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Once More on the Lefschetz Fixed Point Theorem

by

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Summary. An abstract version of the Lefschetz fixed point theorem is presented. Then several generalizations of the classical Lefschetz fixed point theorem are obtained.

0. Introduction. In 1923 S. Lefschetz proved his famous theorem, now known as the Lefschetz fixed point theorem. Originally, the theorem was formulated for compact manifolds. Later, in 1928 H. Hopf gave a new proof for self-mappings of polyhedra. In 1967, A. Granas extended the Lefschetz fixed point theorem to the case of compact maps of absolute neighbourhood retracts. Until now, it has been proved for compact absorbing contraction and condensing mappings (see [4], [6], [7], [9], [10], [11], [13], [21]). In the present paper some further generalizations of this theorem are given.

1. Preliminaries. We restrict our considerations to metric spaces. Following K. Borsuk [2] we define:

DEFINITION 1.1. A space X is called an *absolute neighbourhood retract* ($X \in$ ANR) provided for every space Y and for every homeomorphism $h : X \to Y$ such that $h(X)$ is a closed subset of Y there exists an open neighbourhood U of $h(X)$ in Y and a continuous map (called a retraction) $r : U \to h(X)$ such that $r(u) = u$ for every $u \in h(X)$, i.e. $h(X)$ is a retract of U; X is called an *absolute retract* ($X \in$ AR) provided the above holds true for $U = Y$, i.e. $h(X)$ is a retract of Y.

In other words, $X \in$ ANR ($X \in$ AR) if and only if X has the neighbourhood extension property (resp. extension property) (cf. [2], [9]).

To understand better how large the classes of ANR and AR are, we recall:

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Proposition 1.2 ([2], [9]).

(1) $X \in$ ANR *if and only if there exists a normed space E and an open subset W of E such that X is homeomorphic to a retract of W;*
(2) $X \in$ AR *if and only if there exists a normed space E and a convex subset W of E such that X is homeomorphic to a retract of W.*

In particular, any open subset in a normed space or any finite polyhedron is an ANR; respectively any convex subset of an arbitrary normed space is an AR. Note that any AR is contractible and any ANR is locally contractible.

We shall consider the category of pairs of metric spaces and continuous mappings. By a pair of spaces (X, X_0) we understand a pair consisting of a metric space X and a subset $X_0 \subset X$. A pair of the form $(X, \emptyset)$ will be identified with X. By a map $f : (X, X_0) \to (Y, Y_0)$ we understand a continuous map from X to Y such that $f(X_0) \subset Y_0$. We then write $f_X : X \to Y$ and $f_{X_0} : X_0 \to Y_0$ for the mappings induced by f.

Let H be the Čech homology functor with compact carriers ([1], [7]) and coefficients in the field $\mathbb{Q}$ of rational numbers, from the category of all pairs of spaces and all maps between such pairs, to the category of graded vector spaces over and linear maps of degree zero. Thus

$$H(X, X_0) = \{H_q(X, X_0)\}$$

is a graded vector space, $H_q(X, X_0)$ being the q-dimensional Čech homology space with compact carriers and rational coefficients. For a map $f : (X, X_0) \to (Y, Y_0)$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$, where $f_{*q} : H_q(X, X_0) \to H_q(Y, Y_0)$.

A nonempty space X is called *acyclic* provided:

(i) $H_q(X) = 0$ for all $q \geq 1$,
(ii) $H_0(X) \approx \mathbb{Q}$.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Define $N(u) = \{x \in E : u^n(x) = 0 \text{ for some } n\}$, where u^n is the nth iterate of u, and set $\widetilde{E} = E/N(u)$. As $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u *admissible* provided $\dim \widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a *Leray endomorphism* if

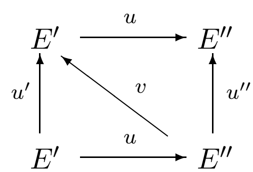
(i) all u_q are admissible,
(ii) almost all $\widetilde{E}_q$ are zero.

