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Notes on Retracts of Coset Spaces

by

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Summary. We study retracts of coset spaces. We prove that in certain spaces the set of points that are contained in a component of dimension less than or equal to n, is a closed set. Using our techniques we are able to provide new examples of homogeneous spaces that are not coset spaces. We provide an example of a compact homogeneous space which is not a coset space. We further provide an example of a compact metrizable space which is a retract of a homogeneous compact space, but which is not a retract of a homogeneous metrizable compact space.

1. Introduction. If G is a topological group acting transitively on a space Z, then for every $z \in Z$ we let $\gamma_z : G \to Z$ be defined by $\gamma_z(g) = gz$. A space Z is called a *coset space* provided that there is a topological group G with closed subgroup H such that Z and $G/H = \{gH : g \in G\}$ are homeomorphic. It is easy to show that Z is a coset space if and only if there is a topological group G acting transitively on Z such that for some $z \in Z$ (equivalently: for all $z \in Z$) the function $\gamma_z : G \to Z$ is open.

In the present paper we are primarily interested in retracts of coset spaces. In [9] van Mill proved that coset spaces satisfy a certain "strong" homogeneity condition. Below we show that a weaker form of this property is preserved under taking retractions and therefore it is valid for retracts of coset spaces. We will use this property to prove some results for retracts of coset spaces, which leads to interesting examples. One of our main results is that if a σ-compact space is a retract of a coset space, then the set of all points that are contained in a component of dimension less than or equal to n is a closed set.

It is well known that coset spaces are homogeneous. Conversely, Ungar [11] proved that if Z is homogeneous, separable, metrizable and locally

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compact then Z is a coset space. This is a consequence of the Effros theorem on transitive actions of Polish groups on Polish spaces (Effros [3]; see also van Mill [10]). In [6] Ford gave an example of a homogeneous space which is not a coset space. Ford's example is neither metrizable nor locally compact. In [9] van Mill gave an example of a metrizable homogeneous space that is not a coset space. Of course, this example cannot be locally compact, it is however σ-compact. We will improve Ford's example in the other direction: we give an example of a compact homogeneous space which is not (a retract of) a coset space.

We further present an example of a compact metrizable space which is a retract of a homogeneous compact space, but which is not a retract of a homogeneous *metrizable* compact space. In fact we show that this example is not a retract of a coset space and thus by Ungar's results in [11], it follows that it is not a retract of a homogeneous metrizable compact space.

Results on retracts of compact homogeneous spaces were obtained earlier by Motorov (cf. Arkhangel'skiĭ [1]). He was able to show that certain spaces are not a retract of a compact and homogeneous space. For example, the well-known $\sin 1/x$-curve in the plane is such an example. Using our results we are able to show that the $\sin 1/x$-curve is not a retract of a coset space.

Uspenskiĭ has shown in [12] that for every space X there is a space W such that $X \times W \approx W$ and W is homogeneous. So if W is Uspenskiĭ's space associated with the $\sin 1/x$-curve, then W is yet another example of a homogeneous space which is not a coset space.

2. A weak form of Ungar's theorem. We assume that all spaces are Tikhonov. Let $\mathcal{U}$ be a cover of the space Z. If $A \subseteq Z$ and $f : A \to Z$ then we say that f is *limited by* $\mathcal{U}$ provided that for every $z \in A$ there is an element $U \in \mathcal{U}$ containing both z and $f(z)$. The following theorem can be found in van Mill [9, Theorem 2.1]. For completeness, we include the proof.

