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Smooth Extensions of Bernoulli Shifts

by

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Summary. For homographic extensions of the one-sided Bernoulli shift we construct σ-finite invariant and ergodic product measures. We apply the above to the description of invariant product probability measures for smooth extensions of one-sided Bernoulli shifts.

0. Introduction. Let σ be the one-sided (p,q)-Bernoulli shift on the space $\Omega = \{0,1\}^{\mathbb{N}}$, $\mathbb{N} = \{0,1,2,\ldots\}$, with the (p,q)-measure μ_p on $(\Omega,\mathcal{B})$, where $\mathcal{B}$ is the Borel product σ-algebra. Let us consider two transformations T_0, T_1 of the interval $[0,1]$ onto itself such that $T_i \in C^2[0,1]$, $T_i' > 0$, $T_i(0) = 0$, $T_i(1) = 1$ for $i = 0,1$ and $T_0 \geq I$, $T_1 \leq I$ where $I(x) = x$ for $x \in [0,1]$. We define the transformation

$$T(\omega,x) = (\sigma(\omega), T_{\omega(0)}^{-1}(x)).$$

This transformation is the realization of the random map $T(x) = T_0^{-1}(x)$ with probability p and $T(x) = T_1^{-1}(x)$ with probability q. Let Λ denote the Lebesgue measure on $[0,1]$. It will cause no confusion if we use the same letter to designate the Lebesgue measure on $\mathbb{R}^+$. Let M_p denote the set of T-invariant measures such that $m|\mathcal{B} \times [0,1] = \mu_p$ for $m \in M_p$. The product measures in M_p allow us to describe the distribution of almost every trajectory of random map. Therefore our purpose is to get a description of such measures in M_p. Some results on this topic have been obtained in [K] for transformations $T_i = (1-\varepsilon_i)x + \varepsilon_i g(x)$, $i = 0,1$, where $g \in C^2[0,1]$, $g(0) = 0$, $g(1) = 1$, $(1-\sup g')^{-1} < \varepsilon_0, \varepsilon_1 < (1-\inf g')^{-1}$. Here we additionally assume that there exists exactly one point x_0 for which $g'(x_0) = 1$ and

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either $g'(x) < 1$ for $x < x_0$ or $g'(x) > 1$ for $x < x_0$. Furthermore, the above mentioned paper contains conditions which ensure that M_p consists of product measures and that it contains no absolutely continuous measures.

The present paper extends these results as follows. In Section 1 we consider commuting homographic transformations T_i, $i = 0, 1$. For the homographic extensions of the one-sided Bernoulli shift we determine σ-finite invariant and ergodic equivalent product measures. Next, we describe the cluster points of $(\mathcal{A}^n I)_{n\in\mathbb{N}}$, where $\mathcal{A}$ is the operator acting on the set of distribution functions of probability measures determined by p and T_i, $i = 0, 1$. In Section 2 we consider the transformations T_i (the same as in [K]), $i = 0, 1$, and the operator $\mathcal{A}_T$ determined by p and T_i, $i = 0, 1$. We prove (Theorem 3) that if $\lim_{n\to\infty} \mathcal{A}_T^n I = 0$ or 1 then the set of product measures in M_p is $\operatorname{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}$. Next, using the results of Section 1 we show (for some instances) how the cluster points of $(\mathcal{A}_T^n I)_{n\in\mathbb{N}}$ may be determined. In particular we extend the description of M_p for the example from [K].

1. Homographic extensions of Bernoulli shifts. Let us consider the transformation of the unit interval of the form

$$T(x) = \frac{ax+b}{cx+d} \quad \text{where } a, b, c, d \in \mathbb{R}.$$

Under the assumptions:

$$\begin{gathered} T : [0,1] \to [0,1], \quad T(0) = 0, \quad T(1) = 1, \\ T'(0) > 1,\ T'(1) < 1 \quad \text{or} \quad T'(0) < 1,\ T'(1) > 1 \end{gathered}$$

we have

$$T(x) = T_\lambda(x) = \frac{x}{\lambda x + 1 - \lambda}, \quad \text{where} \quad \lambda \in (-\infty, 1) \setminus \{0\}.$$

Moreover, $T_\lambda \ge I$ for $\lambda \in (0,1)$ and $T_\lambda \le I$ for $\lambda \in (-\infty, 0)$. The one-parameter family T_λ of homographic maps commutes and

$$T_{\lambda_0} \circ T_{\lambda_1} = T_{1-(1-\lambda_0)(1-\lambda_1)}, \qquad T_\lambda^{-1} = T_{-\lambda/(1-\lambda)}.$$

