Smooth Extensions of Bernoulli Shifts

by

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Summary. For homographic extensions of the one-sided Bernoulli shift we construct \( \sigma \)-finite invariant and ergodic product measures. We apply the above to the description of invariant product probability measures for smooth extensions of one-sided Bernoulli shifts.

0. Introduction. Let \( \sigma \) be the one-sided \((p, q)\)-Bernoulli shift on the space \( \Omega = \{0, 1\}^\mathbb{Z}, \mathbb{Z} = \{0, 1, 2, \ldots\} \), with the \((p, q)\)-measure \( \mu_p \) on \((\Omega, \mathcal{B})\), where \( \mathcal{B} \) is the Borel product \( \sigma \)-algebra. Let us consider two transformations \( T_0, T_1 \) of the interval \([0, 1]\) onto itself such that \( T_i \in C^2[0, 1], T_i' > 0, T_i(0) = 0, T_i(1) = 1 \) for \( i = 0, 1 \) and \( T_0 \geq I, T_1 \leq I \) where \( I(x) = x \) for \( x \in [0, 1] \). We define the transformation

\[
T(\omega, x) = (\sigma(\omega), T_{\omega(0)}^{-1}(x)).
\]

This transformation is the realization of the random map \( T(x) = T_0^{-1}(x) \) with probability \( p \) and \( T(x) = T_1^{-1}(x) \) with probability \( q \). Let \( \Lambda \) denote the Lebesgue measure on \([0, 1]\). It will cause no confusion if we use the same letter to designate the Lebesgue measure on \( \mathbb{R}^+ \). Let \( M_p \) denote the set of \( T \)-invariant measures such that \( m[\mathcal{B} \times [0, 1]] = \mu_p \) for \( m \in M_p \). The product measures in \( M_p \) allow us to describe the distribution of almost every trajectory of random map. Therefore our purpose is to get a description of such measures in \( M_p \). Some results on this topic have been obtained in [K] for transformations \( T_i = (1-\varepsilon_i)x+\varepsilon_ig(x), i = 0, 1, \) where \( g \in C^2[0, 1], g(0) = 0, g(1) = 1, (1-\sup g')^{-1} < \varepsilon_0, \varepsilon_1 < (1-\inf g')^{-1}. \) Here we additionally assume that there exists exactly one point \( x_0 \) for which \( g'(x_0) = 1 \) and

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either $g'(x) < 1$ for $x < x_0$ or $g'(x) > 1$ for $x < x_0$. Furthermore, the above mentioned paper contains conditions which ensure that $M_p$ consists of product measures and that it contains no absolutely continuous measures.

The present paper extends these results as follows. In Section 1 we consider commuting homographic transformations $T_i$, $i = 0, 1$. For the homographic extensions of the one-sided Bernoulli shift we determine $\sigma$-finite invariant and ergodic equivalent product measures. Next, we describe the cluster points of $(A^n I)_{n \in \mathbb{N}}$, where $A$ is the operator acting on the set of distribution functions of probability measures determined by $p$ and $T_i$, $i = 0, 1$. In Section 2 we consider the transformations $T_i$ (the same as in [K]), $i = 0, 1$, and the operator $A_T$ determined by $p$ and $T_i$, $i = 0, 1$. We prove (Theorem 3) that if $\lim_{n \to \infty} A^n_T I = 0$ or 1 then the set of product measures in $M_p$ is $\text{conv}\{\mu_p \times \delta_0, \mu_p \times \delta_1\}$. Next, using the results of Section 1 we show (for some instances) how the cluster points of $(A^n_T I)_{n \in \mathbb{N}}$ may be determined. In particular we extend the description of $M_p$ for the example from [K].

1. Homographic extensions of Bernoulli shifts. Let us consider the transformation of the unit interval of the form

$$ T(x) = \frac{ax + b}{cx + d} \quad \text{where } a, b, c, d \in \mathbb{R}. $$

Under the assumptions:

$$ T : [0, 1] \to [0, 1], \quad T(0) = 0, \quad T(1) = 1, $$

$$ T'(0) > 1, \quad T'(1) < 1 \quad \text{or} \quad T'(0) < 1, \quad T'(1) > 1 $$

we have

$$ T(x) = T_\lambda(x) = \frac{x}{\lambda x + 1 - \lambda}, \quad \text{where } \lambda \in (-\infty, 1) \setminus \{0\}. $$