THEOREM 2.1. *Let Z be a coset space. Then for every open cover $\mathcal{U}$ of Z and every compact $K \subseteq Z$ there is an open cover $\mathcal{V}$ of Z with the following property: for all $V \in \mathcal{V}$ and $x, y \in V$ there is a homeomorphism $h : Z \to Z$ such that $h(x) = y$ and $h \restriction K$ is limited by $\mathcal{U}$.*

Proof. Let G be a topological group acting transitively on Z such that for every $z \in Z$ the function $\gamma_z : G \to Z$ is open. For $z \in K$ let V_z be an open neighbourhood of e in G such that $\gamma_z[V_z^2]$ is contained in an element of $\mathcal{U}$. There is a finite $F \subseteq K$ such that

$$K \subseteq \bigcup_{z \in F} \gamma_z[V_z].$$

Let $V = \bigcap_{z \in F} V_z$, and let W be a symmetric open neighbourhood of e in G such that $W^2 \subseteq V$. Put $\mathcal{V} = \{\gamma_z[W] : z \in Z\}$. Then $\mathcal{V}$ is an open cover of Z,

and we claim that it is as desired. To see this pick arbitrary $z, p, q \in Z$ such that $p, q \in \gamma_z[W]$. There are $h, g \in W$ such that $hz = p$ and $gz = q$. Then $\xi = gh^{-1} \in W^2$ and $\xi p = q$. So it suffices to prove that if $\alpha \in W^2$ and $y \in K$ are arbitrary then there exists $U \in \mathcal{U}$ containing both y and αy. Pick $z \in F$ such that $y \in \gamma_z[V_z] \subseteq \gamma_z[V_z^2]$. Then there is an element $f \in V_z$ such that $fz = y$. Since $\alpha y = (\alpha f)z \in \gamma_z[V_z^2]$ and $\gamma_z[V_z^2]$ is contained in an element of $\mathcal{U}$, this completes the proof. ∎

As a corollary we prove the following result for retracts of coset spaces. We use this corollary to prove some of the main results in this paper.

COROLLARY 2.2. *Let X be a retract of a coset space. Let $K \subseteq X$ be compact and suppose that $\mathcal{U}$ is an open cover of X. Then there is an open cover $\mathcal{V}$ of X with the following property: for all $V \in \mathcal{V}$ and $x, y \in V$ there is a continuous function $f : X \to X$ such that $f(x) = y$ and $f\restriction K$ is limited by $\mathcal{U}$.*

Proof. Let $r : Z \to X$ be a retraction where Z is a coset space. We apply Theorem 2.1 to the cover $\{r^{-1}[U] : U \in \mathcal{U}\}$ and the compact set $K \subseteq Z$. We find a cover $\mathcal{W}$ of Z with the given properties. We let $\mathcal{V} = \{W \cap X : W \in \mathcal{W}\}$. Clearly, $\mathcal{V}$ is an open cover of X. If $x, y \in V$ for some $V \in \mathcal{V}$, then $x, y \in W$ for some $W \in \mathcal{W}$, so there is a homeomorphism $h : Z \to Z$ such that $h(x) = y$ and $h\restriction K$ is limited by $\{r^{-1}[U] : U \in \mathcal{U}\}$. If we define $f : X \to X$ by $f(z) = r(h(z))$ for $z \in X$, then it is clear that $f(x) = y$ and it is easily verified that $f\restriction K$ is limited by $\mathcal{U}$. ∎

For compact metric spaces we may restate the previous result as follows.

COROLLARY 2.3. *Let (X, ϱ) be a compact metric space and suppose that X is a retract of a coset space. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\varrho(x, y) < \delta$ there is a continuous map $f : X \to X$ such that $f(x) = y$ and f moves no point of X more than ε.*

Proof. Apply Corollary 2.2 to the cover $\mathcal{U}$ consisting of all $\varepsilon/2$-balls in X to obtain an open cover $\mathcal{V}$ of X. The number δ is any Lebesgue number for $\mathcal{V}$. ∎

This last result is a weak form of a theorem due to Ungar [11], which states that in a compact and homogeneous metric space X, for every $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $\varrho(x, y) < \delta$ there is a homeomorphism h of X such that $h(x) = y$ and h moves no point of X more than ε.

