Fat $P$-sets in the Space $\omega^*$

by

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Summary. We prove that—consistently—in the space $\omega^*$ there are no $P$-sets with the $\kappa$-cc and any two fat $P$-sets with the $\kappa^+\text{-cc}$ are coabsolute.

1. Introduction. A closed subset $A \subseteq X$ of a topological space $X$ is said to be a $P$-set if the inclusion $A \subseteq \text{int}\bigcap R$ holds for any countable family $R$ of open neighborhoods of $A$. A point $x \in X$ is a $P$-point if the one-element set $\{x\}$ is a $P$-set. Here, we consider the case of $X = \omega^* = \beta[\omega] \setminus \omega$, the remainder of the Stone–Čech compactification of $\omega$ ($= \text{the nonnegative integers with the discrete topology}$). Hence, we may assume that the family $R$ mentioned above consists of open-closed neighborhoods of $A$. The existence of $P$-points follows e.g. from the continuum hypothesis (see [6]), but it is not provable in set theory. In fact, Shelah constructed a model in which $\kappa = \omega_2$ and there are no $P$-points (see [7]).

Let $\kappa$ be a cardinal. We say that a set $A$ has the $\kappa$-cc (the $\kappa$ (anti-)chain condition) if every disjoint family of relatively open subsets of $A$ has power less than $\kappa$. In [3] a construction is presented of a model in which $\kappa = \omega_2$ and there are no $P$-sets with the $\omega_1$-cc in the remainder $\omega^*$ (see also [4] for all the references). In this paper we strengthen this theorem by describing a model in which $\kappa = \omega_2$ and there are no $P$-sets with the $\kappa$-cc. Moreover, in the constructed model any two fat (see below for the definition) $P$-sets with the $\kappa^+\text{-cc}$ are alike in the sense that they are coabsolute (i.e. their Gleason spaces are homeomorphic).

The case of non-fat sets requires a different forcing and will be considered elsewhere.

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2. Basic facts and definitions. If $A$, $B$ are subsets of $\omega$ then $A \subseteq \ast B$ means $A \setminus B$ is finite. Closed subsets of $\omega^*$ can be identified with filters on $\omega$. Indeed, any closed set in $\omega^*$ can be written as
\[
\bigcap \{ \overline{A} \setminus \omega : A \in F \}
\]
where $F$ is a filter on $\omega$ ($\overline{A}$ is the closure of $A \subseteq \omega$ in $\beta[\omega]$), and vice versa. $P$-filters (filters corresponding to $P$-sets) are then characterized by the following condition:

For every family $\{ A_i : i \in \omega \} \subseteq F$ there is an $A \in F$ such that $A \subseteq \ast A_i$ for all $i \in \omega$.

Equivalently, $P$-ideals $I$ (ideals dual to $P$-filters) have the following property:

For every $\{ B_i : i \in \omega \} \subseteq I$ there is a $B \in I$ such that $B_i \subseteq \ast B$ for all $i \in \omega$.

An ideal $I$ (and its dual filter $F$) is called fat if it satisfies the following condition:

If $\{ B_i : i \in \omega \} \subseteq I$ is such that $\lim_i \min B_i = \infty$, then there is an infinite $Z \subseteq \omega$ such that $\bigcup_{i \in Z} B_i \in I$.

In [5] it is proved that every $P$-set having the $\text{c-cc}$ is fat.

For the rest of this section let us fix a fat $P$-filter $F$ and its dual $I$.

The filter $F$ determines a forcing $\mathbb{P} = \mathbb{P}(F)$ in the following way: Let $T$ consist of all trees $t = (t, \leq_t)$, which are suborderings of $(\omega, \leq)$ whose domains $t$ are infinite elements of $I$. Fix a tree $t_0$ order isomorphic to the Cantor tree $\mathcal{B}$ (i.e. a full binary tree of height $\omega$). Moreover we can assume that $\min t_0 = 0$. So, $\mathcal{B}$ can be treated as the set of all finite zero-one sequences. Moreover we can assume that if $h : \{0, 1\}^{<\omega} \rightarrow (t_0, \leq t_0)$ is the required isomorphism then the following condition holds:

If $x_1$, $x_2$ are different elements of $\mathcal{B}$, $\text{length}(x_1 \cap x_2) = l$ and $x_1(l) < x_2(l)$ then $h(x_1|_{\{0,1,\ldots,l\}}) < h(x_2|_{\{0,1,\ldots,l\}})$.