Moreover, $T_\lambda \geq I$ for $\lambda \in (0, 1)$ and $T_\lambda \leq I$ for $\lambda \in (-\infty, 0)$. The one-parameter family $T_\lambda$ of homographic maps commutes and

$$ T_{\lambda_0} \circ T_{\lambda_1} = T_{1 - (1 - \lambda_0)(1 - \lambda_1)}, \quad T_{\lambda}^{-1} = T_{-\lambda/(1 - \lambda)}. $$

Let $\sigma$ be the one-sided $(p, q)$-Bernoulli shift. Take $T_{\lambda_0}, T_{\lambda_1}$ for $\lambda_0 \in (0, 1)$ and $\lambda_1 \in (-\infty, 0)$ and define the transformation

$$ T(\omega, x) = (\sigma(\omega), T_{\lambda_0}^{-1}(x)). \quad (1) $$

Let $\mathcal{D}$ be the set of distribution functions of probability measures on $[0, 1]$. Define the operator $A$ on $\mathcal{D}$ as follows:

$$ AF(x) = p F(T_{\lambda_0}(x)) + q F(T_{\lambda_1}(x)) \quad \text{for } F \in \mathcal{D}. $$

Let $\nu_F$ denote the measure determined by $F$. \hfill
FACT 1 ([K]). The measure $\mu_p \times \nu_F$ is $T$-invariant if and only if $\mathcal{A}F = F$.

We compute $\mathcal{A}^n I$:

$$\mathcal{A}^n I = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} T_{1-(1-\lambda_0)^k(1-\lambda_1)^{n-k}}$$

$$= \frac{x}{1-x} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left[ (1-\lambda_0)^k (1-\lambda_1)^{n-k} + \frac{x}{1-x} \right]^{-1}$$

for $x \in (0, 1)$. Consequently,

$$\mathcal{A}^n I(x) \leq \frac{x}{1-x} \left( \frac{p}{1-\lambda_0} + \frac{q}{1-\lambda_1} \right)^n \quad \text{for } x \in (0, 1).$$

Since

$$\frac{p}{1-\lambda_0} + \frac{q}{1-\lambda_1} < 1 \iff p < \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}$$

we get

FACT 2.

$$\lim_{n \to \infty} \mathcal{A}^n I(x) = 0 \quad \text{for } x \in [0, 1) \quad \text{and} \quad p < \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}.$$ 

FACT 3. $\mathcal{A}I \geq I$ for $p \geq \frac{\lambda_1}{\lambda_0 - \lambda_0}$ and $\mathcal{A}I \leq I$ for $p \leq \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}$.

Proof. Observe that $\mathcal{A}I \geq I$ if and only if

$$d(x) = \frac{p}{\lambda_0 x + 1 - \lambda_0} + \frac{q}{\lambda_1 x + 1 - \lambda_1} \geq 1$$

for every $x \in [0, 1]$. In particular

$$d(0) \geq 1 \iff p \geq \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} \quad \text{and} \quad d(1) = 1.$$ 

Now $d$ attains its minimum exactly at

$$x_0 = 1 + \left( \sqrt{\frac{-\lambda_0 p}{\lambda_1 q}} - 1 \right) \left( \lambda_0 - \lambda_1 \sqrt{\frac{-\lambda_0 p}{\lambda_1 q}} \right)^{-1}.$$ 

Since $x_0 \geq 1$ if and only if $p \geq \frac{\lambda_1}{\lambda_1 - \lambda_0}$, we get $\mathcal{A}I \geq I$ for $p \geq \frac{\lambda_1}{\lambda_1 - \lambda_0}$. The proof of the case $\mathcal{A}I \leq I$ is similar. \(\blacksquare\)

We now state a result that will be of use later.

LEMMA 1.

$$\frac{d}{dx}(\mathcal{A}^n I)(x) \leq \frac{2}{(1-x)x} \quad \text{for } x \in (0, 1) \quad \text{and} \quad n \in \mathbb{N}.$$ 

Proof. Let

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x)^{-1}$$

where $a$ and $b$ are real numbers. We have

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x)^{-1}$$

for $x \in (0, 1)$ and $n \in \mathbb{N}$. The proof of the case $\mathcal{A}I \leq I$ is similar. \(\blacksquare\)
where $a = 1 - \lambda_0$, $b = 1 - \lambda_1$. Then
\[ f'_n(x) = -\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x)^{-2}. \]

Therefore
\[ |f'_n(x)| \leq \frac{1}{2x} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x/2)^{-1} \]
and consequently
\[ (3) \quad |f'_n(x)| \leq \frac{1}{2x} f_n \left( \frac{x}{2} \right). \]