3. Applications. Whenever X is a topological space, and $\mathcal{R}$ is a partition of X, then by $X/\mathcal{R}$ we denote the quotient space associated to $\mathcal{R}$, and $\pi : X \to X/\mathcal{R}$ is the corresponding quotient map. Whenever $x \in X$, by R_x we denote the unique element of $\mathcal{R}$ that contains x. Note that $R_x =$

$\pi^{-1}[\pi(x)]$. We say that the partition $\mathcal{R}$ is an *invariant partition* if the following holds: for every continuous function $f : X \to X$ and for all $R, Q \in \mathcal{R}$, if $f[R] \cap Q \neq \emptyset$ then $f[R] \subseteq Q$. Examples of invariant partitions are $\mathcal{C}$ and $\mathcal{P}$ where $\mathcal{C}$ is the family of all components in X, and $\mathcal{P}$ is the family of all path-components in X. We will always use $\mathcal{C}$ and $\mathcal{P}$ for these families. In particular we use C_x (P_x) for the (path)-component containing x.

In this section we will prove results that are valid for invariant partitions in retracts of coset spaces.

THEOREM 3.1. *Suppose X is a retract of a coset space and $\mathcal{R}$ is an invariant partition of X. Then $\pi : X \to X/\mathcal{R}$ is an open map.*

Proof. Let $U \subseteq X$ be open. We will show that $\pi^{-1}[\pi[U]]$ is open. Assume to the contrary that this set is not open. Then there is an $x \in \pi^{-1}[\pi[U]]$ such that $V \not\subseteq \pi^{-1}[\pi[U]]$ for every neighbourhood V of x. Define $A = X \setminus \pi^{-1}[\pi[U]]$. By assumption we have $x \in \overline{A}$.

Since $\pi(x) \in \pi[U]$ we have $U \cap R_x \neq \emptyset$. So we may choose $y \in U \cap R_x$. Let K be the compact set $\{y\}$. Let $\mathcal{U} = \{U, W\}$ where $W = X \setminus \{y\}$.

It follows from Corollary 2.2 that we may find a continuous function $f : X \to X$ with the property that $f(x) \in A$ and $\{f(y), y\} \subseteq U$. Let $f(x) = a$. By invariance of $\mathcal{R}$ it follows that $f(y) \in R_a$. However, since $a \in A$ we have $a \notin \pi^{-1}[\pi[U]]$ and therefore $R_a \cap \pi^{-1}[\pi[U]] = \emptyset$. In particular it follows that $R_a \cap U = \emptyset$. But we have just shown that $f(y) \in R_a \cap U$, which is a contradiction. ∎

It is a well-known fact of dimension theory that every finite collection of closed subsets of a normal space admits an open swelling (see for example Engelking [4, Theorem 3.1.1]). The following lemma is a corollary to this result; we give a sketch of the simple proof.

LEMMA 3.2. *Let $\mathcal{F}$ be a finite collection of closed subsets of a normal space X. Then there is an open cover $\mathcal{V}$ of X such that for all $F, G \in \mathcal{F}$ and $U, V \in \mathcal{V}$ the following holds*:

$(*)$ *If $F \cap G = \emptyset$, $F \cap U \neq \emptyset$ and $G \cap V \neq \emptyset$ then $U \cap V = \emptyset$.*

Proof. Let $\mathcal{U} = \{U_F : F \in \mathcal{F}\}$ be an open swelling of the family $\mathcal{F}$. For our purposes it suffices to know that whenever $F, G \in \mathcal{F}$ are disjoint then so are U_F and U_G. For every $x \in \bigcup \mathcal{F}$ we set $W_x = \bigcap\{U_F : x \in F \in \mathcal{F}\}$ and $G_x = \bigcup\{F \in \mathcal{F} : x \notin F\}$. One easily verifies that the cover $\mathcal{V}$ given by $\{W_x \setminus G_x : x \in \bigcup \mathcal{F}\} \cup \{X \setminus \bigcup \mathcal{F}\}$ satisfies $(*)$. ∎