Let σ be the one-sided (p,q)-Bernoulli shift. Take T_{λ_0}, T_{λ_1} for $\lambda_0 \in (0,1)$ and $\lambda_1 \in (-\infty, 0)$ and define the transformation

$$T(\omega, x) = (\sigma(\omega), T^{-1}_{\lambda_{\omega(0)}}(x)). \tag{1}$$

Let $\mathcal{D}$ be the set of distribution functions of probability measures on $[0,1]$. Define the operator $\mathcal{A}$ on $\mathcal{D}$ as follows:

$$\mathcal{A}F(x) = pF(T_{\lambda_0}(x)) + qF(T_{\lambda_1}(x)) \quad \text{for } F \in \mathcal{D}.$$

Let ν_F denote the measure determined by F.

FACT 1 ([K]). *The measure* $\mu_p \times \nu_F$ *is* T*-invariant if and only if* $\mathcal{A}F = F$.

We compute $\mathcal{A}^n I$:

$$\mathcal{A}^n I = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} T_{1-(1-\lambda_0)^k(1-\lambda_1)^{n-k}}$$
$$= \frac{x}{1-x} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left[(1-\lambda_0)^k (1-\lambda_1)^{n-k} + \frac{x}{1-x}\right]^{-1}$$

for $x \in (0,1)$. Consequently,

$$\mathcal{A}^n I(x) \le \frac{x}{1-x}\left(\frac{p}{1-\lambda_0} + \frac{q}{1-\lambda_1}\right)^n \quad \text{for } x \in (0,1). \tag{2}$$

Since

$$\frac{p}{1-\lambda_0} + \frac{q}{1-\lambda_1} < 1 \iff p < \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}$$

we get

FACT 2.

$$\lim_{n\to\infty} \mathcal{A}^n I(x) = 0 \quad \textit{for } x \in [0,1) \quad \textit{and} \quad p < \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}.$$

FACT 3. $\mathcal{A}I \ge I$ *for* $p \ge \frac{\lambda_1}{\lambda_1 - \lambda_0}$ *and* $\mathcal{A}I \le I$ *for* $p \le \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}$.

Proof. Observe that $\mathcal{A}I \ge I$ if and only if

$$d(x) = \frac{p}{\lambda_0 x + 1 - \lambda_0} + \frac{q}{\lambda_1 x + 1 - \lambda_1} \ge 1$$

for every $x \in [0,1]$. In particular

$$d(0) \ge 1 \iff p \ge \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} \text{ and } d(1) = 1.$$

Now d attains its minimum exactly at

$$x_0 = 1 + \left(\sqrt{\frac{-\lambda_0 p}{\lambda_1 q}} - 1\right)\left(\lambda_0 - \lambda_1 \sqrt{\frac{-\lambda_0 p}{\lambda_1 q}}\right)^{-1}.$$

Since $x_0 \ge 1$ if and only if $p \ge \frac{\lambda_1}{\lambda_1 - \lambda_0}$, we get $\mathcal{A}I \ge I$ for $p \ge \frac{\lambda_1}{\lambda_1 - \lambda_0}$. The proof of the case $\mathcal{A}I \le I$ is similar. ∎

We now state a result that will be of use later.

LEMMA 1.

$$\frac{d}{dx}(\mathcal{A}^n I)(x) \le \frac{2}{(1-x)x} \quad \textit{for } x \in (0,1) \textit{ and } n \in \mathbb{N}.$$

Proof. Let

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x)^{-1}$$

where $a = 1 - \lambda_0$, $b = 1 - \lambda_1$. Then

$$f_n'(x) = -\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x)^{-2}.$$

Therefore

$$|f_n'(x)| \le \frac{1}{2x} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (a^k b^{n-k} + x/2)^{-1}$$

and consequently

$$|f_n'(x)| \le \frac{1}{2x} f_n\left(\frac{x}{2}\right). \tag{3}$$

Combining the equalities

$$f_n(x) = \frac{1}{x} \mathcal{A}^n I\left(\frac{x}{1+x}\right) \quad \text{and} \quad \mathcal{A}^n I(x) = \frac{x}{1-x} f_n\left(\frac{x}{1-x}\right)$$

with (3) we get

$$\frac{d}{dx} \mathcal{A}^n I(x) = \frac{1}{(1-x)^2} f_n\left(\frac{x}{1-x}\right) + \frac{x}{1-x} \frac{1}{(1-x)^2} f_n'\left(\frac{x}{1-x}\right)$$

and

$$\frac{d}{dx} \mathcal{A}^n I(x) \le \frac{1}{(1-x)x} \left(\mathcal{A}^n I(x) + \mathcal{A}^n I\left(\frac{x}{2-x}\right)\right).$$

This gives

$$\frac{d}{dx} \mathcal{A}^n I(x) \le \frac{2}{(1-x)x} \quad \text{for } x \in (0,1)$$

and the proof is complete. ■

The next lemma ensures the existence of a σ-finite T-invariant measure.

