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Optics in Croke–Kleiner Spaces

by

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Summary. We explore the interior geometry of the CAT(0) spaces $\{X_\alpha : 0 < \alpha \leq \pi/2\}$, constructed by Croke and Kleiner [Topology 39 (2000)]. In particular, we describe a diffraction effect experienced by the family of geodesic rays that emanate from a basepoint and pass through a certain singular point called a triple point, and we describe the shadow this family casts on the boundary. This diffraction effect is codified in the *Transformation Rules* stated in Section 3 of this paper. The Transformation Rules have various applications. The earliest of these, described in Section 4, establishes a topological invariant of the boundaries of all the X_α's for which α lies in the interval $[\pi/2(n+1), \pi/2n)$, where n is a positive integer. Since the invariant changes when n changes, it provides a partition of the topological types of the boundaries of Croke–Kleiner spaces into a countable infinity of distinct classes. This countably infinite partition extends the original result of Croke and Kleiner which partitioned the topological types of the Croke–Kleiner boundaries into two distinct classes. After this countably infinite partition was proved, a finer partition of the topological types of the Croke–Kleiner boundaries into uncountably many distinct classes was established by the second author [J. Group Theory 8 (2005)], together with other applications of the Transformation Rules.

1. Introduction. The CAT(0) spaces X_α $(0 < \alpha \leq \pi/2)$ constructed in [4] and known as *Croke–Kleiner spaces* are remarkable for the fact that each space admits a geometric action (i.e. cocompact, properly discontinuous, and by isometries) by the same group G, and yet the boundaries of these spaces are not necessarily homeomorphic. In [4] it was proved that at least two of the Croke–Kleiner spaces have topologically distinct boundaries. Corollary 4.6 of this paper implies that there is at least a countable infinity of distinct topological types among the boundaries of the Croke–Kleiner spaces X_α. After the proofs in this paper were completed, it was shown in [7] that there are uncountably many distinct topological types among the

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boundaries of the Croke–Kleiner spaces. Until recently(1), the Croke–Kleiner spaces provided the only known example of this phenomenon, and it marks a significant deviation from the strict boundary rigidity enjoyed by hyperbolic groups. A hyperbolic group has a unique boundary, in the sense that if a finitely generated group G acts geometrically on two hyperbolic spaces X and X', then there is an induced quasi-isometry from X to X' that extends to a G-equivariant homeomorphism between their boundaries. (For a full treatment of the uniqueness of hyperbolic group boundaries, see [6].) Clearly, the connection between the large-scale algebraic structure of the group and the large-scale geometric structure of a space on which it acts is considerably looser in the CAT(0) theory.

Despite the apparent simplicity of the Croke–Kleiner spaces, their boundaries are complex objects whose properties are not completely understood at this juncture. One source of this complexity is the way that a geodesic ray can diffract upon passage through a *triple point* (a point which is the intersection of three planes). Most points cast 0-dimensional shadows on the boundary in the sense that the collection of geodesic rays issuing from a common basepoint and passing through the point comprises a 0-dimensional subset of the boundary. However, a geodesic passing through a triple point can thereafter proceed in a one-dimensional set of directions. This phenomenon is quantified by the Transformation Rules (Proposition 3.1), and exploited in Section 4 to demonstrate that Croke–Kleiner boundaries exhibit at least a countable infinity of distinct topological types.