THEOREM 3.3. *Suppose X is a retract of a coset space and $\mathcal{R}$ is an invariant partition of X. Suppose further that all elements of $\mathcal{R}$ are σ-compact. Let $n < \omega$ and consider the set A consisting of all points a in X such that $\dim R_a \leq n$. Then A is a closed subset of X.*

Proof. Assume that $p \in \overline{A}$. We will show that $p \in A$. Since R_p is σ-compact it is Lindelöf and therefore R_p is normal. It follows from the countable closed sum theorem (cf. [4, Theorem 3.1.8]) that it suffices to show that any compact subset K of R_p satisfies $\dim K \leq n$.

Fix a compact set K in R_p. We will prove that every family of $n+1$ pairs of disjoint closed subsets of K is inessential. So let $\{(A_i, B_i) : 1 \leq i \leq n+1\}$ be such a family. Let $\mathcal{F}$ be the family consisting of all compact sets A_i and B_i for $1 \leq i \leq n+1$. The collection $\mathcal{F}$ is a family of closed subsets of the normal space βX, so we may apply the previous lemma to obtain an open cover $\mathcal{V}$ of βX with property $(*)$. Restricting the cover $\mathcal{V}$ to X, we obtain an open cover $\mathcal{U}$ of X with property $(*)$. In particular it follows that whenever $A_i \cap U \neq \emptyset$ and $B_i \cap V \neq \emptyset$ for some $U, V \in \mathcal{U}$ then $U \cap V = \emptyset$. By Corollary 2.2 and the fact that $p \in \overline{A}$, there is a continuous map f of X which maps p onto a for some $a \in A$ and $f \restriction K$ is limited by $\mathcal{U}$. By invariance of $\mathcal{R}$, we have $f[R_p] \subseteq R_a$.

By compactness, the collection $\Gamma = \{(f[A_i], f[B_i]) : 1 \leq i \leq n+1\}$ is a family of $n+1$ pairs of closed subsets of R_a. We will show that it is also a collection of pairs of *disjoint* subsets of R_a. So let $f(z) \in f[A_i]$ and $f(w) \in f[B_i]$, where $z \in A_i$ and $w \in B_i$. Then there are $U, V \in \mathcal{U}$ such that $\{z, f(z)\} \subseteq U$ and $\{w, f(w)\} \subseteq V$. By $(*)$ it follows that $U \cap V = \emptyset$ so $f(z) \neq f(w)$. It follows that $f[A_i] \cap f[B_i] = \emptyset$.

Since Γ is a family of $n+1$ pairs of disjoint closed subsets of R_a and $a \in A$, it follows that it is an inessential family in R_a. By continuity of f, we conclude that the original family $\{(A_i, B_i) : 1 \leq i \leq n+1\}$ is inessential in K. Thus we have shown that $\dim K \leq n$. This completes the proof. ∎

For applications of the previous theorem, we note that every component of a given space is closed. It follows that if X is σ-compact, then every element of $\mathcal{C}$ is σ-compact as well.

We will use the following theorem to show that the $\sin 1/x$-curve is not a retract of a coset space.

THEOREM 3.4. *Suppose X is a retract of a coset space and $\mathcal{R}$ is an invariant partition of X. Let $R, Q \in \mathcal{R}$ be such that $\overline{R} \cap Q \neq \emptyset$. Then $R \subseteq \overline{Q}$.*

Proof. Let $R, Q \in \mathcal{R}$ with $\overline{R} \cap Q \neq \emptyset$. Fix $z \in R$ and let U be an arbitrary neighbourhood of z in X. Apply Corollary 2.2 to the compact set $K = \{z\}$ and the cover $\mathcal{U} = \{U, W\}$ of X where $W = X \setminus \{z\}$, to obtain a cover $\mathcal{V}$ with the stated properties. Pick $y \in \overline{R} \cap Q$. Since $\mathcal{V}$ covers X there is a set $V \in \mathcal{V}$ with $y \in V$. Since $y \in \overline{R}$, we have $R \cap V \neq \emptyset$, so let $x \in R \cap V$. By the properties of $\mathcal{V}$ we may find a continuous function $f : X \to X$ such that $f(x) = y$ and $\{z, f(z)\} \subseteq U$. By invariance of $\mathcal{R}$ it follows that $f[R] \subseteq Q$ and therefore it follows that $f(z) \in Q$. We have shown that $f(z) \in U \cap Q$.

