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Ulm–Kaplansky invariants of $S(KG)/G$

by

P. V. DANCHEV

Presented by Stanisław KWAPIEŃ

Summary. Let G be an infinite abelian p-group and let K be a field of the first kind with respect to p of characteristic different from p such that $s_p(K) = \mathbb{N}$ or $s_p(K) = \mathbb{N} \cup \{0\}$. The main result of the paper is the computation of the Ulm–Kaplansky functions of the factor group $S(KG)/G$ of the normalized Sylow p-subgroup $S(KG)$ in the group ring KG modulo G. We also characterize the basic subgroups of $S(KG)/G$ by proving that they are isomorphic to $S(KB)/B$, where B is a basic subgroup of G.

I. Introduction. Throughout the present paper, G is an abelian p-group (possibly infinite), K is a field of the first kind with respect to p of characteristic different from p, and KG is the group algebra of G over K. The aim of this note is to describe the Ulm–Kaplansky invariants and the basic subgroups of the quotient group $S(KG)/G$, where $S(KG)$ is the Sylow p-subgroup of the normed group $V(KG)$ of units of KG, provided that the spectrum of K contains $\mathbb{N}$, the set of natural numbers.

Notions and notation used are mostly standard and follow [F]. In particular, for any abelian p-group G, we denote by $G[p]$ its socle, by G^1 the first Ulm subgroup of G and by $B = B_G$ a basic subgroup of G. We write also $\exp(G) = \alpha$ if $G^{p^\alpha} = 1$ but $G^{p^\beta} \neq 1$ for all $\beta < \alpha$, where α is an arbitrary ordinal number; actually $\exp(G) = \mathrm{length}(G)$.

Moreover, for a field K, the symbols $\mathrm{const}_p(K)$ and $s_p(K)$ are reserved for the constant and the spectrum of K with respect to p, respectively (see [M1] for more details). In the essential part of all that follows, we will assume that $s_p(K) \supseteq \mathbb{N}$, so either $s_p(K) = \mathbb{N}$ or $s_p(K) = \mathbb{N} \cup \{0\}$.

A key role in the theory of abelian groups and the representation theory of commutative group algebras is played by the Ulm–Kaplansky functions

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defined in [F] in the following manner:

$$f_\alpha(G) = \mathrm{rank}(G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p]), \tag{1}$$

where α is an arbitrary ordinal.

It is well known and documented (see, for instance, [F]) that these cardinal invariants classify up to isomorphism the classes of all torsion-complete abelian p-groups and totally projective abelian p-groups plus some of their modifications.

T. Mollov [M3] calculated these functions for $S(KG)$. Nevertheless, his computations are not sufficient for the full description of the algebraic structure of $S(KG)$ as the following examples show. In [D0] we proved that $S(KG)$ is totally projective if and only if G is a direct sum of cyclic groups, whereas in [D7] it was established that $S(KG)$ is torsion-complete precisely when G is bounded. Thus, in both situations, the properties of $S(KG)$ of being torsion-complete and totally projective are not preserved by the group structure of G. However, if G is inseparable totally projective it follows from [D0] that $S(KG)$ is simply presented but not totally projective, while if G/G^1 is unbounded torsion-complete then $S(KG)$ is not torsion-complete and is not simply presented; in that respect, concerning other major sorts of abelian groups, the reader can also consult [D1], [D2], [D3], [D5], [D8] for the terminology.

Mollov has also claimed that the Ulm–Kaplansky invariants are the best characterizing instruments for the separable abelian p-groups. By what we have just stated above his conclusion is obviously wrong.

Mollov [M1] obtained an important formula for the isomorphism class of $S(KG)$ whenever G is infinite, namely:

$$S(KG) \cong S^1(KG) \times S(K(G/G^1)), \tag{2}$$

where $S^1(KG) \cong \bigoplus_{|G|} \mathbb{Z}(p^\infty)$ when $G^1 \neq 1$, while $S^1(KG) = 1 \Leftrightarrow G^1 = 1$.

