BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 53, No. 2, 2005

COMPUTATIONAL ALGEBRA

Reduction of Power Series in a Polydisc with Respect to a Gröbner Basis

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Summary. We deal with a reduction of power series convergent in a polydisc with respect to a Gröbner basis of a polynomial ideal. The results are applied to proving that a Nash function whose graph is algebraic in a "large enough" polydisc, must be a polynomial. Moreover, we give an effective method for finding this polydisc.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $x_0 \in \Omega$. We say that a holomorphic function $f : \Omega \to \mathbb{C}$ is a *Nash function at* x_0 if there exists an open neighborhood $U \subset \Omega$ of x_0 and a nonzero polynomial $F : \mathbb{C}^{n+1} \to \mathbb{C}$ such that the graph Γ_f of f over U is contained in the zero set of F. We call f a *Nash function in* Ω if it is a Nash function at each $x \in \Omega$. The family of Nash functions in Ω is denoted by $\mathcal{N}(\Omega)$.

A subset X of $\mathbb{C}^n$ is said to be *algebraic in* Ω if $X \cap \Omega = \overline{X} \cap \Omega$ where $\overline{X}$ is the Zariski closure of X.

Remark 1.1 (see [9, Remark 1.2]). Let $\Omega \subset \mathbb{C}^n$ be a domain and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then the following statements are equivalent:

(i) $f \in \mathcal{N}(\Omega)$,
(ii) there exists an irreducible polynomial $F : \mathbb{C}^{n+1} \to \mathbb{C}$, unique up to a multiplicative scalar, such that $F(x, f(x)) = 0$ for $x \in \Omega$.

Theorem 1.2 (see [9, Theorem 1.3]). *Every entire Nash function is a polynomial.*

The proof in [9] is elementary. Theorem 1.2 can also be deduced from Serre's graph theorem ([8]). An elementary proof of the affine version of

2000 *Mathematics Subject Classification*: 13P10, 32C07.
Key words and phrases: Gröbner bases, polynomial ideals, power series, Nash function.

Serre's graph theorem, based on the theory of Gröbner bases, can be found in [1, Theorem 4.2].

The main result of this paper is Theorem 3.4 which gives a reduction of convergent (in a "large enough" polydisc) power series with respect to a Gröbner basis of a given polynomial ideal.

The results obtained are applied to prove Theorem 4.5 which states that if f is a Nash function in Ω and the graph Γ_f of f is algebraic in a "large enough" polydisc contained in $\Omega \times \mathbb{C}$ then f is a polynomial. Moreover, the theory of Gröbner bases may be used to find the "large" polydisc.

2. Notation and basic facts. Let $\mathbb{K}$ be the field of complex ($\mathbb{C}$) or real ($\mathbb{R}$) numbers. We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{R}_+$ the set of positive real numbers. For convenience of the readers we recall some facts; we follow the notation of [1]. The basic algebraic structures involved in this paper are the polynomial ring $\mathcal{R} = \mathbb{K}[X] = \mathbb{K}[X_1, \ldots, X_n]$, the ring $\mathbb{K}[[X]] = \mathbb{K}[[X_1, \ldots, X_n]]$ of formal power series and the rings

$$E_r := \{f \in \mathbb{K}[[X]] : f \text{ is absolutely convergent at the point } r\}$$

corresponding to $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. Note that if $f \in E_r$ then f is absolutely uniformly convergent in the closure of the polydisc

$$P_r := \{(x_1, \ldots, x_n) \in \mathbb{K}^n : |x_1| < r_1, \ldots, |x_n| < r_n\}.$$

Let $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{K}[[X]]$ the *support* of f is defined to be

$$\operatorname{supp} f = \{\alpha : c_\alpha \neq 0\}.$$

For a set $F \subset \mathbb{K}[[X]]$ we put $\operatorname{supp} F = \bigcup_{f \in F} \operatorname{supp} f$.

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in E_r$, where $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$. The space E_r with the norm

$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| r^\alpha \tag{1}$$

is a Banach space (for details see e.g. [5]). For a given nonempty subset $\mathcal{D} \subseteq \mathbb{N}^n$,

$$E_r(\mathcal{D}) := \{f \in E_r : \operatorname{supp} f \subseteq \mathcal{D}\}$$

is a Banach subspace of E_r. The spaces $\mathcal{R}$ and

$$\mathcal{R}(\mathcal{D}) := \{f \in \mathcal{R} : \operatorname{supp} f \subseteq \mathcal{D}\}$$

are dense subspaces of E_r and $E_r(\mathcal{D})$, respectively.