Since U was an arbitrary neighbourhood of z, we have shown that $z \in \overline{Q}$. Since z was arbitrary we have shown that $R \subseteq \overline{Q}$. ∎

COROLLARY 3.5. *Suppose X is a retract of a coset space and $\mathcal{R}$ is an invariant partition of X. Let $R, Q \in \mathcal{R}$. The following are equivalent*:

(1) $\overline{R} \cap Q \neq \emptyset$,
(2) $\overline{Q} \cap R \neq \emptyset$,
(3) $\overline{R} = \overline{Q}$.

Proof. It suffices to show the equivalence of (1) and (3). It is clear that (3)⇒(1), so assume (1), i.e. $\overline{R} \cap Q \neq \emptyset$. By the previous theorem it follows that $R \subseteq \overline{Q}$ and thus $\overline{R} \subseteq \overline{Q}$. In particular $\overline{Q} \cap R \neq \emptyset$ and again it follows that $\overline{Q} \subseteq \overline{R}$. Thus $\overline{R} = \overline{Q}$. ∎

For compact metric spaces (X, ϱ) we can also prove the following result; details of the proof will appear elsewhere. For compact metric spaces, Theorem 3.3 follows from this result when $\mathcal{R} = \mathcal{C}$.

THEOREM 3.6. *Suppose (X, ϱ) is a compact metric space which is a retract of a coset space. Let $A \subseteq X$ be a subset of X with $p \in \overline{A}$. Then there is a sequence $(C_n)_n$ of components of elements of A such that C_p is homeomorphic to the inverse limit of some inverse sequence $\{C_n, h_m^n\}_{m<n<\infty}$. Furthermore, there is a homeomorphism $z : C_p \to C_\infty$ such that for every $x \in C_p$ we have*

$$x = \lim_{n\to\infty} z(x)_n.$$

4. Examples of homogeneous spaces that are not coset spaces. Using the techniques developed in the previous section, we now provide examples. Our first example improves Ford's example [6] considerably as it is an example of a compact homogeneous space which is not a retract of a coset space. Secondly, we present an example of a compact metrizable space which is not a retract of a homogeneous metrizable compact space, but which is a retract of a homogeneous compact space. Our strategy is to show that the space we construct is not a retract of a coset space, and consequently by the results of Ungar [11] it follows that the space is not a retract of a homogeneous metrizable compact space. Finally, we show that the $\sin 1/x$-curve is not a retract of a coset space.

EXAMPLE 4.1. Our example is an adaptation of an example by J. van Mill [8] (see also Hart and Ridderbos [7] for an alternative description). We use the method of resolutions (see Fedorchuk [5] and Watson [13] for details). The underlying set of the space X is given by $\mathsf{C} \times \mathbb{S}^1$. Here $\mathsf{C} = 2^\omega$ is the usual Cantor set and $\mathbb{S}^1$ the circle in the plane. We topologize X as follows.

Whenever $s \in 2^{<\omega}$, so s is a finite sequence of zeros and ones, we put

$$[s] = \{x \in \mathsf{C} : s \subseteq x\}.$$

The family $\{[s] : s \in 2^{<\omega}\}$ is the canonical base for the topology on C. Given $x \in \mathsf{C}$ and $n \in \omega$ we put $U_{x,n} = [x\restriction n]$, the nth basic neighbourhood of x, and $C_{x,n} = U_{x,n} \setminus U_{x,n+1}$. Note that $C_{x,n}$ is of the form $U_{y,n+1}$ for some suitably chosen $y \in \mathsf{C}$.