For such u, we define the (generalized) *Lefschetz number* $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q \operatorname{tr}(\widetilde{u}_q),$$

where tr is the ordinary trace (cf. [7] or [9]). The following important property of the Leray endomorphism is a consequence of the well-known formula $\mathrm{tr}(u \circ v) = \mathrm{tr}(v \circ u)$ for the trace.

PROPOSITION 1.3. *Assume that, in the category of graded vector spaces, the following diagram commutes:*



Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case, $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u : E \to E$ of a graded vector space E is called *weakly nilpotent* if for every $q \geq 0$ and every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly nilpotent endomorphism $u : E \to E$, we have $N(u) = E$, we get:

PROPOSITION 1.4. *If $u : E \to E$ is a weakly nilpotent endomorphism, then $\Lambda(u) = 0$.*

Let $f : (X, X_0) \to (X, X_0)$ be such that $f_* : H(X, X_0) \to H(X, X_0)$ is a Leray endomorphism. Then we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic, then $\Lambda(f)$ is defined if $\Lambda(g)$ is; and, in this case, $\Lambda(f) = \Lambda(g)$.

Observe that if X is an acyclic space, or, in particular, contractible, then for every $f : X \to X$ the endomorphism $f_* : H(X) \to H(X)$ is a Leray endomorphism and $\Lambda(f_*) = 1$.

Consequently, if $X \in \mathrm{AR}$, or in particular if X is a convex subset in a normed space, then for every continuous map $f : X \to X$ the Lefschetz number $\Lambda(f) = \Lambda(f_*)$ is 1.

We have the following lemma (see [3], [5], [7]).

LEMMA 1.5. *Let $f : (X, X_0) \to (X, X_0)$ be a map of pairs. If two of the endomorphisms $f_* : H(X, X_0) \to H(X, X_0)$, $(f_X)_* : H(X) \to H(X)$, $(f_{X_0})_* : H(X_0) \to H(X_0)$ are Leray endomorphisms, then so is the third; in that case,*

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_{X_0}).$$

We shall also use the following proposition:

PROPOSITION 1.6. *Assume that for a mapping $f : X \to X$ the Lefschetz number $\Lambda(f)$ is defined and let p be a prime number. Then $\Lambda(f^p)$ is defined and $\Lambda(f) \equiv \Lambda(f^p) \bmod p$.*

For the proof see [12] or [9].

2. Lefschetz mappings. It is convenient to introduce the following notion.

DEFINITION 2.1. A continuous map $f : X \to X$ is called a *Lefschetz map* provided the generalized Lefschetz number $\Lambda(f)$ is defined and $\Lambda(f) \neq 0$ implies that the set $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$ is nonempty.

In 1969, A. Granas [8] (see also [4], [14]) proved:

THEOREM 2.2. *Let $X \in \mathrm{ANR}$ and let $f : X \to X$ be a continuous and compact map* (*i.e.*, *$\overline{f(X)}$ is a compact set*). *Then f is a Lefschetz map.*

We shall need the Kuratowski measure of noncompactness (see [11], [7]). Let X be a complete metric space and A be a bounded subset of X. We let

$$\gamma(A) = \inf\{r > 0 : \text{there exists a finite covering of } A \text{ by subsets of diameter at most } r\}.$$

Then $\gamma(A)$ is called the *measure of noncompactness* of A.

For a map $f : X \to X$, a compact subset $A \subset X$ is called an *attractor* provided for any open neighbourhood U of A in X and for every $x \in X$ there exists $n = n_x$ such that $f^n(x) \in U$. In what follows we shall denote the family of mappings $X \to X$ with compact attractor by $\mathrm{CA}(X)$.

Note that if $f : X \to X$ has a compact attractor A, then $\mathrm{Fix}(f) \subset A$.