From elementary facts concerning power series we can deduce the following lemma.

LEMMA 2.1. *Let $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$. If $f_\alpha \in E_r(\mathcal{D})$ for $\alpha \in \mathbb{N}^n$ and*

$$\sum_{\alpha \in \mathbb{N}^n} \|f_\alpha\|_r < \infty,$$

then the series $\sum_{\alpha \in \mathbb{N}^n} f_\alpha$ is convergent to an $f \in E_r(\mathcal{D})$.

Let $\prec$ be a fixed admissible term ordering in $\mathbb{N}^n$ (see [1]). Then, by definition, $X^\alpha \prec X^\beta$ if $\alpha \prec \beta$. If $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathcal{R}$, $f \neq 0$, then the *exponent*, *leading coefficient*, *initial term* and *tail* of f are defined to be

$$\begin{aligned} \exp_\prec f &:= \max_\prec \{\alpha : \alpha \in \operatorname{supp} f\}, \\ \operatorname{lc}_\prec f &:= c_{\exp_\prec f}, \\ \operatorname{in}_\prec f &:= \operatorname{lc}_\prec f X^{\exp_\prec f}, \\ \operatorname{tail}_\prec f &:= f - \operatorname{in}_\prec f, \end{aligned}$$

respectively.

For $F \subset \mathcal{R}$ we define

$$\Delta_F := \begin{cases} \bigcup_{f \in F} (\exp_\prec f + \mathbb{N}^n) & \text{if } F \not\subseteq \{0\}, \\ \emptyset & \text{if } F \subseteq \{0\}, \end{cases} \qquad \mathcal{D}_F := \mathbb{N}^n \setminus \Delta_F.$$

Let $I \subset \mathcal{R}$ be a nonzero ideal and let $\prec$ be an admissible term ordering. A finite subset $G \subset I$ is called a *Gröbner basis of I with respect to* $\prec$ if $\Delta_G = \Delta_I$.

The reader is expected to be familiar with fundamental facts of the theory of Gröbner bases (for example presented in [3], [4] or [6]).

3. Reduction of holomorphic functions in a polydisc. We start with the following lemma important in what follows.

LEMMA 3.1. *Let $F \subset \mathcal{R}$ be a finite set and let $\prec$ be an admissible term ordering. Then there exists $r_0 = (r_{01}, \ldots, r_{0n}) \in \mathbb{R}^n_+$ such that*

$$\|\operatorname{in}_\prec f\|_{r_0} > \|\operatorname{tail}_\prec f\|_{r_0} \quad \text{for } f \in F. \tag{2}$$

Proof. By Bayer's Lemma ([2], see also [1]) there exists a linear form

$$L = \sum_{i=1}^n \ell_i X_i \quad \text{with } \ell_i \in \mathbb{N}_+,\ i = 1, \ldots, n,$$

such that, for any $\alpha, \beta \in \operatorname{supp} F$, if $\alpha \prec \beta$ then $L(\alpha) < L(\beta)$.

Now we define a new admissible term ordering $\prec_L$ as follows:

$$\alpha \prec_L \beta \ \Leftrightarrow\ (L(\alpha) < L(\beta) \text{ or } L(\alpha) = L(\beta) \text{ and } \alpha \prec \beta).$$

Observe that the restrictions of the orderings $\prec_L$ and $\prec$ to $\operatorname{supp} F$ coincide. Put $\varrho_t = (t^{\ell_1}, \ldots, t^{\ell_n})$, $t \in \mathbb{R}$. Since F is finite, there exists $t_0 \in \mathbb{R}_+$ such that

$$\|\mathrm{in}_{\prec_L} f\|_{\varrho_{t_0}} = |\mathrm{lc}_{\prec_L} f| t_0^{L(\exp_{\prec_L} f)} > \|\mathrm{tail}_{\prec_L} f\|_{\varrho_{t_0}} \quad \text{for } f \in F.$$