We define the relation which partially orders the family $T$ as follows: $(t, \leq_t) \leq (s, \leq_s)$ if and only if $(s, \leq_s)$ is a subordering of $(t, \leq_t)$ and each branch of $(t, \leq_t)$ contains cofinally a (unique) branch of $(s, \leq_s)$.

For any $t \in T$ and $n \in \omega$ let $t^{(n)} = t \setminus \{0, \ldots, n\}$ denote the subtree of $t$ obtained from $t$ by deleting the elements $0, \ldots, n$.

Now define $\mathbb{P} = \mathbb{P}(F)$ as the set of all pairs $p = (t^p, f^p)$, where $t^p \in T$ and $t^p \leq t_0^{(n)}$ for some $n$ and a tree $t_0$, and $f^p : t^p \rightarrow \{0, 1\}$. The ordering on $\mathbb{P}$ is defined as follows:

$p \leq q \equiv t^p \leq t^q$ and $f^p \supseteq f^q$. 

Thus, the sets of branching points of the trees are exactly the same. (In fact, these points are precisely the branching points of the fixed tree \( t_0 \).) It follows that each branch of the tree \( t^p \) is uniquely coded by a branch of \( t_0 \) (and \( \mathcal{B} \)).

To see how \( \mathbb{P} \) works, suppose that \( G \subseteq \mathbb{P} \) is a generic filter. For any branch \( b \) of \( t_0 \) in \( V \) let \( b^p \) be the branch of \( t^p \) which almost contains \( b \) cofinally; put \( t^G = \bigcup_{p \in G} t^p \), \( f^G = \bigcup_{p \in G} f^p \) and \( b^G = \bigcup_{p \in G} b^p \). Clearly, \( t^G \) is a tree (but \( t^G \not\in I \)), \( f^G : t^G \rightarrow \{0, 1\} \) and \( b^G \) is a branch of \( t^G \) extending \( b \). Define

\[
X_b = \{ i \in \omega : i \in b^G \text{ and } f^G(i) = 1 \}. 
\]

By assumption, the sets \( t^p \) belong to \( I \) and therefore the set \( (\omega \setminus t^p) \cap A \) is infinite, for each \( A \in F \) and \( p \in \mathbb{P} \). It follows that the sets

\[
E_{n, \varepsilon}^{A,b} = \{ p \in \mathbb{P} : \exists i \geq n [ i \in A \cap b^p \text{ and } f^p(i) = \varepsilon ] \}
\]

are dense for each \( A \in F \), \( n \in \omega \), \( \varepsilon = 0, 1 \) and any branch \( b \) of \( t_0 \) (in \( V \)). Hence, \( X_b \) intersects infinitely each set \( A \in F \), and the complement \( b^G \setminus X_b \) has the same property. In [3], it is proved that the countable product \( \mathbb{P}^\omega(F) = \mathbb{P}(F) \times \mathbb{P}(F) \times \cdots \) is always \( \omega \)-proper and \( \omega^\omega \)-bounding (i.e. each function \( f : \omega \rightarrow \omega \) from \( V[G] \) is majorized by a function \( g : \omega \rightarrow \omega \) from \( V \)). Thus \( \omega_1 \) is not collapsed and since distinct branches \( b^G \) are almost disjoint we may say that the forcing \( \mathbb{P} \) adds uncountably many almost disjoint Gregorieff-like sets \( X_b \). In particular, the Suslin number of the set \( \bigcap \{ A \setminus \omega \setminus A \in F \} \) determined by \( F \) will be uncountable in \( V[G] \).

A similar notion of forcing is used in [3] to construct a model in which there are no closed ccc \( P \)-sets (in \( \omega^* \)). Since \( \omega_1 \) is not collapsed, to obtain an uncountable family of relatively open subsets of \( \bigcap F \) it is enough to add a new element \( X_b \) for every branch \( b^V \) of \( \mathcal{B}^V \) (i.e. every branch which belongs to the ground model). Since, starting with \( c = \omega_1 \), we iterate the forcing \( \mathbb{P}^\omega(F) \omega_2 \) times, in the resulting model we have \( c = \omega_2 \). So to ensure that each closed \( P \)-set is \( c \)-ccc we have to add new elements of the type \( X_b \) for \( \omega_2 \) (new) branches of \( \mathcal{B} \).