2. Preliminaries. Before reviewing the Croke–Kleiner construction, we recall some preliminary definitions; see [2] for a full treatment. A CAT(0) *space* (X, d) is a complete metric space in which any two points can be joined by a geodesic arc, and whose metric d satisfies the following condition. Given any geodesic triangle $\triangle abc$ in X and a comparison triangle $\triangle a'b'c'$ in the Euclidean plane, then $d(p,q) \leq d_{\mathbb{E}^2}(p',q')$ for every pair of points p, q in $\triangle abc$ and their counterparts p', q' in $\triangle a'b'c'$. An important consequence of this property is that the metric d is *convex*, which roughly means that once a pair of geodesics in X begins to diverge, they must diverge at a rate that is at least linear. In particular, every pair of points is joined by a unique geodesic. The boundary of a CAT(0) space X is defined as follows. Choose a basepoint $p \in X$ and let $\mathcal{R}(p)$ denote the set of unit-speed geodesic rays issuing from p. For simplicity, assume that every ray in $\mathcal{R}(p)$ has domain $[0, \infty)$. To every ray in $\mathcal{R}(p)$, associate an endpoint $\sigma(\infty)$. For $r > 0$ and $\epsilon > 0$, two rays $\sigma, \tau \in \mathcal{R}(p)$ are (r, ϵ)-*close* if $d(\sigma(r), \tau(r)) < \epsilon$. This defines a topology, called the *cone topology*, on the set $\{\sigma(\infty) : \sigma \in \mathcal{R}(p)\}$. This space

(1) C. Mooney, PhD Thesis, UWM, 2008.

is the *boundary* ∂X of X, and up to homeomorphism it is independent of the choice of basepoint p. See [2, II.8.8].

For any two geodesic segments or rays σ and τ with a common point x in a CAT(0) space X, we denote by $\angle_x(\sigma, \tau)$ the Aleksandrov angle at x between σ and τ. Note that a set of the form $\{\tau(\infty) : \tau \in \mathcal{R}(p) \text{ and } \angle_p(\sigma, \tau) < \delta\}$, where $\sigma \in \mathcal{R}(p)$ and $\delta > 0$, is open in ∂X. (See [2], particularly II.1.6 and II.1.7.)

The space X_α constructed by Croke–Kleiner is the universal cover of a *torus complex* $\overline{X}_\alpha$, constructed as follows. Let T_0, T_1, T_2 be three flat tori. Let a_1, a_2 be a pair of geodesic loops in T_0 that generate $\pi_1(T_0)$ and that meet in a single point in T_0 at an angle α, $0 < \alpha \leq \pi/2$. Note that any angle $\alpha \in (0, \pi/2]$ and any choice of lengths for the geodesic loops a_1 and a_2 can be realized by taking the orbit space of the action on $\mathbb{E}^2$ by the group generated by two translations whose axes cross at an angle of α and which move points through distances equal to the prescribed lengths of the two geodesics. Next choose closed geodesics $b_i \subset T_i$, $i = 1, 2$, such that $\text{length}(b_i) = \text{length}(a_i)$, and let $\overline{X}_\alpha$ be the union of T_0, T_1, and T_2, with a_i identified with b_i. For $i = 1, 2$, let $\overline{Y_i} = T_0 \cup T_i \subset \overline{X}_\alpha$. The space X_α is the universal cover of $\overline{X}_\alpha$. By a *plane of* X_α, we mean a component of the preimage of T_0, T_1, or T_2.

The following facts about X_α were established in [4]:

1. **Blocks:** A component of the preimage of $\overline{Y_i}$ in X_α is called a *block*. Each block is a copy of the universal cover of $\overline{Y_i}$, and hence is isometric to the metric product of a simplicial valence-4 tree with $\mathbb{R}$.

2. **Walls:** Each block is a tree of planes of two types. A plane of the type that covers T_0 is referred to as a *wall* of the block. Each wall is common to exactly two blocks (one covering $\overline{Y_1}$ and the other covering $\overline{Y_2}$), which are called *adjacent* blocks. Any two blocks of X_α are either disjoint or adjacent with a wall as their only intersection. Blocks and walls are convex subsets of X_α.
   The *nerve* of X_α is the (non-locally finite) graph that has one vertex for every block of X_α and which has the property that vertices are adjacent exactly when the corresponding blocks are adjacent. The nerve of X_α is in fact a tree. We can equip the nerve with a metric topology by defining each edge to have length one.

