaces. Fundamentals and Recent Advances*, Dekker, New York, 1999.

[KST] M. Kato, K.-S. Saito and T. Tamura, *Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, Math. Inequal. Appl. 7 (2004), 429–437.

[LM] A. Lasota and J. Myjak, *Attractors of multifunctions*, Bull. Polish Acad. Sci. Math. 48 (2000), 319–334.

[L1] K. Leśniak, *Extremal sets as fractals*, Nonlinear Anal. Forum 7 (2002), 199–208.

[L2] —, *Towards computing Lifshits constant for hyperspaces*, Fixed Point Theory 4 (2003), 159–163.

[LF] E. Llorens Fuster, *Moduli and constants... what a show!*, http://www.uv.es/llorens, 2006; extended version of: *Some moduli and constants related to metric fixed point theory*, in: Handbook of Metric Fixed Point Theory, W. A. Kirk and B. Sims (eds.), Kluwer, Dordrecht, 2001, 133–175.

[M] D. Miklaszewski, *The Role of Various Kinds of Continuity in The Fixed Point Theory of Set-Valued Mappings*, Lecture Notes in Nonlinear Anal. 7, Juliusz Schauder Center for Nonlinear Studies, Nicolaus Copernicus Univ., Toruń, 2005.

[O] E. A. Ok, *Fixed set theory for closed correspondences with applications to self-similarity and games*, Nonlinear Anal. 56 (2004), 309–330.

[WW] J. Wiśnicki and J. Wośko, *On relative Hausdorff mesures of noncompactness and relative Chebyshev radii in Banach spaces*, Proc. Amer. Math. Soc. 124 (1996), 2465–2474.

K. Leśniak
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: much@mat.uni.torun.pl

Received December 19, 2006 (7572)