Since $\mathrm{in}_{\prec} f = \mathrm{in}_{\prec_L} f$ and $\mathrm{tail}_{\prec} f = \mathrm{tail}_{\prec_L} f$ for any $f \in F$, it follows that $r_0 := \varrho_{t_0}$ satisfies

$$\|\mathrm{in}_{\prec} f\|_{r_0} > \|\mathrm{tail}_{\prec} f\|_{r_0} \quad \text{for } f \in F. \tag{3}$$ ∎

Let $\prec$ be an admissible term ordering. We say that $g \in \mathcal{R}$ *reduces to* $g' \in \mathcal{R}$ *modulo* $F \subset \mathcal{R}$, written $g \xrightarrow{F} g'$, if there exist $f \in F$, $\gamma \in \mathbb{N}^n$, $c_\gamma \in \mathbb{K} \setminus \{0\}$ such that

$$g' = g - c_\gamma X^\gamma f \quad \text{and} \quad \gamma + \exp_{\prec} f \in \operatorname{supp} g \setminus \operatorname{supp} g'.$$

That reduction is called a *simple reduction step*.

Lemma 3.2. *Let $F \subset \mathcal{R}$ be a finite set, $\prec$ an admissible term ordering, and let r_0 be as in Lemma 3.1. Then there exists $\varepsilon > 0$ such that*

$$\|g'\|_{r_0} + \varepsilon \|c_\gamma X^\gamma\|_{r_0} \le \|g\|_{r_0} \tag{4}$$

for any simple reduction step $g \xrightarrow{F} g' = g - c_\gamma X^\gamma f$.

Proof. The proof is similar to the proof of Lemma 3.3 from [1]. Since F is finite, there exists $\varepsilon > 0$ such that

$$\|\mathrm{in}_{\prec} f\|_{r_0} \ge \|\mathrm{tail}_{\prec} f\|_{r_0} + \varepsilon \quad \text{for } f \in F. \tag{5}$$

We set $\alpha := \gamma + \exp_{\prec} f$. Then g can be decomposed as $g = c_\alpha X^\alpha + p$ with $\alpha \notin \operatorname{supp} p$ and $c_\alpha = c_\gamma \,\mathrm{lc}_{\prec} f$. Consequently,

$$\begin{aligned} \|g\|_{r_0} &= \|p\|_{r_0} + \|c_\alpha X^\alpha\|_{r_0} = \|p\|_{r_0} + \|c_\gamma X^\gamma \,\mathrm{in}_{\prec} f\|_{r_0} \\ &= \|p\|_{r_0} + \|c_\gamma X^\gamma\|_{r_0} \|\mathrm{in}_{\prec} f\|_{r_0}. \end{aligned}$$

By (5) it follows that

$$\|g\|_{r_0} \ge \|p\|_{r_0} + \|c_\gamma X^\gamma\|_{r_0} (\|\mathrm{tail}_{\prec} f\|_{r_0} + \varepsilon). \tag{6}$$

Applying the triangle inequality to the equation

$$\begin{aligned} g' &= g - c_\gamma X^\gamma f = p + c_\alpha X^\alpha - c_\gamma X^\gamma \,\mathrm{in}_{\prec} f - c_\gamma X^\gamma \,\mathrm{tail}_{\prec} f \\ &= p - c_\gamma X^\gamma \,\mathrm{tail}_{\prec} f \end{aligned}$$

and then using (6) yields

$$\begin{aligned} \|g'\|_{r_0} &\le \|p\|_{r_0} + \|c_\gamma X^\gamma \,\mathrm{tail}_{\prec} f\|_{r_0} = \|p\|_{r_0} + \|c_\gamma X^\gamma\|_{r_0} \|\mathrm{tail}_{\prec} f\|_{r_0} \\ &\le \|g\|_{r_0} - \varepsilon \|c_\gamma X^\gamma\|_{r_0}, \end{aligned}$$

which completes the proof. ∎

Proposition 3.3. *Let $\prec$ be an admissible term ordering. Let $G \subset \mathcal{R}$ be a Gröbner basis of an ideal I and let r_0 be as in Lemma 3.1. Then there* *exists* $\varepsilon > 0$ *such that*