3. **Block boundaries:** Given a block B, its boundary ∂B embeds in ∂X_α and is homeomorphic to the suspension of a Cantor set. The two suspension points, called *poles* of the block, are the common endpoints of the lifts in B of either a_1 or a_2. A *longitude* of the block is an arc in ∂B joining the two poles, i.e. the suspension of a point in the Cantor set. If two blocks are adjacent with common wall W, then their boundaries meet exactly in ∂W. If they are at distance two in

the nerve, then their boundaries meet exactly in the two poles of the block between them. If they are at distance three or more in the nerve, then their boundaries are disjoint.

Let $\sigma : J \to X_\alpha$ be an injective map whose domain J is a closed, connected subset of $\mathbb{R}$. We say that σ *enters a plane* V if there are values $a < b$ in J such that $\sigma([a, b]) \subset V$, and that σ *enters a block* B if it enters a nonwall plane of B.

4. **Block itineraries:** Choose a basepoint p_0 that lies in a nonwall plane V_0 of a block B_0, away from all walls of B_0. Then for every $\sigma \in \mathcal{R}(p_0)$ we can unambiguously define the *block itinerary* $\{B_0, B_2, B_3, \ldots\}$ of σ, where B_i is the ith block that σ enters. Since blocks are convex, a block itinerary can contain no repetitions.
5. **Topological invariance:** Note that the metric on X_α depends on α. Thus for $\alpha, \beta \in (0, \pi/2]$, X_α and X_β are homeomorphic but not isometric as CAT(0) spaces, and hence their boundaries are not necessarily homeomorphic. However, any homeomorphism from ∂X_α to ∂X_β must take block boundaries to block boundaries, poles to poles, and longitudes to longitudes.

At a point $p \in X_\alpha$, the *link* of p is the metric space Link(p) of unit tangent vectors or *germs* of geodesic rays that emanate from p. If σ is such a ray, let $g(\sigma) \in$ Link(p) denote its germ. More generally, if σ is a geodesic such that $\sigma(a) = p$ and $\sigma([a, b]) \subset V$ for some plane V, let $\hat{\sigma}$ be the unique geodesic ray in V emanating from p and containing $\sigma([a, b])$, and define $g(\sigma)$ to be $g(\hat{\sigma})$. If two elements a and b of Link(p) point into the same plane of X_α, then the distance between them, $A(a, b)$, is simply the angle between them. If a and b point into different planes of X_α, then the distance $A(a, b)$ between them is defined to be the minimum value of all sums of the form $A(c_0, c_1) + A(c_1, c_2) + \ldots + A(c_{k-1}, c_k)$ where $a = c_0, c_1, \ldots, c_k = b$ are elements of Link(p) such that c_{i-1} and c_i point into the same plane of X_α for $1 \leq i \leq k$.

Points of X_α exhibit three types of links. If p belongs to only one plane of X_α, then its link is a circle of radius one. If p lies on the intersection of exactly two planes, its link is the union of two circles of radius one intersecting each other in a pair of points that are diametrically opposed on each circle. If p is the intersection of three planes, then Link(p) is a union of three circles C, D, and C' of radius one that intersect in the following way. C and D intersect in a pair of diametrically opposed points a and b. Similarly, D and C' intersect in a pair of diametrically opposed points x and y. In D, the distance between a and x is α. In each case, the diametrically opposed intersection points are germs of rays issuing from p and asymptotic with a pair of poles of a block. See Figure 1.

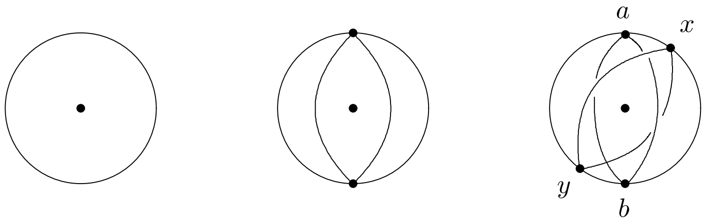

Fig. 1. Three types of links

We shall use the following fundamental principle regarding links in a CAT(0) space. Let $a < b < c$ be real numbers, and suppose $\sigma : [a, c] \to X_\alpha$ is a map with $\sigma(b) = p$ such that $\sigma|_{[a,b]}$ and $\sigma|_{[b,c]}$ are geodesics. Then σ is a geodesic exactly when the angle between the two geodesics at p is greater than or equal to π. More precisely, define a geodesic $\sigma^* : [a, b] \to X$ by $\sigma^*(t) = \sigma(a + b - t)$. ($\sigma^*$ is $\sigma|_{[a,b]}$ run backwards.) Then σ is a geodesic if and only if $A(g(\sigma^*), g(\sigma|_{[b,c]})) \geq \pi$. (See [5, Lemma 1.39, p. 386].)