LEMMA 2. *For every*

$$p \ne \frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)}$$

there exists $s \ne 0$ *such that*

$$\mathcal{A} F_p = F_p \quad \textit{for } F_p = \left(\frac{x}{1-x}\right)^s.$$

Proof. We first observe that the following identity holds:

$$\frac{T_\lambda(x)}{1 - T_\lambda(x)} = \frac{1}{1-\lambda} \frac{x}{1-x}$$

for $x \in (0,1)$ and $\lambda \in (-\infty, 1)$. From this it follows that

$$\mathcal{A}\left(\frac{x}{1-x}\right)^s = p\,\frac{1}{(1-\lambda_0)^s}\left(\frac{x}{1-x}\right)^s + q\,\frac{1}{(1-\lambda_1)^s}\left(\frac{x}{1-x}\right)^s = \left[p\,\frac{1}{(1-\lambda_0)^s} + q\,\frac{1}{(1-\lambda_1)^s}\right]\left(\frac{x}{1-x}\right)^s.$$

Therefore,

$$\text{(4)}\qquad \mathcal{A}\left(\frac{x}{1-x}\right)^s = \left(\frac{x}{1-x}\right)^s \;\Leftrightarrow\; p\,\frac{1}{(1-\lambda_0)^s} + q\,\frac{1}{(1-\lambda_1)^s} = 1 \;\Leftrightarrow\; p = \frac{(1-\lambda_1)^s - 1}{\left(\frac{1-\lambda_1}{1-\lambda_0}\right)^s - 1}.$$

Let

$$\gamma(s) = \frac{(1-\lambda_1)^s - 1}{\left(\frac{1-\lambda_1}{1-\lambda_0}\right)^s - 1} \qquad \text{for } s \neq 0$$

and

$$\gamma(0) = \lim_{s\to 0}\gamma(s) = \frac{\ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)}.$$

The function γ is continuous, strictly decreasing on $\mathbb{R}$ and $\lim_{s\to-\infty}\gamma(s) = 1$, $\lim_{s\to\infty}\gamma(s) = 0$. This shows that for every $p \neq \gamma(0)$ there exists exactly one $s \neq 0$ such that (4) holds, which completes the proof. ∎

Let T be given by (1) and $m = \mu_p \times \nu_{F_p}$. By using the fact that μ_p is Bernoulli it is easy to check that

$$\mu_p \times \nu_{F_p}(T^{-1}(B \times [0,x])) = \mu(B)\mathcal{A}F_p(x)$$

for every $B \in \mathcal{B}$ and $x \in [0,1]$. Hence by Lemma 2 we see that m is a σ-finite infinite T-invariant measure equivalent to $\mu_p \times \Lambda$. The following theorem ensures the ergodicity of (T,m):

Theorem 1. *If*

$$\text{(5)}\qquad \frac{\ln(1-\lambda_1)}{\ln(1-\lambda_0)} \textit{ is an irrational number and } p \neq \frac{ln(1-\lambda_1)}{\ln\left(\frac{1-\lambda_1}{1-\lambda_0}\right)}$$

then the dynamical system (T,m) is ergodic.

Proof. Let B be a T-invariant set. By the theorem of Morita [M], $B = \Omega \times A$ for a measurable set A. Here $T_i(A) = A$ for $T_i = T_{\lambda_i}$, $i = 0,1$. Suppose $\Lambda(A) > 0$. Our goal is to show that $\Lambda(A) = 1$. To do this we introduce the set $C = J(A)$ where

$$J(x) = \frac{1-x}{x}.$$

Since

$$J(T_0^m T_1^n(x)) = (1-\lambda_0)^m(1-\lambda_1)^n J(x)$$

we have

$$(1-\lambda_0)^m(1-\lambda_1)^n C = C.$$

Let $\alpha = \ln(1-\lambda_0)/\ln(1-\lambda_1)$. By the assumption the set $\{m\alpha+n : m,n\in\mathbb{Z}\}$ is dense in $\mathbb{R}$ and consequently the set

$$E = \{(1-\lambda_0)^m(1-\lambda_1)^n : m,n\in\mathbb{Z}\}$$

is dense in $\mathbb{R}^+$. Suppose on the contrary that $\Lambda(A) < 1$. Then $\Lambda(\mathbb{R}^+\setminus C) > 0$. Let x and $y \neq 0$ be density points of $\mathbb{R}^+\setminus C$ and C, respectively. For $\varepsilon < 1/2$ there exists $\delta > 0$ such that for any interval Q, if $\Lambda(Q) < \delta$ and $x \in Q$ (resp. $y \in Q$) then

$$\Lambda(Q\cap\mathbb{R}^+\setminus C) \ge (1-\varepsilon)\Lambda(Q) \quad (\text{resp. } \Lambda(Q\cap C) \ge (1-\varepsilon)\Lambda(Q)). \tag{6}$$