Combining the equalities
\[ f_n(x) = \frac{1}{x} A^n I \left( \frac{x}{1+x} \right) \quad \text{and} \quad A^n I(x) = \frac{x}{1-x} f_n \left( \frac{x}{1-x} \right) \]
with (3) we get
\[ \frac{d}{dx} A^n I(x) = \frac{1}{(1-x)^2} f_n \left( \frac{x}{1-x} \right) + \frac{x}{1-x} \frac{1}{(1-x)^2} f'_n \left( \frac{x}{1-x} \right) \]
and
\[ \frac{d}{dx} A^n I(x) \leq \frac{1}{(1-x)x} \left( A^n I(x) + A^n I \left( \frac{x}{2-x} \right) \right). \]

This gives
\[ \frac{d}{dx} A^n I(x) \leq \frac{2}{(1-x)x} \quad \text{for} \ x \in (0,1) \]
and the proof is complete. \(\blacksquare\)

The next lemma ensures the existence of a \(\sigma\)-finite \(T\)-invariant measure.

**Lemma 2.** For every
\[ p \neq \frac{\ln(1-\lambda_1)}{\ln(\gamma_0)} \]
there exists \(s \neq 0\) such that
\[ AF_p = F_p \quad \text{for} \ F_p = \left( \frac{x}{1-x} \right)^s. \]

**Proof.** We first observe that the following identity holds:
\[ \frac{T_\lambda(x)}{1 - T_\lambda(x)} = \frac{1}{1 - \lambda} \frac{x}{1 - x} \]
for \( x \in (0, 1) \) and \( \lambda \in (-\infty, 1) \). From this it follows that
\[
A \left( \frac{x}{1-x} \right)^s = p \frac{1}{(1-\lambda_0)^s} \left( \frac{x}{1-x} \right)^s + q \frac{1}{(1-\lambda_1)^s} \left( \frac{x}{1-x} \right)^s
\]
\[
= \left[ p \frac{1}{(1-\lambda_0)^s} + q \frac{1}{(1-\lambda_1)^s} \right] \left( \frac{x}{1-x} \right)^s.
\]
Therefore,
\[
A \left( \frac{x}{1-x} \right)^s = \left( \frac{x}{1-x} \right)^s \Leftrightarrow p \frac{1}{(1-\lambda_0)^s} + q \frac{1}{(1-\lambda_1)^s} = 1
\]
\[
\Leftrightarrow p = \frac{(1-\lambda_1)^s - 1}{(1-\frac{1}{1-\lambda_0})^s - 1}.
\]
Let
\[
\gamma(s) = \frac{(1-\lambda_1)^s - 1}{(1-\frac{1}{1-\lambda_0})^s - 1} \quad \text{for} \ s \neq 0
\]
and
\[
\gamma(0) = \lim_{s \to 0} \gamma(s) = \frac{\ln(1-\lambda_1)}{\ln(1-\frac{1}{1-\lambda_0})}.
\]
The function \( \gamma \) is continuous, strictly decreasing on \( \mathbb{R} \) and \( \lim_{s \to -\infty} \gamma(s) = 1 \), \( \lim_{s \to \infty} \gamma(s) = 0 \). This shows that for every \( p \neq \gamma(0) \) there exists exactly one \( s \neq 0 \) such that (4) holds, which completes the proof. \( \blacksquare \)

Let \( T \) be given by (1) and \( m = \mu_p \times \nu_{F_p} \). By using the fact that \( \mu_p \) is Bernoulli it is easy to check that
\[
\mu_p \times \nu_{F_p}(T^{-1}(B \times [0, x])) = \mu(B) AF_p(x)
\]
for every \( B \in \mathcal{B} \) and \( x \in [0, 1] \). Hence by Lemma 2 we see that \( m \) is a \( \sigma \)-finite infinite \( T \)-invariant measure equivalent to \( \mu_p \times \Lambda \). The following theorem ensures the ergodicity of \((T, m)\):

**Theorem 1.** If
\[
\frac{\ln(1-\lambda_1)}{\ln(1-\lambda_0)} \text{ is an irrational number and } p \neq \frac{\ln(1-\lambda_1)}{\ln(1-\frac{1}{1-\lambda_0})}
\]
then the dynamical system \((T, m)\) is ergodic.