We want to label the poles of each block "north" and "south" in a coherent way so that the Aleksandrov angle between north poles of adjacent blocks is α and the angle between the north pole of one block and the south pole of an adjacent block is $\pi - \alpha$, where the angle is computed in the wall common to the two blocks. This can be done by arbitrarily labeling the poles of our base block B_0 "north" and "south", and then inductively labeling poles of blocks at successively larger distances from B_0 in the nerve. Note that to compute the Aleksandrov angle between two points Q and Q' lying in a plane boundary ∂V, one can take geodesic rays emanating from an arbitrary basepoint $p_V \in V$ and asymptotic with Q and Q' in the cone topology associated to $\mathcal{R}(p_0)$.

Let $\sigma : J \to X_\alpha$ be an injective map whose domain J is a closed, connected subset of $\mathbb{R}$, and whose image $\alpha(J)$ intersects each plane of X_α in either the empty set, a single point, a line segment, a ray, or a line. Suppose that α enters a plane V of a block B. Then we assign an angle of inclination $\theta(\sigma, V, B)$ to σ in V relative to B. This angle is a real number that is determined modulo 2π. Suppose $0 \leq a < b$ are values such that $\sigma([a, b]) \subset V$. Let τ be the ray emanating from $\sigma(a)$ that terminates at the north pole of B. Then the angle between $\sigma|_{[a,b]}$ and τ equals $|\theta(\sigma, V, B)| \bmod 2\pi$. In every plane V, there are two continuous choices for the function $\theta(\sigma, V, B)$; one is the negative of the other. If V is a nonwall of B, then we are free to make either choice for $\theta(\sigma, V, B)$. However, if V is a wall of B, with $V = B \cap B'$, then we impose the following restriction on the choice of $\theta(\sigma, V, B)$ that determines it uniquely. If the ray τ' terminates at the north pole of B', then $\theta(\tau', V, B) \equiv \alpha \bmod 2\pi$. (The incorrect choice would result in $\theta(\tau, V, B) \equiv -\alpha \bmod 2\pi$.) By continuity, the angles of inclination of all other rays traveling in V are now determined. Note that if a wall W is common to blocks B and B' and σ travels in W, then $\theta(\sigma, W, B) = \alpha - \theta(\sigma, W, B')$.

3. Transformation rules. In general, when a geodesic ray passes from one plane to another in X_α, the intersection of these planes is a line and the angle of inclination that the ray makes with this line in the successive planes is preserved. However, a ray may travel from a nonwall V of a block B directly into a nonwall V' of an adjacent block B' without traveling for positive time in the intervening wall $W = B \cap B'$. To do this, it must travel through a *triple point*, a singular point which is the only intersection of V and V'. A triple point is a lift of the intersection of the geodesic loops a_1 and a_2 in the torus T_0 . In this case, the ray may continue in a one-dimensional array of different directions in V'. Rule 3 of the following proposition details the transformation of this angle of inclination when the ray passes through a triple point.