Let Q be such that $y \in Q$, $(x/y)\Lambda(Q) < \delta$ and $\Lambda(Q) < \delta$. By the density of E in $\mathbb{R}^+$ there exists $\beta \in E$ such that $\beta\Lambda(Q) < \delta$ and $x \in \beta Q$. Thus we get

$$\Lambda(\beta Q\cap\mathbb{R}^+\setminus C) > (1-\varepsilon)\beta\Lambda(Q)$$

and also

$$\begin{aligned}\Lambda(\beta Q\cap C) &= \Lambda(\beta Q\cap\beta C) = \beta\Lambda(Q\cap C)\\ &\ge (1-\varepsilon)\beta\Lambda(Q) \quad \text{by (6)},\end{aligned}$$

which is impossible. ∎

Corollary 1. *If λ_0,λ_1,p satisfy* (5) *then (T,m) has no product absolutely continuous invariant probability measure.*

Proof. Suppose on the contrary that such a measure exists. Then T is conservative and ergodic. Hence by the unicity of invariant measure ([A, Th. 1.5.6]) the measure ν_{F_p} is finite, which is impossible. ∎

Corollary 2. *If λ_0,λ_1,p satisfy* (5) *then the cluster points of*

$$\Big(\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{A}^k I\Big)_{n\in\mathbb{N}}$$

are constant functions on $(0,1)$.

Proof. By Lemma 1 the family of functions $\mathcal{A}^n I$, $n\in\mathbb{N}$, is uniquely Lipschitzian on $[\varepsilon,1-\varepsilon]$ for every $0<\varepsilon<1$. Therefore any cluster point F^* of

$$\Big(\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{A}^n I\Big)_{n\in\mathbb{N}}$$

is Lipschitzian on $(0,1)$. Hence ν_{F^*} is a convex combination of $\delta_{\{0\}}$, $\delta_{\{1\}}$ and an absolutely continuous measure. Corollary 1 implies that $F^* = \text{const}$ on $(0,1)$. ∎

The following theorem describes the convergence of $(\mathcal{A}^n I)_{n\in\mathbb{N}}$.

THEOREM 2. (i) *If*

$$p \le \frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1}$$

and λ_0, λ_1, p *satisfy* (5) *in the case of equality*, *then*

$$\lim_{n\to\infty} \mathcal{A}^n I = 0 \quad \textit{almost uniformly on } [0,1).$$

(ii) *If*

$$p \ge \frac{\lambda_1}{\lambda_1 - \lambda_0}$$

and λ_0, λ_1, p *satisfy* (5) *in the case of equality*, *then*

$$\lim_{n\to\infty} \mathcal{A}^n I = 1 \quad \textit{almost uniformly on } (0,1].$$

Proof. (i) The first part of (i) follows from Fact 2. Suppose now $p = \lambda_1(\lambda_0 - 1)/(\lambda_0 - \lambda_1)$ and λ_0, λ_1, p satisfy (5). By Fact 3, $\mathcal{A}^{n+1} I \le \mathcal{A}^n I \le I$ for $n = 1, 2, \ldots$, and by Lemma 1, $(\mathcal{A}^n I)_{n\in\mathbb{N}}$ is uniquely Lipschitzian on $[\varepsilon, 1-\varepsilon]$ for every $0 < \varepsilon < 1$. Combining these with Corollary 2 we get $\lim_{n\to\infty} \mathcal{A}^n I = 0$ on $[0,1)$ almost uniformly.