**Proof.** Let \( B \) be a \( T \)-invariant set. By the theorem of Morita [M], \( B = \Omega \times A \) for a measurable set \( A \). Here \( T_i(A) = A \) for \( T_i = T_{\lambda_i}, i = 0, 1 \). Suppose \( \Lambda(A) > 0 \). Our goal is to show that \( \Lambda(A) = 1 \). To do this we introduce the set \( C = J(A) \) where
\[
J(x) = \frac{1-x}{x}.
\]
Since
\[
J(T_0^m T_1^n(x)) = (1-\lambda_0)^m (1-\lambda_1)^n J(x)
\]
we have
\[(1 - \lambda_0)^m (1 - \lambda_1)^n C = C.\]
Let \(\alpha = \ln(1 - \lambda_0)/\ln(1 - \lambda_1).\) By the assumption the set \(\{m\alpha + n : m, n \in \mathbb{Z}\}\)
is dense in \(\mathbb{R}\) and consequently the set
\[E = \{(1 - \lambda_0)^m (1 - \lambda_1)^n : m, n \in \mathbb{Z}\}\]
is dense in \(\mathbb{R}^+.\) Suppose on the contrary that \(\Lambda(A) < 1.\) Then \(\Lambda(\mathbb{R}^+ \setminus C) > 0.\)
Let \(x\) and \(y \neq 0\) be density points of \(\mathbb{R}^+ \setminus C\) and \(C,\) respectively. For \(\varepsilon < 1/2\) there exists \(\delta > 0\) such that for any interval \(Q,\) if \(\Lambda(Q) < \delta\) and \(x \in Q\)
(resp. \(y \in Q\)) then
\[(6) \quad \Lambda(Q \cap \mathbb{R}^+ \setminus C) \geq (1 - \varepsilon)\Lambda(Q) \quad \text{(resp. } \Lambda(Q \cap C) \geq (1 - \varepsilon)\Lambda(Q)).\]
Let \(Q\) be such that \(y \in Q,\) \((x/y)\Lambda(Q) < \delta\) and \(\Lambda(Q) < \delta.\) By the density
of \(E\) in \(\mathbb{R}^+\) there exists \(\beta \in E\) such that \(\beta\Lambda(Q) < \delta\) and \(x \in \beta Q.\) Thus we get
\[\Lambda(\beta Q \cap \mathbb{R}^+ \setminus C) > (1 - \varepsilon)\beta\Lambda(Q)\]
and also
\[\Lambda(\beta Q \cap C) = \Lambda(\beta Q \cap \beta C) = \beta\Lambda(Q \cap C)\]
\[\geq (1 - \varepsilon)\beta\Lambda(Q) \quad \text{by (6),}\]
which is impossible. ■

**Corollary 1.** If \(\lambda_0, \lambda_1, p\) satisfy (5) then \((T, m)\) has no product absolutely continuous invariant probability measure.

**Proof.** Suppose on the contrary that such a measure exists. Then \(T\)
is conservative and ergodic. Hence by the unicity of invariant measure
([A, Th. 1.5.6]) the measure \(\nu_{F_p}\) is finite, which is impossible. ■

**Corollary 2.** If \(\lambda_0, \lambda_1, p\) satisfy (5) then the cluster points of
\[\left(\frac{1}{n} \sum_{k=0}^{n-1} A^k I\right)_{n \in \mathbb{N}}\]
are constant functions on \((0, 1).\)

**Proof.** By Lemma 1 the family of functions \(A^n I, n \in \mathbb{N},\) is uniquely
Lipschitzian on \([\varepsilon, 1 - \varepsilon]\) for every \(0 < \varepsilon < 1.\) Therefore any cluster point \(F^*\)
of
\[\left(\frac{1}{n} \sum_{k=0}^{n-1} A^n I\right)_{n \in \mathbb{N}}\]
is Lipschitzian on \((0, 1).\) Hence \(\nu_{F^*}\) is a convex combination of \(\delta_{\{0\}}, \delta_{\{1\}}\)
and an absolutely continuous measure. Corollary 1 implies that \(F^* = \text{const}\)
on \((0, 1).\) ■

The following theorem describes the convergence of \((A^n I)_{n \in \mathbb{N}}.\)
Theorem 2. (i) If
\[ p \leq \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} \]
and \( \lambda_0, \lambda_1, p \) satisfy (5) in the case of equality, then
\[ \lim_{n \to \infty} A^n I = 0 \quad \text{almost uniformly on } [0, 1). \]

(ii) If
\[ p \geq \frac{\lambda_1}{\lambda_1 - \lambda_0} \]
and \( \lambda_0, \lambda_1, p \) satisfy (5) in the case of equality, then
\[ \lim_{n \to \infty} A^n I = 1 \quad \text{almost uniformly on } (0, 1]. \]

Proof. (i) The first part of (i) follows from Fact 2. Suppose now \( p = \lambda_1(\lambda_0 - 1)/(\lambda_0 - \lambda_1) \) and \( \lambda_0, \lambda_1, p \) satisfy (5). By Fact 3, \( A^{n+1} I \leq A^n I \leq I \) for \( n = 1, 2, \ldots \), and by Lemma 1, \((A^n I)_{n \in \mathbb{N}}\) is uniquely Lipschitzian on \([\varepsilon, 1 - \varepsilon]\) for every \( 0 < \varepsilon < 1 \). Combining these with Corollary 2 we get \( \lim_{n \to \infty} A^n I = 0 \) on \([0, 1)\) almost uniformly.