We observe that the above isomorphism trivially holds for separable abelian p-groups. That is why it is necessary to find another approach for the complete description of the structure of $S(KG)$. In [D5] we stated a conjecture, named the *Direct Factor Conjecture*, that if A is a separable abelian p-group, then $S(KA)/A$ is a direct sum of cyclic groups, and so A will be a direct factor of $S(KA)$ with a complementary factor which is a direct sum of cyclic groups provided that $s_p(K) \supseteq \mathbb{N}$. If the second half of the conjecture is true, Mollov's preceding isomorphic characterization may be written as

$$S(KG) \cong \begin{cases} \bigoplus_{|G|} \mathbb{Z}(p^\infty) \times G/G^1 \times S(K(G/G^1))/(G/G^1), & G^1 \neq 1; \\ G \times S(KG)/G, & G^1 = 1, \end{cases} \tag{3}$$

where, in both formulas, the last quotient is a direct sum of cyclic groups.

Therefore, to solve the problem of isomorphic classification of $S(KG)$ completely, it is enough to know the structure of this quotient. We indicate that the Ulm–Kaplansky functions serve to classify the direct sums of cyclic groups ([F]). Thus, it is the purpose of this paper to compute these invariants for $S(KG)/G$ in an explicit form, only in terms of K and G or their sections.

II. Main results

LEMMA 0 ([D6]). *If P is a pure subgroup of the abelian p-group C, then $(C/P)[p] = C[p]P/P$.*

LEMMA 1. *Assume P is a pure subgroup of the abelian p-group C. Then, for each $n \geq 0$, $(PC^{p^n})[p] = P[p]C^{p^n}[p]$.*

Proof. Evidently, the left hand side contains the right hand side. For the converse inclusion, let $x \in (PC^{p^n})[p]$, so $x = ac^{p^n}$ for $a \in P$ and $c \in C$ with $a^{-p} = c^{p^{n+1}}$. Then $a^{-p} \in P \cap C^{p^{n+1}} = P^{p^{n+1}}$ and $a^{-p} = b^{p^{n+1}}$ for some $b \in P$. Finally, $ab^{p^n} \in P[p]$ and $x = ab^{p^n}(cb^{-1})^{p^n} \in P[p]C^{p^n}[p]$, and the desired equality follows. ∎

LEMMA 2 ([D7]). *The abelian p-group G is pure in $S(KG)$ provided that $s_p(K)$ contains all naturals.*

Proof. Given $g \in G \cap S^{p^n}(KG)$ for any $n \in \mathbb{N}$, we have $g \in F \cap S^{p^n}(KF)$ where F is a finite subgroup of G. Set $F' = \bigoplus_\omega F$, so F' and $S(KF')$ are infinite bounded. Because the nonzero Ulm–Kaplansky invariants of $S(KF')$ are equal to $|F'|$ (see [M1] or [M3]) and because $f_i(S(KF')) = 0$ implies $f_i(F') = 0 \Leftrightarrow f_i(F) = 0$ for all $i \geq 0$ since $\exp(S(KF')) = \exp(F') = \exp(F)$, we derive that the Ulm–Kaplansky functions of $F \times S(KF')$ and $S(KF')$ are equal. Since these two groups are both bounded, we have $S(KF') \cong F \times S(KF')$, and thus a lemma due to May ([May, Lemma 2]) shows that F is a direct factor of $S(KF)$, whence it is its pure subgroup. Finally, $g \in F^{p^n} \subseteq G^{p^n}$, which yields the asserted property. ∎

REMARK. The foregoing lemma on purity strengthens an analogous assertion due to Karpilovsky [K] when $K = \mathbb{Q}$, the field of all rationals (see also [D7]). Another nontrivial example of a field with spectrum $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$ is the cyclotomic extension $\mathbb{Q}(\xi_{q^i})$ of the field $\mathbb{Q}$, where ξ_{q^i} is a primitive q^i-root of unity whenever i is a non-negative integer and q is a prime.