Note that since the set \( t^G \) does not belong to the ground model its characteristic function is a new branch in the binary tree \( \mathcal{B} \).

3. Construction of the model. Let us fix a ground model \( V \) in which \( c = \omega_1 \), \( 2^{\omega_1} = \omega_2 \) and the diamond principle holds:

There is a sequence \( \langle T_\alpha : \alpha < \omega_2 \text{ and } \text{cf}(\alpha) = \omega_1 \rangle \) such that for every \( Y \subseteq H(\omega_2) \) the set \( \{ \alpha < \omega_2 : \text{cf}(\alpha) = \omega_1 \text{ and } Y \cap H_\alpha = T_\alpha \} \) is stationary.

Here, \( H(\omega_2) \) denotes the family of all sets of hereditary power less than \( \omega_2 \); \( H(\omega_2) = \bigcup_{\alpha<\omega_2} H_\alpha \) is a standard decomposition into a continuously in-
creasing chain with \( \text{card}(H_\alpha) = \omega_1 \). (As \( V \) we can take for example the constructible universe.)

Define by induction a countable support iteration \( \langle P_\alpha : \alpha < \omega_2 \rangle \) as follows:

- \( P_\alpha = \text{Lim}_{\beta < \alpha} P_\beta \) (limit with countable supports) for limit \( \alpha \),
- \( P_{\alpha+1} = P_\alpha * P_\alpha(T_\alpha) \) if \( \text{cf}(\alpha) = \omega_1 \) and \( P_\alpha \models \text{"} T_\alpha \text{ is a fat } P\text{-filter} \),
- \( P_{\alpha+1} = P_\alpha * Q_\alpha \) in all remaining cases (\( Q_\alpha \) is a \( P_\alpha \)-name of an arbitrary forcing whose countable support iteration is proper and \( \omega^\omega \)-bounding).

We can assume that \( P_\alpha \subseteq H(\omega_2) \). In view of [7, Ch. V, Theorem 4.3] the forcing \( P_{\omega_2} \) is proper and \( \omega^\omega \)-bounding. Let \( G \) be a generic filter.

4. Main theorem. The model constructed in the previous section satisfies the following.

**Theorem 1.** In the model \( V[G] \) there are no \( P \)-sets with the \( c \)-cc.

**Proof.** Suppose the opposite and derive a contradiction. Thus, there is a fat \( P \)-filter \( F \in V[G] \). Clearly, we have

\[ V[G] \models \text{"} c = (\omega_2)^V \text{" and } V[G][\alpha] \models \text{"} c = \omega_1 \text{"} \text{ for all } \alpha < \omega_2. \]

Set \( F_\alpha = F \cap V[G][\alpha] \). We claim that

The set \( C_1 = \{ \alpha < \omega_2 : F_\alpha \in V[G][\alpha] \} \) is an \( \omega_1 \)-cub

(i.e. unbounded and closed under limits of length \( \omega_1 \)). Choose a canonical name \( F \subseteq H(\omega_2) \) of \( F \) consisting of some pairs \( \langle x, p \rangle \) (in fact, canonical names for pairs, which are elements of \( H(\omega_2) \)), where \( x \) is a \( P_{\omega_2} \)-name of a subset of \( \omega \) and \( p \in P_{\omega_2} \). Denote by \( F(x) \) the set \( \{ p : \langle x, p \rangle \in F \} \). Since card \( F(x) \) is at most \( \omega_1 \), the set

\[ C_2 = \{ \alpha < \omega_2 : \forall x \left[ x \in \left( V(P_\alpha) \right) \rightarrow F(x) \subseteq P_\alpha \right] \} \]

is easily seen to be an \( \omega_1 \)-cub. Plainly, the set \( F_\alpha = F \cap \left( V(P_\alpha) \times P_\alpha \right) \) is a \( P_\alpha \)-name and we have


whence \( F_\alpha \in V[G][\alpha] \) for all \( \alpha \in C_2 \).