(ii) We first show a new estimate of \( \frac{d}{dx} A^n I \) which is more useful for our aim. Let us compute
\[ \frac{d}{dx} A^n I(x) = \frac{1}{x^2} \sum_{k=0}^{n} \binom{n}{k} [p(1 - \lambda_0)]^k[q(1 - \lambda_1)]^{n-k} T_{1-(1-\lambda_0)k(1-\lambda_1)^n-k}(x). \]

Hence
\[ \frac{d}{dx} A^n I(x) \leq \frac{1}{x^2} [p(1 - \lambda_0) + q(1 - \lambda_1)]^n \]
for \( x \in (0, 1] \). Now, if \( \lambda_1/(\lambda_1 - \lambda_0) < p \) or equivalently \( p(1 - \lambda_0) + q(1 - \lambda_1) < 1 \) then \( \frac{d}{dx} A^n I(x) \to 0 \) uniformly on \((\varepsilon, 1]\) for every \( 0 < \varepsilon < 1 \). By Fact 3, \( I \leq A^n I \leq A^{n+1} I \) for \( n = 1, 2, \ldots \). This and the above imply \( \lim_{n \to \infty} A^n I = 1 \) almost uniformly on \((0, 1]\). Let \( p = \lambda_1/(\lambda_1 - \lambda_0) \) and suppose \( \lambda_0, \lambda_1, p \) satisfy (5). Since \( \frac{d}{dx} A^n I(x) \leq 1/x^2 \) for \( x \in (0, 1], n = 1, 2, \ldots \), \((A^n I)_{n \in \mathbb{N}}\) is uniquely Lipschitzian on \((\varepsilon, 1]\) for every \( 0 < \varepsilon < 1 \). Combining this with Corollary 2 we get \( \lim_{n \to \infty} A^n I = 1 \) on \([0, 1)\) almost uniformly. 

2. Product measures for smooth extensions of Bernoulli shifts.

Let us consider two transformations of the unit interval
\[ T_0(x) = (1 - \varepsilon_0)x + \varepsilon_0 g(x), \]
\[ T_1(x) = (1 + \varepsilon_1)x - \varepsilon_1 g(x), \]
where \( g \in C^2[0,1], g(0) = 0, g(1) = 1, (1 - \sup g')^{-1} < \varepsilon_0, -\varepsilon_1 < (1 - \inf g')^{-1} \). We also suppose that there exists exactly one point \( x_0 \) for
which $g'(x_0) = 1$ and either $g'(x) < 1$ for $x < x_0$ or $g'(x) > 1$ for $x < x_0$. Let

$$A_T F = pF(T_0) + qF(T_1) \quad \text{for } F \in \mathcal{D}.$$  

We are thus led to the following strengthening of Lemma 4 of [K].

**Theorem 3.** If $\lim_{n \to \infty} A^n_T I = 0$ (resp. $\lim_{n \to \infty} A^n I = 1$) for $x \in [0, 1)$ (resp. $x \in (0, 1]$) then the set of product measures in $M_p$ is $\text{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}$.

We will give the proof only for the case $\lim_{n \to \infty} A^n I = 0$, the second case is similar. We precede the proof by a lemma.

**Lemma 3.** If $\lim_{n \to \infty} A^n_T I = 0$ for $x \in [0, 1)$ then for any $G \in C[0, 1]$ we have $\lim_{n \to \infty} A^n_T G = G(0)$.

**Proof.** We first observe that $\lim_{n \to \infty} A^n_T (x^k) = 0$ on $[0, 1)$ for $k = 1, 2, \ldots$. This is an immediate consequence of the following implication:

$$x^k \leq x \Rightarrow A^n_T (x^k) \leq A^n_T I \quad \text{for } x \in [0, 1] \text{ and } n = 1, 2, \ldots.$$  

Hence, $\lim_{n \to \infty} A^n_T w = w(0)$ on $[0, 1)$ for any polynomial $w$. Since $A_T$ is a contraction in the supremum norm and the set of polynomials is dense in $C[0, 1]$, we get

$$\lim_{n \to \infty} A^n_T G = G(0) \quad \text{in } [0, 1)$$  

for any $G \in C[0, 1]$.  