It is well known that $\mathbb{S}^1$ has a point d with a dense positive semi-orbit under some homeomorphism η of $\mathbb{S}^1$, i.e. the set $\{\eta^n(d) : n \in \omega\}$ is dense is $\mathbb{S}^1$. We define $d_n = \eta^n(d)$ for $n \in \omega$. For every $x \in \mathsf{C}$ we define the resolution maps $f_x : \mathsf{C} \setminus \{x\} \to \mathbb{S}^1$ by $f_x(y) = d_n$ iff $y \in C_{x,n}$.

Now we define basic open sets of $X = \mathsf{C} \times \mathbb{S}^1$ as follows. Whenever $x \in \mathsf{C}$, U_x is a neighbourhood of x in C and $W \subseteq \mathbb{S}^1$ is open, we define

$$U_x \otimes W = (\{x\} \times W) \cup \bigcup\{\{x'\} \times \mathbb{S}^1 : x' \in U_x \cap f_x^{-1}[W]\}.$$

Topologized in this way, X is the resolution of the Cantor set into circles by the maps f_x. The space X is compact and Hausdorff (see for example [13]). Carefully following the argument of van Mill (cf. [8] and [7]) one can show that X is homogeneous. Unlike the space constructed in [8], it is homogeneous even in ZFC. Homogeneity follows from the inequality $\omega < \mathfrak{p}$ which is valid in ZFC. In [8] the inequality $\omega_1 < \mathfrak{p}$ is needed to prove homogeneity, since the weight of the uncountable torus is ω_1. We have replaced the uncountable torus with the circle and the weight of this space is ω.

We will show that the projection $\pi : X \to X/\mathcal{C}$ is not an open mapping. The components of X are precisely the sets $\{x\} \times \mathbb{S}^1$, thus we may identify $X/\mathcal{C}$ with the set C. Consider a basic open set of the form $U_x \otimes W$ in X. Then

$$\pi[U_x \otimes W] = \{x\} \cup \{x' \in \mathsf{C} : x' \in U_x \cap f_x^{-1}[W]\}.$$

Then $\pi^{-1}[\pi[U_x \otimes W]]$ is the set

$$(\{x\} \times \mathbb{S}^1) \cup \bigcup\{\{x'\} \times \mathbb{S}^1 : x' \in U_x \cap f_x^{-1}[W]\}.$$

Whenever W is not dense in $\mathbb{S}^1$, this set is not open in the resolution topology. This follows from the observation that if $V \cap W = \emptyset$ and V is open in $\mathbb{S}^1$, then $U_x \otimes V$ is an open neighbourhood of some point of $\{x\} \times \mathbb{S}^1$, but

$$U_x \cap f_x^{-1}[V] \not\subseteq U_x \cap f_x^{-1}[W].$$

Since $\pi : X \to X/\mathcal{C}$ is not an open mapping but $\mathcal{C}$ is an invariant partition of X, it follows from Theorem 3.1 that X is not a retract of a coset space. ∎

We now present an application of Theorem 3.3. We construct a space Y which is not a retract of a coset space. In particular, since every homogeneous metrizable compact space is a coset space, it follows that Y is not a retract of a homogeneous metrizable compact space. We will however prove the

surprising property that Y is a retract of a homogeneous compact space, in fact Y is a retract of the space X constructed in the previous example.

EXAMPLE 4.2. The space X is as in the previous example; it is a homogeneous compact space which is not a retract of a coset space. We define the subspace Y of X as follows. Let e be the point of C with all coordinates zero. We abbreviate $U_{e,n}$ and $C_{e,n}$ by U_n and C_n respectively. For every $n < \omega$ we pick $x_n \in C_n$. The space Y is given as the union of A and B where

$$A = \{e\} \times \mathbb{S}^1, \qquad B = \{(x_n, d_n) : n < \omega\}.$$

The space Y inherits the topology of X, but this coincides with the topology that Y inherits from the usual cartesian product of the Cantor set and the circle in the plane. One can easily verify this. It suffices to note that B is a discrete subspace of X, and $(U_e \otimes W) \cap Y = (U_e \times W) \cap Y$ whenever U_e is an open neighbourhood of e in C and W is an open subset of $\mathbb{S}^1$.