The following arguments show that the restriction on the spectrum of the coefficient field to have no gaps cannot be ignored.

EXAMPLE. When $s_p(K)$ does not coincide with $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$, G is not pure in $S(KG)$. In fact, if we assume that B is pure in $S(KB)$, from [F, Vol. II, p. 94, property Γ, or p. 102, Corollary 81.3] we deduce that $f_i(B) \leq f_i(S(KB))$ for all $i \geq 0$. But, on the other hand, because there is some $i \in$

$\mathbb{N}\cup\{0\}$ so that $i+1 \notin s_p(K)$, by what we shall prove below, $f_i(S(KB)) = 0$ while $f_i(B)$ may not be zero. Indeed, choose $1 \neq x \in S^{p^i}(KG)[p]$, hence $x = y^{p^i}$ and $y^{p^{i+1}} = 1$ where $y \in S(KG)$. Thus $y \in S(KF)$ where F is a finite subgroup of G. We observe that y belongs to a direct sum of cyclic groups of orders p^j, where $j \in s_p(K)$. But $i+1 \notin s_p(K)$, whence in this direct sum there are no direct summands of order p^{i+1} and $j > i+1$. Then $y = z_1^{p^{s_1}} z_2^{p^{s_2}} \cdots z_k^{p^{s_k}}$, where $s_1, \ldots, s_k \geq 1$ and $k \in \mathbb{N}$. So, $y^{p^i} \in S^{p^i}(KF)$ does imply $y^{p^i} \in S^{p^{i+1}}(KF)$, hence $x \in S^{p^{i+1}}(KG)[p]$, which means that $f_i(S(KG)) = 0$. This substantiates our claim.

Lemma 3 ([D5]). *The abelian p-group G is nice in $S(KG)$ provided G is separable.*

Combining Lemmas 2 and 3, we directly obtain the following consequence.

Corollary 1. *The abelian p-group G is balanced in $S(KG)$ provided G is separable and $s_p(K) \supseteq \mathbb{N}$.*

T. Mollov proved in [M3] that if P is a pure subgroup of the separable p-group G, then $S(KP)$ is pure in $S(KG)$. We shall now establish the following expansion.

Lemma 4. *Assume G is separable with a pure subgroup P such that G/P is divisible. Then $GS(KP)$ is pure in $S(KG)$.*

Proof. Since $G = PG^{p^n}$ for any natural number n, the modular law from [F] and [M3, Proposition 1] ensure that

$$\begin{aligned} [GS(KP)] \cap S^{p^n}(KG) &= [G^{p^n} S(KP)] \cap S^{p^n}(KG) \\ &= G^{p^n}[S(KP) \cap S^{p^n}(KG)] \\ &= G^{p^n} S^{p^n}(KP) = [GS(KP)]^{p^n}, \end{aligned}$$

as required. ∎

As an immediate consequence, we deduce the following.

Corollary 2. *$GS(KB)$ is pure in $S(KG)$ provided G is separable.*

The following technical result demonstrates that the restriction in Lemma 3 and Corollary 1 that G be separable can be dropped.

Proposition 1. *For every abelian p-group G,*

$$S(KG)/[GS^1(KG)] \cong S(K(G/G^1))/(G/G^1).$$

In particular, G is always a nice p-subgroup of $S(KG)$ and so $(S(KG)/G)^1$ is divisible. If in addition $s_p(K) \supseteq \mathbb{N}$, then G is balanced in $S(KG)$.