In addition we may assert that the \( F_\alpha \)'s are fat \( P \)-filters, because they are so on an \( \omega_1 \)-cub subset of \( C_1 \). Indeed, if \( R = \{ A_i : i \in \omega \} \subseteq F_\alpha \), then there is an \( A \in F \) such that \( A \subseteq^* A_i \) for all \( i \in \omega \). There is a \( \beta < \omega_2 \) such that \( A \in V[G][\beta] \) (cf. [7, Ch. V, 4.4]). Since there are at most \( c = \omega_1 \) countable subfamilies \( R \) of \( F_\alpha \) in \( V[G][\alpha] \), there must be an ordinal \( \alpha^* \in C_1 \) such that in \( V[G][\alpha^*] \) we can find some lower bounds \( A \in F \) for all such \( R \)'s. Now define inductively \( \alpha_0 = \alpha, \alpha_{\xi+1} = \alpha_\xi^* \) and \( \alpha_\lambda = \sup_{\xi < \lambda} \alpha_\xi \) for limit \( \lambda \).

If \( \beta = \sup_{\xi < \omega_1} \alpha_\xi \), then \( \beta \in C_1 \) and \( F_\beta \subseteq V[G][\beta] \) is a \( P \)-filter. Obviously,
the set
\[ C_1' = \{ \beta < \omega_2 : F_\beta \text{ is a } P\text{-filter} \} \]
is $\omega_1$-closed, which proves that $C_1'$ is an $\omega_1$-cub.

In a similar way we show that $F_\beta$ is fat on an $\omega_1$-cub subset $\subseteq C_1$, which
proves the claim.

A standard reasoning also shows that the set
\[ C_3 = \{ \alpha < \omega_2 : F \cap H_\alpha = F_\alpha \} \]
is an $\omega_1$-cub. Hence, $C = C_1 \cap C_3$ is also an $\omega_1$-cub. By the diamond principle
applied to $Y = F \subseteq H(\omega_2)$ there is an $\alpha < \omega_2$, with $\text{cf}(\alpha) = \omega_1$
such that $T_\alpha$ is a $\mathbb{P}_\alpha$-name of the fat $P$-filter $F_\alpha$. By the definition of the iteration we
force with $\mathbb{P}^{\omega}(F_\alpha)$ at stage $\alpha$, i.e. $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \ast P^{\omega}(F_\alpha)$.

To finish the proof fix a $p \in \mathbb{P}_{\omega_2}$. Then the $\alpha$th term $p(\alpha)$ can be written
as $p(\alpha) = \langle p_n(\alpha) : n \in \omega \rangle$. Let $b$ be a branch in the binary tree $B$ (in $V[G|\alpha]$)
and denote by $b^{G}_{n,\alpha}$ the corresponding branch in $t^{G}_{n,\alpha} = \bigcup_{p \in G} t^{p}_{n,\alpha}$,
where $p|_{\alpha} \vdash "p(\alpha) = \langle t^{p}_{n,\alpha}, f^{p}_{n,\alpha} \rangle"$. Define
\[ X^{\alpha}_{n,\alpha}(b) = \{ k \in b^{G}_{n,\alpha} : f^{G}_{n,\alpha}(k) = 1 \} \]

Note that for any $\xi > \alpha$ with $\text{cf}(\xi) = \omega_1$, $t^{G}_{m,\xi} = \bigcup_{p \in G} t^{p}_{m,\xi}$ determines a
new branch in the binary tree and thus a new branch $b^{m,\xi}_{n,\alpha}$ in the tree $t^{G}_{n,\alpha}$. Indeed, we can identify the set $t^{G}_{n,\alpha}$ with its characteristic function. Such a function is just a (new) branch in the binary tree, say $b$. Let $b^{m,\xi}_{n,\alpha}$ be the corresponding branch of the tree $t^{G}_{n,\alpha}$. (Of course, this branch does not belong to the model $V[G|\alpha]$.) Define
\[ X^{m,\xi}_{n,\alpha} = \{ k \in b^{m,\xi}_{n,\alpha} : f^{G}_{m,\alpha}(k) = 1 \} \]
(Note that the sets $X^{m,\xi}_{n,\alpha}$ for $\alpha \leq \xi$ and $m \in \omega$ form a family of $c$ almost
disjoint Gregorieff-like sets centered with $F_\alpha$.)