(ii) We first show a new estimate of $\frac{d}{dx}\mathcal{A}^n I$ which is more useful for our aim. Let us compute

$$\frac{d}{dx}\mathcal{A}^n I(x) = \frac{1}{x^2}\sum_{k=0}^{n}\binom{n}{k}[p(1-\lambda_0)]^k[q(1-\lambda_1)]^{n-k} T^2_{1-(1-\lambda_0)^k(1-\lambda_1)^{n-k}}(x).$$

Hence

$$\frac{d}{dx}\mathcal{A}^n I(x) \le \frac{1}{x^2}\,[p(1-\lambda_0) + q(1-\lambda_1)]^n$$

for $x \in (0,1]$. Now, if $\lambda_1/(\lambda_1 - \lambda_0) < p$ or equivalently $p(1-\lambda_0) + q(1-\lambda_1) < 1$ then $\frac{d}{dx}\mathcal{A}^n I(x) \to 0$ uniformly on $(\varepsilon, 1]$ for every $0 < \varepsilon < 1$. By Fact 3, $I \le \mathcal{A}^n I \le \mathcal{A}^{n+1} I$ for $n = 1, 2, \ldots$. This and the above imply $\lim_{n\to\infty} \mathcal{A}^n I = 1$ almost uniformly on $(0,1]$. Let $p = \lambda_1/(\lambda_1 - \lambda_0)$ and suppose λ_0, λ_1, p satisfy (5). Since $\frac{d}{dx}\mathcal{A}^n I(x) \le 1/x^2$ for $x \in (0,1]$, $n = 1, 2, \ldots$, $(\mathcal{A}^n I)_{n\in\mathbb{N}}$ is uniquely Lipschitzian on $(\varepsilon, 1]$ for every $0 < \varepsilon < 1$. Combining this with Corollary 2 we get $\lim_{n\to\infty} \mathcal{A}^n I = 1$ on $(0,1]$ almost uniformly. ∎

2. Product measures for smooth extensions of Bernoulli shifts. Let us consider two transformations of the unit interval

$$T_0(x) = (1-\varepsilon_0)x + \varepsilon_0 g(x),$$
$$T_1(x) = (1+\varepsilon_1)x - \varepsilon_1 g(x),$$

where $g \in C^2[0,1]$, $g(0) = 0$, $g(1) = 1$, $(1 - \sup g')^{-1} < \varepsilon_0$, $-\varepsilon_1 < (1 - \inf g')^{-1}$. We also suppose that there exists exactly one point x_0 for which $g'(x_0) = 1$ and either $g'(x) < 1$ for $x < x_0$ or $g'(x) > 1$ for $x < x_0$. Let

$$\mathcal{A}_T F = pF(T_0) + qF(T_1) \quad \text{for } F \in \mathcal{D}.$$

We are thus led to the following strengthening of Lemma 4 of [K].

THEOREM 3. *If* $\lim_{n\to\infty} \mathcal{A}_T^n I = 0$ (*resp.* $\lim_{n\to\infty} \mathcal{A}^n I = 1$) *for* $x \in [0,1)$ (*resp.* $x \in (0,1]$) *then the set of product measures in* M_p *is* $\mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}$.

We will give the proof only for the case $\lim_{n\to\infty} \mathcal{A}^n I = 0$, the second case is similar. We precede the proof by a lemma.

LEMMA 3. *If* $\lim_{n\to\infty} \mathcal{A}_T^n I = 0$ *for* $x \in [0,1)$ *then for any* $G \in C[0,1]$ *we have* $\lim_{n\to\infty} \mathcal{A}_T^n G = G(0)$.

Proof. We first observe that $\lim_{n\to\infty} \mathcal{A}_T^n(x^k) = 0$ on $[0,1)$ for $k = 1,2,\ldots$. This is an immediate consequence of the following implication:

$$x^k \le x \;\Rightarrow\; \mathcal{A}_T^n(x^k) \le \mathcal{A}_T^n I \quad \text{for } x \in [0,1] \text{ and } n = 1,2,\ldots.$$

Hence, $\lim_{n\to\infty} \mathcal{A}_T^n w = w(0)$ on $[0,1)$ for any polynomial w. Since $\mathcal{A}_T$ is a contraction in the supremum norm and the set of polynomials is dense in $C[0,1]$, we get

$$\lim_{n\to\infty} \mathcal{A}_T^n G = G(0) \quad \text{in } [0,1)$$

for any $G \in C[0,1]$. ∎

Proof of Theorem 3. Let G be a distribution function. We define two new ones as follows:

$$G_1(x) = \begin{cases} G(x) & \text{for } x \in [0,1), \\ G(1^-) & \text{for } x = 1, \end{cases}$$

and $G_2(x) = G_1(x^-)$ for $x \in (0,1]$. Here $G_1(x^-)$ means the left-hand limit of G_1 at x. Now G_1 is upper semicontinuous, being nondecreasing and right-continuous. Similarly G_2 is lower semicontinuous. By the definition

$$G_2(x) \le G(x) \le G_1(x) \quad \text{for } x \in [0,1). \tag{7}$$

By Baire's theorem there are sequences of continuous functions $(H_n^2)_{n\in\mathbb{N}}$, $(H_n^1)_{n\in\mathbb{N}}$ such that $H_n^2 \nearrow G_2$ and $H_n^1 \searrow G_1$. Also,

$$\lim_{n\to\infty} H_n^2(0) = G_2(0) = G(0) = G_1(0) = \lim_{n\to\infty} H_n^1(0).$$