**Proof of Theorem 3.** Let $G$ be a distribution function. We define two new ones as follows:

$$G_1(x) = \begin{cases} G(x) & \text{for } x \in [0, 1), \\ G(1^-) & \text{for } x = 1, \end{cases}$$  

and $G_2(x) = G_1(x^-)$ for $x \in (0, 1]$. Here $G_1(x^-)$ means the left-hand limit of $G_1$ at $x$. Now $G_1$ is upper semicontinuous, being nondecreasing and right-continuous. Similarly $G_2$ is lower semicontinuous. By the definition

(7) \hspace{1cm} G_2(x) \leq G(x) \leq G_1(x) \quad \text{for } x \in [0, 1).$$  

By Baire’s theorem there are sequences of continuous functions $(H^2_n)_{n \in \mathbb{N}}$, $(H^1_n)_{n \in \mathbb{N}}$ such that $H^2_n \not\to G_2$ and $H^1_n \not\searrow G_1$. Also,

$$\lim_{n \to \infty} H^2_n(0) = G_2(0) = G(0) = G_1(0) = \lim_{n \to \infty} H^1_n(0).$$  

From Lemma 3 and (7) we conclude that

$$\lim_{n \to \infty} A^n_T G = G(0) \quad \text{for } x \in [0, 1).$$  

Hence, if $A_T G = G$ for $G \in \mathcal{D}$ then $G \equiv G(0)$ on $[0, 1)$, which proves the theorem.  

To make use of the above theorem we apply Theorem 2 to determine the limit of the sequence $(A^n_T I)_{n \in \mathbb{N}}$.  

EXAMPLE 1. Let \( g(x) = T_{\lambda_0}(x) \) for \( \lambda_0 \in (0, 1) \) and
\[
T_\varepsilon(x) = (1 + \varepsilon)x - \varepsilon T_{\lambda_0}(x) \quad \text{for } 0 < \varepsilon < 1/\lambda_0 - 1.
\]
Put \( \mathcal{A}_\varepsilon = \mathcal{A}_T \). We will determine \( \lambda_1 \in (-\infty, 0) \) such that \( \text{sgn}(T_\varepsilon(x) - T_{\lambda_1}(x)) = \text{const} \) for \( x \in (0, 1) \) or equivalently
\[
\text{sgn} \left( (x - 1) \left( \frac{\lambda_1}{\lambda_1 x + 1 - \lambda_1} + \frac{\varepsilon \lambda_0}{\lambda_0 x + 1 - \lambda_0} \right) \right) = \text{const}
\]
for \( x \in (0, 1) \). We have
\[
\begin{align*}
T_\varepsilon(x) &\geq T_{\lambda_1}(x) \iff \lambda_1 \leq \frac{\varepsilon \lambda_0}{\lambda_0(1 + \varepsilon) - 1}, \quad (8) \\
T_\varepsilon(x) &\leq T_{\lambda_1}(x) \iff -\varepsilon \lambda_0 \leq \lambda_1 < 0. \quad (9)
\end{align*}
\]
By (8), \( \mathcal{A}^n I \leq \mathcal{A}_{\varepsilon}^n I \) for \( n \in \mathbb{N} \) and \( \lambda_1 \leq \varepsilon \lambda_0/(\lambda_0(1 + \varepsilon) - 1) \). Therefore, by Theorem 2(ii),
\[
\lim_{n \to \infty} \mathcal{A}_{\varepsilon}^n I = 1 \quad \text{for } p \geq \frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 - \lambda_0}.
\]
If \( p = \frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 - \lambda_0} \) then we assume that \( p, \lambda_0, \lambda_1 = \varepsilon \lambda_0/(\lambda_0(1 + \varepsilon) - 1) \) satisfy (5). Here we use the fact that
\[
\min \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_0} : \lambda_1 \leq \frac{\varepsilon \lambda_0}{\lambda_0(1 + \varepsilon) - 1} \right\}
\]
is achieved for \( \lambda_1 = \varepsilon \lambda_0/(\lambda_0(1 + \varepsilon) - 1) \).