Actually, what we will use is

**Lemma 1.** Assume that $\langle \xi_i : i \in \omega \rangle \in V$ is a sequence of ordinals greater
then or equal to $\alpha$, of cofinality $\omega_1$. Let $\langle n_i : i \in \omega \rangle \in V[G|\alpha]$ and $\langle m_i : i \in \omega \rangle \in V[G[\sup \xi_i]]$ be sequences of natural numbers (the first one is injective).
Suppose that a sequence $\langle b_i : i \in \omega \rangle$ of branches of the binary tree $B$
and a function $g : \omega \to \omega$ belong to $V[G|\alpha]$. Denote by $X_i$ the set $X^{\alpha}_{n_i,\alpha}(b_i)$
if $\xi_i = \alpha$, and $X^{m_i,\xi}_{n_i,\alpha}$ otherwise. Then the set
\[ \bigcap_{i \in \omega} [(\omega \setminus X_i) \cup [0, g(i))] \text{ is in the ideal } I_\alpha \text{ (dual to } F_\alpha). \]

**Proof.** The proof consists of two cases:

**Case 1.** Suppose that $\xi_i = \alpha$ for all $i \in \omega$. It is enough to show that the
assertion of the lemma holds for $V$ and the forcing $\mathbb{P}^{\omega}(F_\alpha)$. So suppose that
$F$ is a fat $P$-filter in $V$. For a given $p = \langle (t_n, f_n) : n \in \omega \rangle$ there is a set $B$ and integers $k_i \in \omega$ such that

$$t_{n_i} \setminus [0, k_i) \subseteq B$$

for all $i \in \omega$.

We may assume that $g(i) < k_i < k_{i+1}$ and $y_i = [k_i, k_{i+1}) \setminus B \neq \emptyset$ for all $i$. Extend each $t_{n_i}$ by adding to the branch $b_i$ all elements of $y_i$ and put $f_{n_i}(k) = 1$ for $k \in y_i$. Then $q$ obtained in this way forces that

$$[(\omega \setminus X_i) \cup [0, g(i))] \cap y_i = \emptyset,$$

and hence

$$\bigcap_{i \in \omega} [(\omega \setminus X_i) \cup [0, g(i))] \cap \bigcup_{i \in \omega} y_i = \emptyset.$$

But $\bigcup_{i \in \omega} y_i = [k_{m_0}, \infty) \setminus B$ and thus

$$q \models \ " \bigcap_{i \in \omega} (\omega \setminus X_i) \cup [0, g(i)) \subseteq \ B",$$

which proves the first case.

**Case 2.** Suppose that there are $\xi_i > \alpha$. Without losing generality we can treat $V[G|\alpha]$ as a ground model. Define $y_i$ as above. Then first of all, for all $i \in \{j : \xi_j = \alpha\}$, extend conditions $(t_{n_i}, f_{n_i})$ by adding new elements to the trees $t_{n_i, \alpha}$, in the way described above.

Let $\xi_i > \alpha$. By definition, $t_{n_i} \leq t_{0}(m)$ for some $m \in \omega$. (In fact, $t_{n_i} \leq t_{0}(m)(\xi_i, n_i)$, since the choice of the tree $t_0$ depends on an ideal $I(\xi_i, n_i)$ considered at stage $\xi_i$ and for the index $n_i$. We can assume that if $I(\xi_1, n_1) \subseteq I(\xi_2, n_2)$ for some $\xi_1 < \xi_2$ or $\xi_1 = \xi_2$ and $n_1 < n_2$, then $t_0(\xi_1, n_1) \subseteq t_0(\xi_2, n_2)$.) We choose a finite sequence of pairwise comparable conditions $p = p_0 \geq p_1 \geq \cdots \geq p_k$ which forces all properties listed below. To simplify notation we will write $h_i, x_i, y_i$ etc. instead of their canonical names.