DEFINITION 2.3. Let X be a complete metric space and $k \in [0, 1)$. A continuous mapping $f : X \to X$ is called a *condensing* (resp. *k-set contraction*) *map* provided that if $A \subset X$ and $\gamma(A) \neq 0$, then $\gamma(f(A)) < \gamma(A)$ (resp. $\gamma(f(A)) \le k\gamma(A)$).

Of course, any compact map is a k-set contraction map for each $k \in (0, 1)$, and a k-set contraction map is a condensing map.

THEOREM 2.4 ([1], [7]). *Let U be an open subset of a Banach space E, and $f : U \to U$ a condensing map which has a compact attractor. Then f is a Lefschetz map.*

For some generalization of Theorem 2.4 see Theorem 3.9 (cf. also [6], [7], [11]).

DEFINITION 2.5. Let $f : X \to X$ be continuous and X_0 a subset of X. We shall say that X_0 *absorbs compact sets* provided for any compact set $K \subset X$ there exists a natural number $n = n_K$ such that $f^n(K) \subset X_0$.

It is easy to prove the following:

PROPOSITION 2.6. *Assume that $f : X \to X$ is a continuous map and X_0 is an open subset of X which absorbs points. If $f(X_0) \subset X_0$, then X_0 absorbs compact sets.*

Now we are able to prove a general version of the Lefschetz fixed point theorem.

THEOREM 2.7 (General version of the Lefschetz fixed point theorem). *Let $f : (X, X_0) \to (X, X_0)$ be continuous. Assume that*:

(1) $f_{X_0} : X_0 \to X_0$ *is a Lefschetz map*,
(2) $f_* : H(X, X_0) \to H(X, X_0)$ *is a weakly nilpotent endomorphism.*

Then f_X is a Lefschetz map.

Proof. First, in view of Proposition 1.4, we have $\Lambda(f) = 0$. Consequently, by Lemma 1.5, $\Lambda(f_X)$ is defined and $\Lambda(f_X) = \Lambda(f_{X_0})$.

Now $\Lambda(f_X) \neq 0$ implies $\Lambda(f_{X_0}) \neq 0$ and by assumption $\mathrm{Fix}(f_{X_0}) \neq \emptyset$. Finally, since $\mathrm{Fix}(f_{X_0}) \subset \mathrm{Fix}(f_X)$, our theorem is proved. ∎

REMARK 2.8 ([5], [7]). Observe that if X_0 absorbs compact sets or X_0 is open and absorbs points then 2.7(2) is automatically satisfied.

We refer to [1], [4], [5], [9], [10], [13], [14] for different formulations of the Lefschetz fixed point theorem.

3. Consequences and applications of Theorem 2.7. In what follows all mappings are assumed to be continuous and all spaces considered are metric.

Following [5] we recall the notion of compact absorbing contractions.

DEFINITION 3.1. A mapping $f : X \to X$ is called a *compact absorbing contraction* (written $f \in \mathrm{CAC}(X)$) provided the following two conditions are satisfied:

(1) there exists an open subset U of X such that $\overline{f(U)} \subset U$ and $\overline{f(U)}$ is compact,
(2) the set U in (1) absorbs points.

First, we indicate how large the CAC class is. Evidently, any compact map $f : X \to X$ is a CAC. In fact, the compact set $\overline{f(X)}$ is an attractor of f and we can take $U = X$. More generally, a map $f : X \to X$ is called *eventually compact* (written $f \in \mathrm{EC}(X)$) provided there exists a natural number n such that $f^n : X \to X$ is compact. Observe that $\overline{f^n(X)}$ is then a compact attractor for f. We shall say that a map $f : X \to X$ is *asymptotically compact* provided that for each $x \in X$ the orbit $\{x, f(x), \ldots, f^n(x), \ldots\}$ is relatively compact and the *core*

$$C_f = \bigcap_{n=1}^{\infty} \overline{f^n(X)}$$

is nonempty compact.