(i) *for any* $f \in \mathcal{R}$ *there exist polynomials* h_g *corresponding to* $g \in G$ *and exactly one polynomial* $f_{\text{red}} \in \mathcal{R}(\mathcal{D}_I)$ *such that*

$$f = \sum_{g \in G} h_g g + f_{\text{red}}, \tag{7}$$

(ii) *the mapping* $\text{red} : \mathcal{R} \ni f \mapsto f_{\text{red}} \in \mathcal{R}(\mathcal{D}_I)$ *is linear*,

(iii) $\|f_{\text{red}}\|_{r_0} + \varepsilon \sum_{g \in G} \|h_g\|_{r_0} \leq \|f\|_{r_0}$ *for* $f \in \mathcal{R}$,

(iv) $\|h_g\|_{r_0} \leq \varepsilon^{-1} \|f\|_{r_0}$ *for* $f \in \mathcal{R}$ *and* $g \in G$,

(v) $\|f_{\text{red}}\|_{r_0} \leq \|f\|_{r_0}$.

Proof. (i) and (ii) follow from the well known Buchberger Algorithm (see e.g. [4, Proposition 1, p. 79]).

To prove (iii) we will use the same method as in the proof of Proposition 3.4(i) in [1]. According to the Buchberger Algorithm, f can be rewritten in the form

$$f = \sum_{\mu=1}^{m} c_\mu X^{\alpha_\mu} g_\mu + f_{\text{red}}$$

with $c_\mu X^{\alpha_\mu}$ which appeared in a simple reduction step of a reduction sequence

$$f \xrightarrow{G} f - c_1 X^{\alpha_1} g_1 \xrightarrow{G} f - \sum_{\mu=1}^{2} c_\mu X^{\alpha_\mu} g_\mu \xrightarrow{G} \cdots \xrightarrow{G} f - \sum_{\mu=1}^{m} c_\mu X^{\alpha_\mu} g_\mu = f_{\text{red}}.$$

Condition (iii) follows by applying Lemma 3.2 to each step of the reduction sequence. Conditions (iv) and (v) are trivial consequences of (iii). ∎

By (v) and since $f_{\text{red}} = 0$ if and only if $f \in I$, the division formula (7) gives a representation of $\mathcal{R}$ as a direct sum

$$\mathcal{R} = I \oplus \mathcal{R}(\mathcal{D}_I)$$

with a continuous projection "red" of $\mathcal{R}$ onto $\mathcal{R}(\mathcal{D}_I)$.

Theorem 3.4. *Let* $\prec$, G, *and* r_0 *be as in Proposition* 3.3. *Then*

(i) *if* $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in E_{r_0}$ *then the series* $\sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha_{\text{red}}$ *is convergent to an* $f_{\text{red}} \in E_{r_0}(\mathcal{D}_I)$,

(ii) *the extended mapping* "red" *gives a continuous projection of* E_{r_0} *onto* $E_{r_0}(\mathcal{D}_I)$,

(iii) $\|f_{\text{red}}\|_{r_0} \leq \|f\|_{r_0}$ *for* $f \in E_{r_0}$,

(iv) *if* $f \in E_{r_0}$ *then* $f_{\text{red}} = 0$ *if and only if* $f \in IE_{r_0}$,

(v) $E_{r_0} = IE_{r_0} \oplus E_{r_0}(\mathcal{D}_I)$ *(direct sum)*.

Proof. (i) follows from condition (v) of Proposition 3.3 and Lemma 2.1. To prove (ii) and (iii) observe that the mapping

$$\text{red} : \mathcal{R} \ni f \mapsto f_{\text{red}} \in \mathcal{R}(\mathcal{D}_I)$$

can be uniquely extended to the Banach space E_{r_0} with preservation of the norm, because it is a densely defined bounded linear mapping.

Since I is dense in IE_{r_0} and $f_{\mathrm{red}} = 0$ for $f \in I$, we have $f_{\mathrm{red}} = 0$ for $f \in IE_{r_0}$, which completes the proof of condition (iv).