Proof. Consider the sequence

$$S(KG) \to S(K(G/G^1)) \to S(K(G/G^1))/(G/G^1),$$

where the first map is a surjective group homomorphism linearly extending the natural homomorphism $G \to G/G^1$ (see [M1]), and the second is the natural map. Consequently, their composition $S(KG) \to S(K(G/G^1))/(G/G^1)$ is an epimorphism. We need to find its kernel. In fact, the kernel of the first map from the sequence is $S^1(KG)$ (see [M1]) whereas for the second it is G/G^1. Therefore, it is easily checked that the required kernel of $S(KG) \to S(K(G/G^1))/(G/G^1)$ is $GS^1(KG)$. In fact, that $GS^1(KG)$ is contained in the kernel is apparent. In order to derive the reverse inclusion, we take $\sum_{g\in G} r_g g$ in the kernel $\subseteq S(KG)$, whence $\sum_{g\in G} r_g gG^1 = aG^1$ for some $a \in G$. Furthermore, $\sum_{g\in G} r_g a^{-1}gG^1 = G^1$ and so by [M1] we infer that $\sum_{g\in G} r_g a^{-1}g \in S^1(KG)$. Finally, $\sum_{g\in G} r_g g = a\sum_{g\in G} r_g a^{-1}g \in GS^1(KG)$, thus finishing the argument.

For the last assertion since G/G^1 is separable, Lemma 3 and the isomorphism obtained give that $GS^1(KG)$ is nice in $S(KG)$. Because $S^1(KG)$ is divisible by [M1], the niceness is equivalent to the identities

$$\begin{aligned} G\Big(\bigcap_{n<\omega} S^{p^n}(KG)\Big) = GS^1(KG) &= \bigcap_{n<\omega}(GS^1(KG)S^{p^n}(KG)) \\ &= \bigcap_{n<\omega}(GS^{p^n}(KG)), \end{aligned}$$

which yields the niceness of G in $S(KG)$ (e.g. [F]). Therefore

$$(S(KG)/G)^1 = S^1(KG)G/G \cong S^1(KG)/(S^1(KG)\cap G),$$

and we are done.

The final part follows from the previous one combined with Lemma 2. ∎

We emphasize that either $s_p(K) = \{\mathrm{const}_p(K), \mathrm{const}_p(K)+1, \ldots\}$ or $s_p(K) = \{0, \mathrm{const}_p(K), \mathrm{const}_p(K)+1, \ldots\}$ or $s_p(K) = \{1, \mathrm{const}_p(K), \mathrm{const}_p(K)+1, \ldots\}$. Thus it is quite possible that either $i \in s_p(K)$ but $i+1 \notin s_p(K)$, or $i \notin s_p(K)$ but $i+1 \in s_p(K)$.

It was argued in [M1] that if $\exp(G) = i \in \mathbb{N}$, then

$$\exp(S(KG)) = \begin{cases} i, & i \in s_p(K); \\ \mathrm{const}_p(K), & i \notin s_p(K), \end{cases}$$

of course $1 \le i \le \mathrm{const}_p(K)$, and if $i < \mathrm{const}_p(K)$ then $i \notin s_p(K)$ or $i = 1 \in s_p(K)$.

We can now attack the following statement.

Proposition 2. *For an abelian p-group G and $s_p(K) \supseteq \mathbb{N}$, we have*

$$\exp(S(KG)/G) = i \in \mathbb{N} \;\Leftrightarrow\; \exp(G) = i \in \mathbb{N}.$$

Proof. Since $G^{p^i} = 1$, by what we have already shown above, $S^{p^i}(KG) = 1$, hence $(S(KG)/G)^{p^i} = 1$. Next, if $(S(KG)/G)^{p^j} = 1$ for some $j < i$, employing Lemma 2 we obtain $S^{p^j}(KG) = G^{p^j}$. Therefore the method used in [D7] leads to $G^{p^j} = 1$, which is false. ∎

We indicate the fact, to be used in the main theorem formulated below, that if the quotient G/G^1 is finite, then in [M2], the invariants $f_i(S(K(G/G^1))) = f_i(S(KG))$ are computed only in terms of K and G or their sections. However, we do not reproduce those results here.