For a fixed isomorphism $h_i : B \rightarrow t_0$ denote by $c_l$ the images $h_i(x_l)$, where $x_l$ is a sequence of length $l$ and range $\{1\}$. Put

$$c_i = c_{l_0}$$

where $l_0 = \{l : c_l \in t_{n_i}\}$.

Thus $c_i$ is the least branching point of $t_{n_i}$ which belongs to the branch $u_i = \bigcup_{l \in \omega} h_i(x_l)$.

We extend $t_{n_i}$ as follows: For all $r \in y_i$ and $r < c_i$ we put $r \leq t_{n_i} c$. There exists the least $l_i \in \omega$ such that

$$y_i \setminus \{0, 1, \ldots, c_i\} \subset [c_i, c_{l_i}) \cap \omega.$$

Thus we add all elements of $y_i$ to the branch corresponding to $u_i$ and put $f_{n_i}(r) = 1$ for each of them. Moreover extend the condition $p(\xi_i)(m_i) = (t_{m_i, \xi_i}, f_{m_i, \xi_i})$ in such a way that

$$p|_{\xi_i} \models \ "\{0, 1, \ldots, l_i\} \subset t_{m_i, \xi_i},"$$
The condition \( p_k \) obtained in this way forces the same as the \( q \) defined in Case 1. So Lemma 1 is proved.

Now, we can finish the proof of Theorem 1. From the \( \epsilon \)-cc assumption it follows that for every \( i \in \omega \) there is an \( \eta_i < \omega_2 \) such that \( \omega \setminus X^{m_i, \xi}_{n_i, \alpha} \in F \) for all \( \xi \geq \eta_i \) with \( \text{cf}(\xi) = \omega_1 \) and \( m_i \in \omega \). Let \( \omega_2 > \xi_i > \eta_j \) with \( \text{cf}(\xi_i) = \omega_1 \) and \( m_i \in \omega \).

Since \( F \) is a \( P \)-set, there is an \( A \in F \) and a function \( g : \omega \to \omega \) such that
\[
A \setminus [0, g(i)) \subseteq \omega \setminus X^{m_i, \xi_i}_{n_i, \alpha} \quad \text{for each} \ i \in \omega,
\]
which implies \( \bigcap_{i \in \omega} (\omega \setminus X^{m_i, \xi_i}_{n_i, \alpha}) \cup [0, g(i)) \in F_\alpha \). Since \( P_{\omega_2} \) is bounding we may assume that \( g \in V \). This contradicts Lemma 1 and the proof is finished. ■

5. Structure of a fat \( P \)-set.\ The coabsoluteness assertion follows easily from the following theorem, which clarifies the structure of any fat \( P \)-set.

**Theorem 2.** In the model \( V^{P_{\omega_2}}[G] \) every fat \( P \)-set \( F \) has a \( \pi \)-base tree of height \( \omega \), each vertex of which splits into \( \epsilon \) elements.

**Proof.** Let \( F \) denote also the corresponding filter. Keeping the notation from the preceding proof we use the sets \( X_i^\gamma \) to construct a dense tree in the factor algebra \( P(\omega)/F \). Fix an enumeration of the branches of the binary tree \( \{b_\gamma : \gamma < \omega_2\} \). Denote by \( X^i_\gamma \) the set \( X^{m_i, \xi_i}_{n_i, \alpha} \) if \( b_\gamma \) is the characteristic function of the tree \( t^{G}_{n_i, \alpha}(\xi_i > \alpha) \), and the set \( X^{\alpha}_{n_i, \alpha}(b_\gamma) \) if \( b_\gamma \) belongs to the model \( V[G|\alpha] \). Notice first that for a given positive element \( a = [A] > \emptyset \) from \( P(\omega)/F \), there is an \( i \in \omega \) such that
\[
a \cdot x^i_\gamma > \emptyset \quad \text{for} \ \epsilon \text{-many} \ \gamma,
\]
where \( x^i_\gamma = [X^i_\gamma] \) denotes the equivalence class mod \( F \) of elements determined by the branch \( b_\gamma \) in the tree \( t^{G}_{\gamma, \alpha} \).