To prove (v) take $f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in E_{r_0}$. According to Proposition 3.3 we have

$$X^\alpha = \sum_{g\in G} h_{g,\alpha} g + X^\alpha_{\mathrm{red}}$$

such that $f_{\mathrm{red}} = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha_{\mathrm{red}}$. Therefore,

$$f = \sum_{\alpha\in\mathbb{N}^n}\sum_{g\in G} c_\alpha h_{g,\alpha} g + f_{\mathrm{red}}.$$

Set $h_g := \sum_{\alpha\in\mathbb{N}^n} c_\alpha h_{g,\alpha}$. From condition (iv) of Proposition 3.3 and Lemma 2.1 it follows immediately that $h_g \in E_{r_0}$. Since red : $\mathcal{R} \ni f \mapsto f_{\mathrm{red}} \in \mathcal{R}(\mathcal{D}_I)$ is the identity mapping on the dense subspace $\mathcal{R}(\mathcal{D}_I)$ of $E_{r_0}(\mathcal{D}_I)$, the extended mapping "red" is the identity mapping on $E_{r_0}(\mathcal{D}_I)$, which completes the proof. ■

4. Applications. Let $I \subset \mathcal{R}$ be a polynomial ideal, $\prec$ be an admissible term ordering, and G be the reduced Gröbner basis of I with respect to $\prec$.

Definition 4.1. We say that a polydisc P_r, $r \in \mathbb{R}^n_+$, is *convenient for reduction with respect to I and $\prec$* if

$$\|\mathrm{in}_\prec g\|_r > \|\mathrm{tail}_\prec g\|_r \quad \text{for } g \in G.$$

Proposition 4.2. *If P_r is a polydisc convenient for reduction with respect to an ideal I and a term ordering $\prec$, then for any $f \in E_r$ there exist a unique $h \in IE_r$ and a unique $f_{\mathrm{red}} \in E_r(\mathcal{D}_I)$ such that $f = h + f_{\mathrm{red}}$.*

Proof. This follows immediately from Lemma 3.1 and Theorem 3.4. ■

Define

$$\mathcal{M}_{I,\prec} := \{r \in \mathbb{R}^n_+ : P_r \text{ is a polydisc convenient for reduction with respect to the ideal } I \text{ and the term ordering } \prec\}.$$

Remark 4.3. Since the functions

$$\mathbb{R}^n_+ \ni r \mapsto \|\mathrm{in}_\prec g\|_r - \|\mathrm{tail}_\prec g\|_r \in \mathbb{R},$$

for $g \in G$ are continuous, the set $\mathcal{M}_{I,\prec}$ is open.

Let $I \subset \mathbb{K}[X,Y] := \mathbb{K}[X_1,\ldots,X_n,Y]$ be an ideal. Let $\prec_Y$ be an *elimination ordering* for Y, i.e. an admissible term ordering in $\mathbb{N}^n \times \mathbb{N}$ such that

$$X^\alpha \prec_Y Y^k \quad \text{for } \alpha \in \mathbb{N}^n,\ k \in \mathbb{N}\setminus\{0\}. \tag{8}$$

Let G be the reduced Gröbner basis of the ideal I with respect to $\prec_Y$.

PROPOSITION 4.4. *If* $r = (r_1, \ldots, r_n, r_{n+1}) \in \mathcal{M}_{I,\prec_Y}$, *then*

$$r_t = (r_1, \ldots, r_n, t) \in \mathcal{M}_{I,\prec_Y} \quad \textit{for } t > r_{n+1}.$$

Proof. Let $g \in G$ and $t > r_{n+1}$. Since $r \in \mathcal{M}_{I,\prec_Y}$, we have

$$\|\mathrm{in}_{\prec_Y} g\|_r > \|\mathrm{tail}_{\prec_Y} g\|_r. \tag{9}$$

If $\mathrm{tail}_{\prec_Y} g$ is independent of Y, the right side of (9) is constant and the left side is nondecreasing with respect to r_{n+1}, which completes the proof.

Otherwise, $\mathrm{tail}_{\prec_Y} g$ depends on Y in a degree k. Then $\mathrm{in}_{\prec_Y} g$ also depends on Y. We have the inequality

$$ar_{n+1}^k > \sum_{j=0}^{k} b_j r_{n+1}^j, \tag{10}$$

where $a = a(r_1, \ldots, r_n)$, $b = b(r_1, \ldots, r_n)$, $a, b_j \geq 0$ for $j = 1, \ldots, k$. Multiplying (10) by $(t/r_{n+1})^k$ we obtain