As observed in [7] or [9], any asymptotically compact map $f : X \to X$ has a compact attractor A equal to C_f.

According to the above we can summarize:

PROPOSITION 3.2.

(1) *Any compact map has a compact attractor.*
(2) *Any eventually compact map has a compact attractor.*
(3) *Any asymptotically compact map has a compact attractor.*

So the class of mappings with compact attractors is quite large.

To explain the connection between mappings with compact attractors and CACs we need one more notion.

A map $f : X \to X$ is called *locally compact* (LC) provided that for every $x \in X$ there exists an open neighbourhood U_x of x in X such that $\overline{f(U_x)}$ is compact.

We have:

PROPOSITION 3.3 ([5]–[7]). *Any locally compact map with a compact attractor is a CAC.*

The above can be illustrated in the following:

$$\mathrm{CA} \cap \mathrm{LC} \subset \mathrm{CAC} \subset \mathrm{CA}.$$

Let us mention the following first application of Theorem 2.7:

THEOREM 3.4 (cf. [5]). *Let $X \in \mathrm{ANR}$ and $f : X \to X$ be a CAC. Then f is a Lefschetz map.*

Proof. We choose an open subset $U \subset X$ according to Definition 3.1. Then $f(U) \subset U$ and $\overline{f(U)} \subset U$ is compact. Therefore, in view of Theorem 2.2, the map $\widetilde{f} : U \to U$, $\widetilde{f}(x) = f(x)$, is a Lefschetz map. Now our claim follows from Remark 2.8 and Theorem 2.7. ∎

Observe that Theorem 3.4 can be formulated in the following slightly more general form:

THEOREM 3.5. *Let $f : X \to X$ be a CAC. Assume that there exists an ANR $A \subset X$ such that $\overline{f(U)} \subset A$, where U is chosen according to Definition 3.1. Then f is a Lefschetz map.*

COROLLARY 3.6. *If $X \in \mathrm{AR}$ and $f : X \to X$ is a CAC, then $\mathrm{Fix}(f) \neq \emptyset$.*

PROBLEM 3.7. *Can one replace CAC by CA in Theorem 3.4?*

Now, we recall the Lefschetz fixed point theorem for condensing mappings.

DEFINITION 3.8. A complete, bounded metric space (X, d) is called a *special ANR* (written $X \in \mathrm{ANR_s}$) provided that there exists an open subset

U of a Banach space E and two continuous mappings $r : U \to X$ and $s : X \to U$ such that:

(1) $r \circ s = \mathrm{id}_X$,
(2) r and s are nonexpansive, i.e., $\gamma(r(B)) \leq \gamma(B)$ and $\gamma(s(A)) \leq \gamma(A)$ for all bounded sets A and B.

We have the following:

THEOREM 3.9 ([6]). *Let $X \in \mathrm{ANR_s}$ and let $f : X \to X$ be a condensing map. Then f is a Lefschetz map.*

Let $f, h : (X, X_0) \to (X, X_0)$ be two given mappings.

DEFINITION 3.10. We shall say that $f_X : X \to X$ is a *generalized compact absorbing contraction* with respect to h (written $f_X \in \mathrm{GCAC}_h(X)$) provided the following conditions are satisfied:

(1) $f_{X_0} : X_0 \to X_0$ is a Lefschetz map,
(2) h_* is an epimorphism or a monomorphism; moreover, if h_* is an epimorphism then for every compact $K \subset X$ there exists $n = n(K)$ such that $f^n(h(K)) \subset X_0$, and if h_* is a monomorphism then for every compact $K \subset X$ there exists $n = n(K)$ such that $h(f^n(K)) \subset X_0$ and $f(h^{-1}(X_0)) \subset h^{-1}(X_0)$.