So as Y is a subspace of the cartesian product $\mathsf{C} \times \mathbb{S}^1$, it follows that Y is a compact metrizable space, so all components in Y are compact. Note that B consists of components all of dimension 0, and A is a component of Y of dimension 1. Since B is dense in Y, it follows from Theorem 3.3 that Y is not a retract of a coset space.

We will show that Y is a retract of X. We define the function $r : X \to Y$ as follows:

$$r(w,z) = \begin{cases} (w,z) & \text{if } w = e, \\ (x_n, d_n) & \text{if } w \in C_n. \end{cases}$$

We show that r is continuous. First note that $r^{-1}[(x_n, d_n)]$ is open in X since this set is $C_n \times \mathbb{S}^1$ and this is just the basic open subset $C_n \otimes \mathbb{S}^1$ of X.

Next we consider basic open subsets V of Y that intersect the set A. Suppose

$$V = (\{e\} \times W) \cup \{(x_n, d_n) : n \geq N \text{ and } d_n \in W\},$$

where $W \subseteq \mathbb{S}^1$ is open and $N < \omega$. We will show that $r^{-1}[V]$ is open in X. First note that

$$r^{-1}[V] = (\{e\} \times W) \cup \bigcup \{C_n \times \mathbb{S}^1 : n \geq N \text{ and } d_n \in W\}.$$

Since sets of the form $C_n \times \mathbb{S}^1$ are open in X, we are done if we can show that the set $U_N \otimes W$, which contains the set $\{e\} \times W$, is contained in $r^{-1}[V]$. The basic open set $U_N \otimes W$ is given by

$$(\{e\} \times W) \cup \bigcup \{\{x'\} \times \mathbb{S}^1 : x' \in U_N \cap f_e^{-1}[W]\}.$$

The set $\{e\} \times W$ is contained in $r^{-1}[V]$, so suppose that $\{x'\} \times \mathbb{S}^1 \subseteq U_N \otimes W$ where $x' \neq e$. Then $x' \in C_n$ for some $n < \omega$. Since $x' \in U_N$, it follows that $n \geq N$. By definition of f_e we have $f_e(x') = d_n$, therefore since $f_e(x') \in W$, it follows that $d_n \in W$. So the set $\{x'\} \times \mathbb{S}^1$ is contained in $r^{-1}[V]$.

We have shown that $r^{-1}[B]$ is open for every $B \in \mathcal{B}$ for some basis $\mathcal{B}$ of Y. It follows that r is a retraction. Note that since Y is a retract of X, it follows once again that X is not a retract of a coset space. ∎

As an application of Corollary 3.5 we show that the $\sin 1/x$-curve is not a retract of a coset space. If W is the associated space constructed by Uspenskiĭ in [12] then W is a homogeneous space which is not a coset space.

In particular it also follows that the $\sin 1/x$-curve is not a retract of a homogeneous metrizable compact space. This result is not new since Motorov has proved the more general theorem stating that the $\sin 1/x$-curve is not a retract of a homogeneous compact space. However our general result does not follow from Motorov's observations, since coset spaces need not be compact.

EXAMPLE 4.3. The $\sin 1/x$-curve in the plane is given as the union of its two path components P_1 and P_2 where

$$P_1 = \{(0, x) : -1 \leq x \leq 1\}, \qquad P_2 = \{(x, \sin 1/x) : 0 < x \leq 1\}.$$

Since $P_1 \subseteq \overline{P}_2$ but $\overline{P}_1 = P_1$ it follows from Corollary 3.5 that this space is not a retract of a coset space. ∎

5. Further examples. In this section we provide some further examples to illustrate some limitations of our results. We will show that the inequalities in Theorem 3.3 cannot be replaced by equality. We provide a space X which is a retract of a compact homogeneous metrizable space, and a dense subset $A \subseteq X$ where $\dim C_a = 1$ for every $a \in A$, but $\dim C_x = 0$ for some $x \in X$. Recall that a compact and homogeneous metrizable space is a coset space.