$$at^k > \sum_{j=0}^{k} b_j t^j.$$

Thus $r_t = (r_1, \ldots, r_n, t) \in \mathcal{M}_{I,\prec_Y}$. ∎

THEOREM 4.5. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain,* $f : \Omega \to \mathbb{C}$ *a Nash function in* Ω, *and* $I \subset \mathbb{C}[X, Y]$ *the ideal of the graph of* f. *Let* $P_r = P_{r'} \times P_{r''}$, *where* $r = (r_1, \ldots, r_{n+1})$, $r' = (r_1, \ldots, r_n)$, $r'' = r_{n+1}$, *be a polydisc convenient for reduction with respect to the ideal* I *and* $\prec_Y$, *an elimination ordering for* Y. *If* $P_{r'} \subset \Omega$ *and the graph* Γ_f *of* f *is algebraic in* P_r *then* f *is a polynomial.*

Proof. Since the Zariski closure of Γ_f is an algebraic set of codimension 1, the reduced Gröbner basis G of I with respect to $\prec_Y$ consists of only one polynomial g of the form

$$g(X, Y) = a_k(X)Y^k + a_{k-1}(X)Y^{k-1} + \cdots + a_0(X),$$

with $k \geq 1$ and $a_k \neq 0$, and so $\mathrm{in}_{\prec_Y} g = X^\alpha Y^k$ with an $\alpha \in \mathbb{N}^n$. Hence $G \cap \mathbb{C}[X] = \emptyset$ and $f = f_{\mathrm{red}}$. Since $Y - f(X)$ vanishes on Γ_f and Γ_f is algebraic in P_r, $Y - f(X) \in I\mathcal{O}(P_r)$, where $\mathcal{O}(P_r)$ is the ring of holomorphic functions in P_r (see e.g. [7, Theorem 4.6]). The set $\mathcal{M}_{I,\prec}$ is open (see Remark 4.3). Thus, we can find $\widetilde{r} \in \mathcal{M}_{I,\prec}$ such that the closure of $P_{\widetilde{r}}$ is contained in P_r and for $P_{\widetilde{r}}$ all the assumptions of Theorem 4.5 are satisfied. Since $I\mathcal{O}(P_r) \subset E_{\widetilde{r}}$, we have $Y - f(X) \in IE_{\widetilde{r}}$ and so $0 = (Y - f)_{\mathrm{red}} = Y_{\mathrm{red}} - f_{\mathrm{red}}$, where "red" is the reduction in $E_{\widetilde{r}}$. On the other hand, Y_{red} is a polynomial. Hence so is $f = f_{\mathrm{red}}$, which completes the proof. ∎

EXAMPLE 4.6. Let $f_k(X) := 1/(X-k)$, $k \in \mathbb{N}$, and let $I := \langle (X-k)Y-1 \rangle$ be the ideal in $\mathbb{C}[X, Y]$ generated by $(X-k)Y-1$. The set $G = \{(X-k)Y-1\}$

is the reduced Gröbner basis of I with respect to any elimination ordering for Y.

1. If $P_{(r_1,r_2)}$ is convenient for reduction then $r_1 > k$. Indeed, according to Definition 4.1,

$$r_1 r_2 > k r_2 + 1,$$

which implies that $r_1 > k$.

2. If $r_1 < k$ then $P_{(r_1,r_2)}$ is not convenient for reduction. To see this, fix $0 < r_1 < k$ and consider the Nash function f_k in $\Omega = \{x \in \mathbb{C} : |x| < k\}$ given by

$$f_k(X) = -\frac{1}{k}\sum_{j=0}^{\infty}\left(\frac{X}{k}\right)^j = \frac{1}{X-k}.$$

The series f_k is absolutely convergent at r_1 and so $f_k \in E_{(r_1,r_2)}$, because f_k is independent of Y. Note that

$$\mathcal{D}_I = \{(i,j) \in \mathbb{N}^2 : ij = 0\},$$

and $Y - f_k \in IE_{(r_1,r_2)}$, by the same argument as in the proof of Theorem 4.5. Then

$$0 \neq Y - f_k(X) \in E_{(r_1,r_2)}(\mathcal{D}_I) \cap IE_{(r_1,r_2)},$$

which contradicts condition (iv) of Theorem 3.4.

Acknowledgments. The author wishes to express her thanks to Professor Tadeusz Winiarski for suggesting the problem and for many stimulating conversations.

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Received March 7, 2005 (7464)