From Lemma 3 and (7) we conclude that

$$\lim_{n\to\infty} \mathcal{A}_T^n G = G(0) \quad \text{for } x \in [0,1).$$

Hence, if $\mathcal{A}_T G = G$ for $G \in \mathcal{D}$ then $G \equiv G(0)$ on $[0,1)$, which proves the theorem. ∎

To make use of the above theorem we apply Theorem 2 to determine the limit of the sequence $(\mathcal{A}_T^n I)_{n\in\mathbb{N}}$.

EXAMPLE 1. Let $g(x) = T_{\lambda_0}(x)$ for $\lambda_0 \in (0,1)$ and

$$T_\varepsilon(x) = (1+\varepsilon)x - \varepsilon T_{\lambda_0}(x) \quad \text{for } 0 < \varepsilon < 1/\lambda_0 - 1.$$

Put $\mathcal{A}_\varepsilon = \mathcal{A}_T$. We will determine $\lambda_1 \in (-\infty, 0)$ such that $\operatorname{sgn}(T_\varepsilon(x) - T_{\lambda_1}(x)) = \text{const}$ for $x \in (0,1)$ or equivalently

$$\operatorname{sgn}\left[(x-1)\left(\frac{\lambda_1}{\lambda_1 x + 1 - \lambda_1} + \frac{\varepsilon\lambda_0}{\lambda_0 x + 1 - \lambda_0}\right)\right] = \text{const}$$

for $x \in (0,1)$. We have

$$T_\varepsilon(x) \ge T_{\lambda_1}(x) \;\Leftrightarrow\; \lambda_1 \le \frac{\varepsilon\lambda_0}{\lambda_0(1+\varepsilon) - 1}, \tag{8}$$

$$T_\varepsilon(x) \le T_{\lambda_1}(x) \;\Leftrightarrow\; -\varepsilon\lambda_0 \le \lambda_1 < 0. \tag{9}$$

By (8), $\mathcal{A}^n I \le \mathcal{A}_\varepsilon^n I$ for $n \in \mathbb{N}$ and $\lambda_1 \le \varepsilon\lambda_0/(\lambda_0(1+\varepsilon)-1)$. Therefore, by Theorem 2(ii),

$$\lim_{n\to\infty} \mathcal{A}_\varepsilon^n I = 1 \quad \text{for } p \ge \frac{\varepsilon}{1+\varepsilon}\,\frac{1}{1-\lambda_0}.$$

If $p = \frac{\varepsilon}{1+\varepsilon}\frac{1}{1-\lambda_0}$ then we assume that p, λ_0, $\lambda_1 = \varepsilon\lambda_0/(\lambda_0(1+\varepsilon)-1)$ satisfy (5). Here we use the fact that

$$\min\left\{\frac{\lambda_1}{\lambda_1 - \lambda_0} : \lambda_1 \le \frac{\varepsilon\lambda_0}{\lambda_0(1+\varepsilon)-1}\right\}$$

is achieved for $\lambda_1 = \varepsilon\lambda_0/(\lambda_0(1+\varepsilon)-1)$.

REMARK. We have $T_\varepsilon(x) \ge T_{\lambda_0}^{-1}(x)$ for $\varepsilon \le 1 - \lambda_0$ because $T_{\lambda_0}^{-1} = T_{-\lambda_0/(1-\lambda_0)}$. Hence, by Theorem 2 of [K],

$$\lim_{n\to\infty} \mathcal{A}_\varepsilon^n I = 1 \quad \text{for } p > \max\left\{\frac{1}{2}, \frac{\varepsilon}{1+\varepsilon}\right\} \text{ and } \varepsilon \le 1 - \lambda_0.$$

By (9), $\mathcal{A}_\varepsilon^n I \le \mathcal{A}^n I$ for $n \in \mathbb{N}$ and $-\varepsilon\lambda_0 \le \lambda_1 < 0$. Therefore, by Theorem 2(i),

$$\lim_{n\to\infty} \mathcal{A}_\varepsilon^n I = 0 \quad \text{for } p \le \frac{\varepsilon(1-\lambda_0)}{1+\varepsilon}.$$

If $p = \varepsilon(1-\lambda_0)/(1+\varepsilon)$ then we furthermore assume that p, λ_0, $\lambda_1 = -\varepsilon\lambda_0$ satisfy (5). Here we use the fact that

$$\max\left\{\frac{\lambda_1(\lambda_0 - 1)}{\lambda_0 - \lambda_1} : -\varepsilon\lambda_0 \le \lambda_1 \le 0\right\}$$

is achieved for $\lambda_1 = -\varepsilon\lambda_0$.