Now, we come to the central result that motivated the writing of the present article (compare with the scheme of proof in [D6] for the modular case). Specifically, we proceed by proving the following.

THEOREM 1. *Let G be an abelian p-group with G/G^1 infinite and let $s_p(K) \supseteq \mathbb{N}$. Then, for all $i \geq 0$,*

$$f_i(S(KG)/G) = \begin{cases} |B|, & i < \exp(G/G^1); \\ 0, & i \geq \exp(G/G^1). \end{cases}$$

If G/G^1 is finite, then

$$f_i(S(KG)/G) = f_i(S(K(G/G^1))) - f_i(G/G^1) = f_i(S(KG)) - f_i(G).$$

Proof. Firstly, we concentrate on an infinite group basis and distinguish two basic cases:

CASE 1: *G is separable.* Since $S(KG)/G$ must be separable by Lemma 3, we restrict our attention to the case when i is a positive integer or zero. Now, because from Lemma 2, G is a pure subgroup of $S(KG)$, we derive that it is pure even in $GS^{p^i}(KG) \subseteq S(KG)$. Therefore, owing to Lemmas 0 and 1, we deduce that

$$\begin{aligned}(S(KG)/G)^{p^i}[p] &= (S^{p^i}(KG)G/G)[p] = (S^{p^i}(KG)G)[p]G/G \\ &= S^{p^i}(KG)[p]G/G.\end{aligned}$$

Similarly, $(S(KG)/G)^{p^{i+1}}[p] = S^{p^{i+1}}(KG)[p]G/G$. Hence

$$\begin{aligned}f_i(S(KG)/G) &= \operatorname{rank}(S^{p^i}(KG)[p]G/G/S^{p^{i+1}}(KG)[p]G/G) \\ &= \operatorname{rank}(S^{p^i}(KG)[p]G/S^{p^{i+1}}(KG)[p]G) \\ &= \operatorname{rank}(S^{p^i}(KG)[p]/S^{p^{i+1}}(KG)[p]G^{p^i}[p]),\end{aligned}$$

since

$$\begin{aligned}&S^{p^i}(KG)[p]G/S^{p^{i+1}}(KG)[p]G \\ &\qquad \cong S^{p^i}(KG)[p]/(S^{p^i}(KG)[p] \cap (GS^{p^{i+1}}(KG)[p])) \\ &\qquad = S^{p^i}(KG)[p]/(S^{p^{i+1}}(KG)[p]G^{p^i}[p])\end{aligned}$$

according to the modular law from [F] and to Lemma 2. As a final step,

$$f_i(S(KG)/G) = \begin{cases} |S^{p^i}(KG)[p]/(S^{p^{i+1}}(KG)[p]G^{p^i}[p])| & \text{if it is infinite;} \\ \log_p |S^{p^i}(KG)[p]/(S^{p^{i+1}}(KG)[p]G^{p^i}[p])| & \text{otherwise.} \end{cases}$$

We consider two possibilities for i.

1. $i < \exp(G)$. Because the cardinality of an epimorphic image of an arbitrary group is less than or equal to the cardinality of the whole group, invoking [M3] we find that

$$|S^{p^i}(KG)[p]/(S^{p^{i+1}}(KG)[p]G^{p^i}[p])| \leq |S^{p^i}(KG)[p]/S^{p^{i+1}}(KG)[p]| = |B|.$$

On the other hand, by [D5] or [D7], $S(KB)/B$ is a direct sum of cyclic groups and B is a direct factor of $S(KB)$. So, we may write $S(KB) \cong B \times S(KB)/B$. Therefore, $f_i(S(KB)/B) = |B|$ via exploiting the facts that $|S(KB)/B| = |B|$ and that the cyclic factors of $S(KB)/B$ of order p are precisely $|B|$, which follows analogously to [M1, Theorem 12]. Moreover, $S(KB)/B \cong S(KB)G/G$, and Corollary 2 along with [F] show that $S(KB)G/G$ is pure in $S(KG)/G$. But, as we have already observed, Lemma 3 implies that $S(KG)/G$ is separable, so furthermore $S(KB)G/G$ is isotype in $S(KG)/G$. Hence, $f_i(S(KB)G/G) \leq f_i(S(KG)/G)$, i.e. $f_i(S(KG)/G) \geq |B|$, as expected. Thus $f_i(S(KG)/G) = |B|$.