REMARK. We have \( T_\varepsilon(x) \geq T_{\lambda_0}^{-1}(x) \) for \( \varepsilon \leq 1 - \lambda_0 \) because \( T_{\lambda_0}^{-1} = T_{-\lambda_0/(1 - \lambda_0)} \). Hence, by Theorem 2 of [K],
\[
\lim_{n \to \infty} \mathcal{A}_{\varepsilon}^n I = 1 \quad \text{for } p > \max \left\{ \frac{1}{2}, \frac{\varepsilon}{1 + \varepsilon} \right\} \quad \text{and } \varepsilon \leq 1 - \lambda_0.
\]
By (9), \( \mathcal{A}_{\varepsilon}^n I \leq \mathcal{A}^n I \) for \( n \in \mathbb{N} \) and \( -\varepsilon \lambda_0 \leq \lambda_1 < 0 \). Therefore, by Theorem 2(i),
\[
\lim_{n \to \infty} \mathcal{A}_{\varepsilon}^n I = 0 \quad \text{for } p \leq \frac{\varepsilon(1 - \lambda_0)}{1 + \varepsilon}.
\]
If \( p = \varepsilon(1 - \lambda_0)/(1 + \varepsilon) \) then we furthermore assume that \( p, \lambda_0, \lambda_1 = -\varepsilon \lambda_0 \) satisfy (5). Here we use the fact that
\[
\max \left\{ \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} : -\varepsilon \lambda_0 \leq \lambda_1 \leq 0 \right\}
\]
is achieved for \( \lambda_1 = -\varepsilon \lambda_0 \).

REMARK. We have \( T_\varepsilon(x) \leq T_{\lambda_0}^{-1}(x) \) for \( \varepsilon \geq 1/(1 - \lambda_0) \). Hence, by [K],
\[
\lim_{n \to \infty} \mathcal{A}_{\varepsilon}^n I = 0 \quad \text{for } p < \min \left\{ \frac{1}{2}, \frac{\varepsilon}{1 + \varepsilon} \right\} \quad \text{and } \varepsilon \geq \frac{1}{1 - \lambda_0}.
\]
We summarize these considerations in the following theorem.
**Theorem 4.** If

\[ p \in \left(0, \frac{\varepsilon}{1 + \varepsilon} (1 - \lambda_0)\right] \cup \left[\frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 - \lambda_0}, 1\right] \]

then the set of product measures in \( M_p \) is

\[ \text{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}. \]

Here

\[ p = \begin{cases} \frac{\varepsilon}{1 + \varepsilon} (1 - \lambda_0) & \text{if } p, \lambda_0, \lambda_1 = -\varepsilon \lambda_0 \text{ satisfy (5)}, \\ \frac{\varepsilon}{1 + \varepsilon} \frac{1}{1 - \lambda_0} & \text{if } p, \lambda_0, \lambda_1 = \frac{\varepsilon \lambda_0}{\lambda_0(1 + \varepsilon) - 1} \text{ satisfy (5)}. \end{cases} \]

For \( p = \varepsilon/(1 + \varepsilon) \),

\[ M_p = \text{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times A\}. \]

**Example 2.** Let \( g(x) = x^2 \) and

\[ T_0(x) = (1 + \varepsilon_0)x - \varepsilon_0x^2, \quad T_1(x) = (1 - \varepsilon_1)x + \varepsilon_1x^2, \]

for \( 0 \leq \varepsilon_i \leq 1, i = 0, 1 \). We will determine \( \lambda_0 \in (0, 1) \) such that

\[ \text{sgn}(T_0(x) - T_{\lambda_0}(x)) = \text{const} \quad \text{for } x \in (0, 1) \]

or equivalently

\[ \text{sgn}[(1 - x)(\varepsilon_0 \lambda_0 x - \varepsilon_0 \lambda_0 + \varepsilon_0 - \lambda_0)] = \text{const}. \]

We have

\[ T_0(x) \geq T_{\lambda_0}(x) \iff \lambda_0 \leq \frac{\varepsilon_0}{1 + \varepsilon_0}, \]

\[ T_0(x) \leq T_{\lambda_0}(x) \iff \varepsilon_0 \leq \lambda_0. \]

Similarly,

\[ \text{sgn}(T_1(x) - T_{\lambda_1}(x)) = \text{const} \quad \text{for } x \in (0, 1) \]

if and only if

\[ \text{sgn}[(x - 1)(\varepsilon_1 \lambda_1 x - \varepsilon_1 \lambda_1 + \varepsilon_1 + \lambda_1)] = \text{const}. \]

Hence

\[ T_1(x) \geq T_{\lambda_1}(x) \iff \lambda_1 \leq -\frac{\varepsilon_1}{1 - \varepsilon_1}, \]

\[ T_1(x) \leq T_{\lambda_1}(x) \iff -\varepsilon_1 \leq \lambda_1. \]