REMARK. We have $T_\varepsilon(x) \le T_{\lambda_0}^{-1}(x)$ for $\varepsilon \ge 1/(1-\lambda_0)$. Hence, by [K],

$$\lim_{n\to\infty} \mathcal{A}_\varepsilon^n I = 0 \quad \text{for } p < \min\left\{\frac{1}{2}, \frac{\varepsilon}{1+\varepsilon}\right\} \text{ and } \varepsilon \ge \frac{1}{1-\lambda_0}.$$

We summarize these considerations in the following theorem.

Theorem 4. *If*

$$p \in \left(0, \frac{\varepsilon}{1+\varepsilon}(1-\lambda_0)\right] \cup \left[\frac{\varepsilon}{1+\varepsilon}\frac{1}{1-\lambda_0}, 1\right)$$

then the set of product measures in M_p *is*

$$\mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}.$$

Here

$$p = \begin{cases} \dfrac{\varepsilon}{1+\varepsilon}(1-\lambda_0) & \textit{if } p,\ \lambda_0,\ \lambda_1 = -\varepsilon\lambda_0 \textit{ satisfy } (5), \\ \dfrac{\varepsilon}{1+\varepsilon}\dfrac{1}{1-\lambda_0} & \textit{if } p,\ \lambda_0,\ \lambda_1 = \dfrac{\varepsilon\lambda_0}{\lambda_0(1+\varepsilon)-1} \textit{ satisfy } (5). \end{cases}$$

For $p = \varepsilon/(1+\varepsilon)$,

$$M_p = \mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda\}.$$

Example 2. Let $g(x) = x^2$ and

$$T_0(x) = (1+\varepsilon_0)x - \varepsilon_0 x^2, \qquad T_1(x) = (1-\varepsilon_1)x + \varepsilon_1 x^2,$$

for $0 \le \varepsilon_i \le 1$, $i = 0, 1$. We will determine $\lambda_0 \in (0,1)$ such that

$$\mathrm{sgn}(T_0(x) - T_{\lambda_0}(x)) = \mathrm{const} \quad \text{for } x \in (0,1)$$

or equivalently

$$\mathrm{sgn}[(1-x)(\varepsilon_0\lambda_0 x - \varepsilon_0\lambda_0 + \varepsilon_0 - \lambda_0)] = \mathrm{const}.$$

We have

$$T_0(x) \ge T_{\lambda_0}(x) \Leftrightarrow \lambda_0 \le \frac{\varepsilon_0}{1+\varepsilon_0}, \tag{10}$$

$$T_0(x) \le T_{\lambda_0}(x) \Leftrightarrow \varepsilon_0 \le \lambda_0. \tag{11}$$

Similarly,

$$\mathrm{sgn}(T_1(x) - T_{\lambda_1}(x)) = \mathrm{const} \quad \text{for } x \in (0,1)$$

if and only if

$$\mathrm{sgn}[(x-1)(\varepsilon_1\lambda_1 x - \varepsilon_1\lambda_1 + \varepsilon_1 + \lambda_1)] = \mathrm{const}.$$

Hence

$$T_1(x) \ge T_{\lambda_1}(x) \Leftrightarrow \lambda_1 \le -\frac{\varepsilon_1}{1-\varepsilon_1}, \tag{12}$$

$$T_1(x) \le T_{\lambda_1}(x) \Leftrightarrow -\varepsilon_1 \le \lambda_1. \tag{13}$$

Denote by $\mathcal{A}_T$ the operator determined by p and T_i, $i = 0, 1$. By (11) and (13), $\mathcal{A}_T^n I \le \mathcal{A}^n I$ for $n \in \mathbb{N}$ and for $(\lambda_0, \lambda_1) \in [\varepsilon_0, 1] \times [-\varepsilon_1, 0]$. Therefore, by Theorem 2(i),

$$\lim_{n\to\infty} A_T^n I = 0 \quad \text{for } p \le \varepsilon_1 \frac{1-\varepsilon_0}{\varepsilon_0+\varepsilon_1}.$$

If $p = \varepsilon_1 \frac{1-\varepsilon_0}{\varepsilon_0+\varepsilon_1}$ then we additionally assume that p, $\lambda_0 = \varepsilon_0$, $\lambda_1 = -\varepsilon_1$ satisfy (5). Here we use the fact that

$$\max\left\{\frac{\lambda_1(\lambda_0-1)}{\lambda_0-\lambda_1} : (\lambda_0,\lambda_1) \in [\varepsilon_0, 1] \times [-\varepsilon_1, 0]\right\}$$

is achieved for $\lambda_0 = \varepsilon_0$, $\lambda_1 = -\varepsilon_1$.