The following property is evident:

PROPOSITION 3.11. $\mathrm{GCAC}_{\mathrm{Id}_{(X,X_0)}}(X) = \mathrm{CAC}(X)$ *provided that X_0 is an open subset of X, $X \in \mathrm{ANR}$ and $f_{X_0} : X_0 \to X_0$ is a compact map.*

Note that if X and X_0 are convex subsets of a normed space, the conditions 3.10(2) and 3.10(3) are satisfied automatically. The same holds true if for example h is homotopic to the identity map of (X, X_0). The following example shows that the class $\mathrm{GCAC}_h(X)$ is larger than $\mathrm{CAC}(X)$.

EXAMPLE 3.12. Let $f_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f_{\mathbb{R}}(x) = \begin{cases} -2x - 3/2 & \text{for } x \leq -1, \\ 1/2 & \text{for } x \in (-1, 1), \\ 2x - 3/2 & \text{for } x \geq 1. \end{cases}$$

Let us consider the map $f : (\mathbb{R}, (-1, 1)) \to (\mathbb{R}, (-1, 1))$, $f(x) = f_{\mathbb{R}}(x)$ for every $x \in \mathbb{R}$, and $h : (\mathbb{R}, (-1, 1)) \to (\mathbb{R}, (-1, 1))$, $h(x) = 0$ for every $x \in \mathbb{R}$. It is easy to see that $f_{\mathbb{R}} \in \mathrm{GCAC}_h(\mathbb{R})$ but $f_{\mathbb{R}} \notin \mathrm{CAC}(\mathbb{R})$.

Now we prove the following:

LEMMA 3.13. *Let $f : (X, X_0) \to (X, X_0)$ be such that $f_X \in \mathrm{GCAC}_h(X)$ with respect to $h : (X, X_0) \to (X, X_0)$. Then $f_* : H(X, X_0) \to H(X, X_0)$ is weakly nilpotent.*

Proof. Let h_* be a monomorphism and let $z \in H(X, X_0)$. We have to prove that there exists $n = n(z)$ such that $(f_*)^n(z) = 0$. First observe that $(f_*)^n = (f^n)_*$. Since we consider homology with compact carriers, we can assume that $\operatorname{supp}(z) \subset K$, where K is a compact subset of X. By assumption there exists $n = n(K)$ such that $h(f^n(K)) \subset X_0$; consequently, in view of our assumption $f(h^{-1}(X_0)) \subset h^{-1}(X_0)$, we have $h(f^{n+p}(K)) \subset X_0$ for every $p \geq 1$. This implies that $(h \circ f^n)_*(z) = 0$. But

$$0 = (h \circ f^n)_*(z) = (h_* \circ (f^n)_*)(z) = h_*((f_*)^n(z)),$$

so $(f_*)^n(z) = 0$.

Now, assume that h_* is an epimorphism and let $z \in H(X, X_0)$. There exists $y \in H(X, X_0)$ such that $h_*(y) = z$ and again we can assume that $\operatorname{supp}(y) \subset K_1$, where K_1 is a compact subset of X. By assumption there exists $m = m(K_1)$ such that $f^m(h(K_1)) \subset X_0$. This implies that $(f^m \circ h)_*(y) = 0$, and so

$$0 = (f^m \circ h)_*(y) = ((f^m)_* \circ h_*)(y) = (f_*)^m(h_*(y)) = (f_*)^m(z). \ \blacksquare$$

Now from Lemma 3.13 and Theorem 2.7 we deduce the following generalization of the Lefschetz fixed point theorem.

THEOREM 3.14. *If $f_X \in \mathrm{GCAC}_h(X)$, then f is a Lefschetz map.*

COROLLARY 3.15. *If $f_X \in \mathrm{GCAC}_h(X)$ and X is an acyclic space* (*in particular*, *if $X \in \mathrm{AR}$*), *then* $\mathrm{Fix}(f_X) \neq \emptyset$.

Corollary 3.15 generalizes the Schauder fixed point theorem. Some other related results will be presented below.