Indeed, suppose the opposite, i.e. for each \( i \in \omega \) there is a \( \beta_i \) such that
\[
A \cap X^i_\gamma \in I \quad \text{for every} \ \gamma \geq \beta_i.
\]
Let \( \gamma > \sup_{j \in \omega} \beta_j \) with \( \text{cf}(\gamma) = \omega_1 \). We then have
\[
A \setminus B_i \subseteq \omega \setminus X^i_\gamma \quad \text{for all} \ i \in \omega,
\]
where the \( B_i \)'s belong to \( I \). Since \( I \) is a \( P \)-ideal there is a set \( B \in I \) such that \( B_i \subseteq \star B \) for all \( i \in \omega \) and hence
\[
A \setminus B \subseteq \bigcap_{i \in \omega} (\omega \setminus X^i_\gamma) \cup [0, g(i)),
\]
where \( g \) may be assumed to be in \( V \), as the forcing is bounding. Now, Lemma 1 implies that \( A \setminus B \) and hence \( A \) are in \( I \), which contradicts the positivity of \( a = [A] \).
Clearly, deleting “small” sets (i.e. those from \( I \)) and renumbering we may assume that we are given a matrix

\[
M = \{ x^i_\gamma : i \in \omega \text{ and } \gamma < c \}
\]

such that every element \( x^i_\gamma \) is positive in \( P(\omega)/F \) and each positive element \( a > 0 \) intersects \( c \) elements \( x^i_\gamma \) in some row \( i \) (depending on \( a \)). Obviously, each row in \( M \) is an antichain. Finally, note that any positive \( a > 0 \) splits into \( c \) elements. To see this, we check that the filter

\[
F_a = \{ X \subseteq \omega : [a] \subseteq [X] \} = \{ X \subseteq \omega : A \setminus X \in I \}
\]

is a fat \( P \)-filter, as \( F \) is, and then apply Theorem 1 to convince ourselves that the algebra \( P(\omega)/F_a \) has an antichain of power \( c \).

Now, define a tree \( T \) as follows. Extend, if necessary, the first row \( \{ x^0_\gamma : \gamma < c \} \) to a maximal antichain \( T_0 = \{ y^0_\beta : \beta < c \} \).

Assume that the levels \( T_0, \ldots, T_{n-1} \) are already defined so that each \( T_i = \{ y^i_\beta : \beta < c \} \) is a maximal antichain. Extend each of the antichains \( \{ y^{n-1}_\beta \cap x^n_\gamma : \gamma < c \} \setminus \{ 0 \} \) to a maximal \( T^3_n \). By the remark above we may assert that \( T^3_n \) always has \( c \) elements. Let \( T_n = \bigcup_{\beta < c} T^3_n \). Clearly, in the resulting tree \( T = \bigcup_{n \in \omega} T_n \) each vertex splits into \( c \) elements and every positive \( a > 0 \) intersects \( c \) elements of some level \( T_n \). Hence, to each \( a > 0 \) we may assign an element \( y_a \in T \) such that \( a \cdot y_a > 0 \) and \( y_a \neq y_b \) whenever \( a \neq b \), as in [1]. Replace any such \( y_a \) by \( a \cdot y_a \) if \( y_a - a > 0 \), and the same for all \( y \leq y_a \). Clearly, the tree \( T \) so modified becomes additionally a \( \pi \)-base. ■

6. Final remarks. From Theorem 2 it follows immediately that for any two fat \( P \)-filters \( F_1 \) and \( F_2 \) in \( V[G] \) the Boolean algebras \( P(\omega)/F_1 \) and \( P(\omega)/F_2 \) have order isomorphic dense subsets. This, in turn, implies that the completions of \( P(\omega)/F_1 \) and \( P(\omega)/F_2 \) are Boolean isomorphic, or, equivalently, the Gleasons \( G(F_1) \) and \( G(F_2) \) are homeomorphic.

Finally, let us comment on other consistent configurations of \( P \)-sets. For example, we can construct a model with \( c > \omega_1 \) in which there are no \( P \)-points and there are two fat \( P \)-sets, one with the \( \omega_1 \)-cc and the other with the \( \omega_2 \)-cc. In connection with Theorem 2 we note that it is consistent to have two fat \( P \)-sets having the \( c^+ \)-cc with distinct Gleasons. The proof is more difficult and will be published elsewhere.

References


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