A special class of retracts of coset spaces is the class of all compact metrizable spaces X for which X^ω is homogeneous; such spaces are power homogeneous. It was noted by Arkhangel'skiĭ (cf. [2]) that if in a power homogeneous space some point has a clopen base, then the space is zero-dimensional. It follows that the previous example is not power homogeneous. Our second example will be a compact metric space X for which X^ω is homogeneous and for some dense set A in X we have $\dim C_a = 2$ for every $a \in A$ whereas $\dim C_x = 1$ for some $x \in X$.

EXAMPLE 5.1. Our first example is a subspace of the plane $\mathbb{R}^2$. It is given by

$$X = \{(0,0)\} \cup \bigcup \{\{1/n\} \times [0, 1/n] : n \in \mathbb{N}\}.$$

It is a trivial observation that X is a retract of the space $Z \times \mathbb{I}$ where Z is the convergent sequence given by $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $\mathbb{I}$ is the usual unit interval. Since $(Z \times \mathbb{I})^\omega$ is homogeneous, it follows that X is a retract of a compact homogeneous metrizable space.

The set $\bigcup\{\{1/n\} \times [0,1/n] : n \in \mathbb{N}\}$ is dense in X and it consists of components of X, all of dimension 1. The set $\{(0,0)\}$ is a component of dimension 0, showing that the inequalities in Theorem 3.3 cannot be replaced by equality. ∎

The previous example does not seem to be very powerful. As announced, we will now provide an example with similar properties, but which is furthermore a compact space which is even power homogeneous.

EXAMPLE 5.2. As in the previous example, Z is the convergent sequence and X is given by

$$X = \{0\} \cup \bigcup_{n\in\mathbb{N}} [1/(2n+1), 1/2n].$$

We endow Z with the usual topology and X is a subspace of $\mathbb{R}$. The space X is clearly homeomorphic to the space described in the previous example.

The example is $X \times \mathbb{I}$. We will prove that this space is power homogeneous. By Q we denote the Hilbert cube $\mathbb{I}^\omega$. The following is our main observation:

PROPOSITION 5.3. *The spaces $Z \times Q$ and $X \times Q$ are homeomorphic.*

Proof. We write $X = \{0\} \cup \bigcup_{n\in\mathbb{N}} I_n$ where $I_n = [1/(2n+1), 1/2n]$. For every $n \in \mathbb{N}$ we fix a homeomorphism $h_n : I_n \to \mathbb{I}$. We define a map $h : X \times Q \to Z \times Q$ as follows. For $(x,y) \in X \times Q$, $h(x,y) = (x,y)$ if $x = 0$ and $h(x,y) = (1/n, w)$ if $x \in I_n$ and w is given by

$$w_m = \begin{cases} y_m & \text{if } m < n, \\ h_n(x) & \text{if } m = n, \\ y_{m-1} & \text{if } m > n. \end{cases}$$

Thus the set $I_n \times Q$ is mapped onto $\{1/n\} \times Q$ and the interval I_n is mapped onto the nth interval in Q. It is not hard to verify that h is a homeomorphism, and this completes the proof. ∎

COROLLARY 5.4. *The space $(X \times \mathbb{I})^\omega$ is homogeneous.*

Proof. By the previous proposition it follows that

$$(X \times \mathbb{I})^\omega \approx (X \times Q)^\omega \approx (Z \times Q)^\omega \approx Z^\omega \times Q.$$

This last space is the product of the Cantor set and the Hilbert cube and is therefore homogeneous. ∎

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