DEFINITION 3.16. A map $f : X \to X$ is called an *acyclically compact absorbing contraction* (written $f \in \mathrm{ACAC}(X)$) provided there are two subsets U and A of X such that:

(1) U is open and nonempty,
(2) $f(U) \subset U$ and the contraction $f_U : U \to U$ is compact,
(3) A is an acyclic set such that $\overline{f(U)} \subset A$ and $U \cap A \in \mathrm{ANR}$,
(4) there exists $n = n(A)$ such that $f^n(A) \subset U$.

First, we prove the following:

PROPOSITION 3.17. *Let U be an open subset of X. If X is a closed convex subset of a Banach space and $f : (X, U) \to (X, U)$ is a map such that $f_X \in \mathrm{CAC}(X)$, then $f_X \in \mathrm{ACAC}(X)$.*

Proof. It is sufficient to observe that $A = \overline{\mathrm{conv}}(f(\overline{U}))$ is a compact AR as the closed convex hull of a compact set in a Banach space (cf. Mazur's lemma [1]), and A satisfies the assumptions of Definition 3.16. ∎

It is easy to see that there are acyclically compact absorbing contractions which are not compact absorbing contractions.

EXAMPLE 3.18. Let $U = (-2,-1) \cup (-1,0)$ and $f : (\mathbb{R}, U) \to (\mathbb{R}, U)$ be defined as follows:

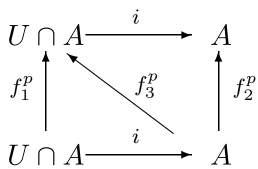
$$f(x) = \begin{cases} x/3 - 1/6 & \text{for } x < 0, \\ 2x - 1/6 & \text{for } x \geq 0, \end{cases}$$

and let $A = (-4, 0)$. It is easy to see that $f_{\mathbb{R}} \in \mathrm{ACAC}(\mathbb{R})$ but $f_{\mathbb{R}} \notin \mathrm{CAC}(\mathbb{R})$.

We prove:

THEOREM 3.19. *If* $f_X \in \mathrm{ACAC}(X)$, *then* $\mathrm{Fix}(f_X) \neq \emptyset$.

Proof. By assumption there exists $n = n(A)$ such that $(f_X)^n(A) \subset U$ and $(f_X)^{n+k}(A) \subset (f_X)^k(U) \subset A$ for all $k \geq 1$, where U is chosen according to Definition 3.16. We can assume that $n+k = p$ is a prime number for some $k \geq 1$. We have the following commutative diagram:



in which f_1^p, f_2^p, f_3^p are the appropriate contractions of f_X^p.

By assumption, f_1^p is a compact map and $U \cap A \in \mathrm{ANR}$, so $\Lambda(f_1^p)$ is defined. By Proposition 1.3,

$$\Lambda(f_1^p) = \Lambda(f_2^p) = 1.$$

Now, Proposition 1.6 yields

$$\Lambda(f_1^p) \equiv \Lambda(f_1) \bmod p,$$

where f_1 is the contraction of f_X to $U \cap A$. By assumptions (see 3.16(2) and 3.16(3)), f_1 is a compact map. Since $\Lambda(f_1) \neq 0$, we have $\mathrm{Fix}(f_1) \neq \emptyset$, and as $\mathrm{Fix}(f_1) \subset \mathrm{Fix}(f_X)$, the proof is complete. ■

REMARK 3.20. Observe that the assertion of Theorem 3.19 is still true if we assume only that $U \cap A$ is a compact AANR (in the sense of Noguchi, see [2] for details) instead of being an ANR.

REMARK 3.21. We note that all the results in this paper remain true if:

(1) we assume that all spaces are Hausdorff topological spaces and in place of ANRs we consider retracts of open sets in Klee admissible spaces (for details see [1]; cf. also [7]);
(2) we consider multivalued admissible mappings in place of single-valued ones (cf. [1] or [7]).

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