Denote by \( A_T \) the operator determined by \( p \) and \( T_i, i = 0, 1 \). By (11) and (13), \( A_T^n I \leq A^n I \) for \( n \in \mathbb{N} \) and for \( (\lambda_0, \lambda_1) \in [\varepsilon_0, 1] \times [-\varepsilon_1, 0] \). Therefore, by Theorem 2(i),

\[ \lim_{n \to \infty} A_T^n I = 0 \quad \text{for } p \leq \varepsilon_1 \frac{1 - \varepsilon_0}{\varepsilon_0 + \varepsilon_1}. \]
If \( p = \frac{1 - \varepsilon_0}{\varepsilon_0 + \varepsilon_1} \), then we additionally assume that \( p, \lambda_0 = \varepsilon_0, \lambda_1 = -\varepsilon_1 \) satisfy (5). Here we use the fact that
\[
\max \left\{ \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} : (\lambda_0, \lambda_1) \in [\varepsilon_0, 1] \times [-\varepsilon_1, 0] \right\}
\]
is achieved for \( \lambda_0 = \varepsilon_0, \lambda_1 = -\varepsilon_1 \).

**Remark.** By Theorem 2 of [K], if \( T_0 \circ T_1 \leq I \) then
\[
\lim_{n \to \infty} A^n I = 0 \quad \text{for } p < \min \left\{ \frac{1}{2}, \frac{\varepsilon_1}{\varepsilon_0 + \varepsilon_1} \right\}.
\]
The condition \( T_0 \circ T_1 \leq I \) is not always satisfied, e.g. for \( \varepsilon_0 = 3/4, \varepsilon_1 = 1 \). If the conditions (10) and (12) hold then
\[
A^n T_I \geq A^n I \quad \text{for } n \in \mathbb{N} \text{ and } (\lambda_0, \lambda_1) \in \left[ 0, \frac{\varepsilon_0}{1 + \varepsilon_0} \right] \times \left( -\infty, -\frac{\varepsilon_1}{1 - \varepsilon_1} \right).
\]
Therefore, by Theorem 2(ii),
\[
\lim_{n \to \infty} A^n T_I = 1 \quad \text{for } p \geq \frac{1 + \varepsilon_0}{\varepsilon_0 + \varepsilon_1}.
\]
If \( p = \frac{1 + \varepsilon_0}{\varepsilon_0 + \varepsilon_1} \), then we furthermore assume that \( p, \lambda_0 = \frac{\varepsilon_0}{1 + \varepsilon_0}, \lambda_1 = -\frac{\varepsilon_1}{1 - \varepsilon_1} \) satisfy (5). Here we use the fact that
\[
\min \left\{ \frac{\lambda_1}{\lambda_1 - \lambda_0} : (\lambda_0, \lambda_1) \in \left[ 0, \frac{\varepsilon_0}{1 + \varepsilon_0} \right] \times \left( -\infty, -\frac{\varepsilon_1}{1 - \varepsilon_1} \right) \right\}
\]
is achieved for \( \lambda_0 = \frac{\varepsilon_0}{1 + \varepsilon_0}, \lambda_1 = -\frac{\varepsilon_1}{1 - \varepsilon_1} \).

**Remark.** By Theorem 2 of [K], if \( T_0 \circ T_1 \geq I \) then
\[
\lim_{n \to \infty} A^n T_I = 1 \quad \text{for } p > \max \left\{ \frac{1}{2}, \frac{\varepsilon_1}{\varepsilon_0 + \varepsilon_1} \right\}.
\]
The condition \( T_0 \circ T_1 \geq I \) is not always satisfied, e.g. for \( \varepsilon_0 = \varepsilon_1 = 1/2 \). We summarize Example 2 in the following theorem:

**Theorem 5.** If
\[
p \in \left( 0, \varepsilon_1 \frac{1 - \varepsilon_0}{\varepsilon_0 + \varepsilon_1} \right] \cup \left[ \varepsilon_1 \frac{1 + \varepsilon_0}{\varepsilon_0 + \varepsilon_1}, 1 \right)
\]
then the set of product measures in \( M_p \) is
\[
\text{conv} \{ \mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}} \}.
\]
Here
\[
p = \begin{cases} \varepsilon_1 \frac{1 - \varepsilon_0}{\varepsilon_0 + \varepsilon_1} & \text{if } p, \lambda_0 = \varepsilon_0, \lambda_1 = -\varepsilon_1 \text{ satisfy (5)}, \\ \varepsilon_1 \frac{1 + \varepsilon_0}{\varepsilon_0 + \varepsilon_1} & \text{if } p, \lambda_0 = \frac{\varepsilon_0}{1 + \varepsilon_0}, \lambda_1 = -\frac{\varepsilon_1}{1 - \varepsilon_1} \text{ satisfy (5)}. \end{cases}
\]
For $p = \varepsilon_1/(\varepsilon_0 + \varepsilon_1)$ we have

$$M_p = \text{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times A\}.$$ 

References


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