2. $i \geq \exp(G)$. It follows automatically from Proposition 2 that we have zero Ulm–Kaplansky invariants.

Case 2: *G is arbitrary.* Since $S^1(KG)$ is divisible by [M1], we deduce that so is its epimorphic image

$$S^1(KG)G/G \cong S^1(KG)/[G \cap S^1(KG)] = S^1(KG)/G^1.$$

Therefore by [F] and Proposition 1,

$$\begin{aligned} S(KG)/G &\cong S^1(KG)G/G \times S(KG)/(GS^1(KG)) \\ &\cong S^1(KG)G/G \times S(K(G/G^1))/(G/G^1). \end{aligned}$$

In conjunction with [F, p. 185, Exercise 8], we get

$$f_i(S(KG)/G) = f_i(S(K(G/G^1))/(G/G^1)).$$

Now, Case 1 is applicable to yield $f_i(S(KG)/G) = |B_{G/G^1}| = |BG^1/G^1| = |B|$, because $BG^1/G^1 \cong B$, when $i < \exp(G/G^1)$, while $f_i(S(KG)/G) = 0$ when $i \geq \exp(G/G^1)$.

The final part when the factor G/G^1 is finite follows like this. In view of [D5] and Lemma 2,

$$S(K(G/G^1)) \cong G/G^1 \times [S(K(G/G^1))/(G/G^1)]$$

since $S(K(G/G^1))$ must be finite. Hence [F] together with the above established isomorphism and equality ratios plus (2) can be employed. ∎

REMARKS. The statement of the main Theorem 7 from [M3] is incorrect as it stands. Indeed, if G is an infinite but bounded direct sum of cyclic groups so that $\exp(G) = \mathrm{const}_p(K)$ it is apparent that $\exp(S(KG)) = \mathrm{const}_p(K)$, hence $f_{\mathrm{const}_p(K)}(S(KG)) = 0$, contrary to Mollov's claim that $f_{\mathrm{const}_p(K)}(S(KG)) = |G| \geq \aleph_0$. This allows us to conclude that if we include for the group basis the extra condition "unbounded", everything will be available. In [D5] we have used Mollov's claim above for a concrete situation, but our conclusions are true since if G is separable then $\exp(G) = \exp(B)$.

The correct formulation of Mollov's claim would be the following ($i \in \mathbb{N}$):

$$f_i(S(KG)) = \begin{cases} |B|, & i+1 \in s_p(K) \text{ but} \\ & i = \exp(G) \notin s_p(K) \text{ or } i < \exp(G) \in s_p(K); \\ 0, & i+1 \notin s_p(K) \text{ or } i \geq \exp(G) \in s_p(K) \text{ or} \\ & s_p(K) \not\ni \exp(G) < i = \mathrm{const}_p(K). \end{cases}$$

The above ideas enable us to compute the Ulm–Kaplansky functions of $S(KG)$ in the general situation when G is inseparable and G/G^1 is infinite. In order to do this, utilizing (2) together with [F], we obtain

$$f_i(S(KG)) = f_i(S(K(G/G^1))) = |B_{G/G^1}| = |B|$$

when $i+1 \in s_p(K)$ and either $i < \exp(G/G^1) \in s_p(K)$ or $i = \exp(G/G^1) \notin s_p(K)$, but $f_i(S(KG)) = 0$ when $i+1 \notin s_p(K)$ or $i \geq \exp(G/G^1) \in s_p(K)$ or $s_p(K) \not\ni \exp(G/G^1) < i = \mathrm{const}_p(K)$. When G/G^1 is finite, everything was done in [M2].