REMARK. By Theorem 2 of [K], if $T_0 \circ T_1 \le I$ then

$$\lim_{n\to\infty} \mathcal{A}^n I = 0 \quad \text{ for } p < \min\left\{\frac{1}{2}, \frac{\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right\}.$$

The condition $T_0 \circ T_1 \le I$ is not always satisfied, e.g. for $\varepsilon_0 = 3/4$, $\varepsilon_1 = 1$.

If the conditions (10) and (12) hold then

$$\mathcal{A}_T^n I \ge \mathcal{A}^n I \quad \text{ for } n \in \mathbb{N} \text{ and } (\lambda_0, \lambda_1) \in \left[0, \frac{\varepsilon_0}{1+\varepsilon_0}\right] \times \left(-\infty, -\frac{\varepsilon_1}{1-\varepsilon_1}\right].$$

Therefore, by Theorem 2(ii),

$$\lim_{n\to\infty} \mathcal{A}_T^n I = 1 \quad \text{ for } p \ge \varepsilon_1 \frac{1+\varepsilon_0}{\varepsilon_0+\varepsilon_1}.$$

If $p = \varepsilon_1 \frac{1+\varepsilon_0}{\varepsilon_0+\varepsilon_1}$ then we furthermore assume that p, $\lambda_0 = \frac{\varepsilon_0}{1+\varepsilon_0}$, $\lambda_1 = -\frac{\varepsilon_1}{1-\varepsilon_1}$ satisfy (5). Here we use the fact that

$$\min\left\{\frac{\lambda_1}{\lambda_1-\lambda_0} : (\lambda_0,\lambda_1) \in \left[0, \frac{\varepsilon_0}{1+\varepsilon_0}\right] \times \left(-\infty, -\frac{\varepsilon_1}{1-\varepsilon_1}\right]\right\}$$

is achieved for $\lambda_0 = \frac{\varepsilon_0}{1+\varepsilon_0}$, $\quad \lambda_1 = -\frac{\varepsilon_1}{1-\varepsilon_1}$.

REMARK. By Theorem 2 of [K], if $T_0 \circ T_1 \ge I$ then

$$\lim_{n\to\infty} \mathcal{A}_T^n I = 1 \quad \text{ for } p > \max\left\{\frac{1}{2}, \frac{\varepsilon_1}{\varepsilon_0+\varepsilon_1}\right\}.$$

The condition $T_0 \circ T_1 \ge I$ is not always satisfied, e.g. for $\varepsilon_0 = \varepsilon_1 = 1/2$.

We summarize Example 2 in the following theorem:

THEOREM 5. *If*

$$p \in \left(0, \varepsilon_1 \frac{1-\varepsilon_0}{\varepsilon_0+\varepsilon_1}\right] \cup \left[\varepsilon_1 \frac{1+\varepsilon_0}{\varepsilon_0+\varepsilon_1}, 1\right)$$

then the set of product measures in M_p is

$$\operatorname{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}\}.$$

Here

$$p = \begin{cases} \varepsilon_1 \dfrac{1-\varepsilon_0}{\varepsilon_0+\varepsilon_1} & \textit{if } p,\ \lambda_0 = \varepsilon_0,\ \lambda_1 = -\varepsilon_1 \textit{ satisfy } (5), \\[2ex] \varepsilon_1 \dfrac{1+\varepsilon_0}{\varepsilon_0+\varepsilon_1} & \textit{if } p,\ \lambda_0 = \dfrac{\varepsilon_0}{1+\varepsilon_0},\ \lambda_1 = -\dfrac{\varepsilon_1}{1-\varepsilon_1} \textit{ satisfy } (5). \end{cases}$$

For $p = \varepsilon_1/(\varepsilon_0 + \varepsilon_1)$ *we have*

$$M_p = \operatorname{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda\}.$$

References

[1] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Math. Surveys Monogr. 50, Amer. Math. Soc., 1997, p. 284.

[2] Z. S. Kowalski, *Invariant measures for smooth extensions of Bernoulli shifts*, Bull. Polish Acad. Sci. Math. 51 (2003), 261–267.

[3] T. Morita, *Deterministic version lemmas in ergodic theory of random dynamical systems*, Hiroshima Math. J. 18 (1988), 15–29.

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