NOTE. Applying our method of proof of Theorem 1, it is not difficult to establish that $f_\alpha(G/A) \leq f_\alpha(G)$ whenever A is a balanced subgroup of the abelian p-group G and α is an ordinal, and also that $f_i(G/G^1) = f_i(G)$ for $i \in \mathbb{N} \cup \{0\}$.

COROLLARY 3. *Suppose that*, *under the assumptions of Theorem* 1, *the Direct Factor Conjecture holds. Then the isomorphic structure of* $S(KG)$ *is completely determined.*

Proof. This follows from (3), Theorem 1 and the fact that the direct sums of cyclic p-groups can be classified via the Ulm–Kaplansky cardinal numbers [F]. ∎

We proved in [D7] that if G is a direct sum of cyclic p-groups, then the same holds for $S(KG)/G$ provided that $s_p(K) \supseteq \mathbb{N}$, and later on in [D5] we have confirmed the same assertion but without the condition on $s_p(K)$. Our main goal here is also to characterize the isomorphic form of a basic subgroup of $S(KG)/G$, a problem first posed in [D4].

PROPOSITION 3. *Under the hypotheses of Theorem* 1,

$$B_{S(KG)/G} \cong S(KB)/B.$$

Proof. If G is finite then $G = B$, and so appealing to [D5] we infer that $S(KG)/G$ must be finite, whence $B_{S(KG)/G} = S(KG)/G = S(KB)/B$.

Let now G be infinite. Referring to the above stated isomorphism from Case 2 and to [F], $B_{S(KG)/G} \cong B_{S(KA)/A}$ where $A = G/G^1$. Since $B_A \cong B$ and $S(KB_A)/B_A \cong S(KB)/B$ (this follows even from the cancellation property when B_A is finite), we may consider A to be infinite, hence so is B. In view of [D5, D7], $S(KB)/B$ is a direct sum of cyclic groups and as we have previously seen, $f_i(S(KB)/B) = |B|$ when $i < \exp(B)$, while $f_i(S(KB)/B) = 0$ when $i \geq \exp(B)$. Moreover, bearing in mind [F], we infer at once that $f_i(B_{S(KG)/G}) = f_i(S(KG)/G)$, and so Theorem 1 guarantees that $f_i(B_{S(KG)/G}) = f_i(S(KB)/B)$ since $\exp(A) = \exp(B)$. Finally, [F] ensures that $B_{S(KG)/G}$ and $S(KB)/B$ must be isomorphic. ∎

PROPOSITION 4. *Suppose that G is separable. Then*

$$S(KB)G/G \subseteq B_{S(KG)/G}.$$

Proof. Before proving the statement, we need one useful technical assertion that follows directly from a result due to L. Kovacs published in [F, p. 167, Theorem 33.4].

CLAIM. *Every pure subgroup which is a direct sum of cyclic groups is contained in a basic subgroup.*

Then the application of Corollary 2 and of the fact that $S(KB)G/G \cong S(KB)/B$ is a direct sum of cyclic groups by [D5] completes the verification of the inclusion. ∎

We end the paper with a short comment.

First, we ask whether the theorem remains true when $s_p(K) \subset \mathbb{N}$. The problem of finding explicitly the basic subgroups of $S(KG)$ and $S(KG)/G$ for a semisimple group ring KG is open as well. Furthermore, the calculation of the Warfield p-invariants for $V(KG)$ and $V(KG)/G$ would be of certain interest and importance (for the modular situation, the reader can see [D3]).

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P. V. Danchev
13, General Kutuzov Street, block 7, floor 2, flat 4
4003 Plovdiv, Bulgaria
E-mail: pvdanchev@yahoo.com
pvdanchev@mail.bg

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