PROPOSITION 3.1 (Transformation Rules). *Let $\sigma : J \to X_\alpha$ be a unit speed injective map whose domain J is a closed, connected subset of $\mathbb{R}$, and whose image $\sigma(J)$ intersects each plane of X_α in either the empty set, a single point, a line segment, a ray, or a line. Then σ is a geodesic if and only if it satisfies the following three rules.*

1. *If σ enters planes V and V' successively, both of which lie in the same block B, then $\theta(\sigma, V, B) \equiv \pm\theta(\sigma, V', B) \bmod 2\pi$.*
2. *If σ enters a wall W that is the intersection of adjacent blocks B and B', then $\theta(\sigma, W, B) \equiv \alpha - \theta(\sigma, W, B') \bmod 2\pi$.*
3. *Suppose σ enters planes V and V' successively, where V and V' are nonwall planes of adjacent blocks B and B' respectively, and σ passes from V to V' via a triple point p. Let β and β' be representatives of $\theta(\sigma, V, B)$ and $\theta(\sigma, V', B')$ respectively that are chosen to lie in the interval $[-\pi, \pi]$.*
   - *If $|\beta| \le \alpha$, then $|\beta'| \in [\alpha - |\beta|, \alpha + |\beta|]$.*
   - *If $\alpha \le |\beta| \le \pi - \alpha$, then $|\beta'| \in [|\beta| - \alpha, |\beta| + \alpha]$.*
   - *If $\pi-\alpha \le |\beta| \le \pi$, then $|\beta'| \in [(\pi-\alpha)-(\pi-|\beta|), (\pi-\alpha)+(\pi-|\beta|)]$.*

Transformation Rule 1 tells us that if a geodesic ray travels from a plane V to another plane V' in the same block, then the geodesic ray may proceed in either of two directions in V'. As an illustration of Transformation Rule 3, consider a geodesic segment $\sigma : [0, t] \to X_\alpha$ from the basepoint p_0 to a triple point p that is the intersection of the nonwall planes V and V' of adjacent blocks B and B' respectively. Suppose $\beta = \theta(\sigma, V, B)$ satisfies $|\beta| \le \alpha$. Then for every real number β' satisfying $|\beta'| \in [\alpha-|\beta|, \alpha+|\beta|]$, there is an extension of σ to a geodesic ray that enters V' with angle of inclination β'.

Proof of Proposition 3.1. Rule 2 holds for any oriented straight line segment in a wall by virtue of the manner in which we labeled poles and angles of inclination. Consequently, Rule 2 holds for σ regardless of whether it is a geodesic.

We will prove that Rules 1 and 3 provide a necessary and sufficient condition for σ to be a geodesic. Since local geodesics are geodesics in CAT(0) spaces (see [2, Prop. 1.4, p. 160]), it suffices to prove that σ is a local geodesic at the points where it moves between planes if and only if it satisfies Rules 1 and 3 at these points. Recall that σ is a local geodesic at a point p where it moves between two planes if and only if the distance in Link(p) between the germs at p of the reversed incoming segment of σ and the outgoing segment of σ is at least π (see [5, Lemma 1.39, p. 386]).

Rule 1 pertains to the case in which there are real numbers $a < b < c$ and planes V and V' lying in the same block B such that $\sigma([a,b]) \subset V$, $\sigma([b,c]) \subset V'$, and $\sigma(b) = p$. In this case, Link(p) is of the second type or third type depending on whether or not p is a triple point.

First suppose p is not a triple point and Link(p) is of the second type, namely a union of two unit circles C and C' which meet at diametrically opposed points n and s that are the germs of geodesic rays emanating from p toward the north and south poles of B respectively. Let $\sigma_1 = \sigma|_{[a,b]}$, let σ_1^* denote σ_1 run backwards, and let $\sigma_2 = \sigma|_{[b,c]}$. Let β and β' be representatives of $\theta(\sigma, V, B)$ and $\theta(\sigma, V', B)$ that are chosen to lie in the interval $[-\pi, \pi]$. Recall that $|\beta|$ is the Aleksandrov angle between σ_1 and a ray approaching the north pole of B. Thus $|\beta| = \pi - A(g(\sigma_1^*), n) = A(s, g(\sigma_1^*))$. Also, $|\beta'| = A(n, g(\sigma_2)) = \pi - A(g(\sigma_2), s)$. (See Figure 2.) Recall that σ is a local geodesic at p if and only if the distance $A(g(\sigma_1^*), g(\sigma_2))$ in Link(p) is at least π.

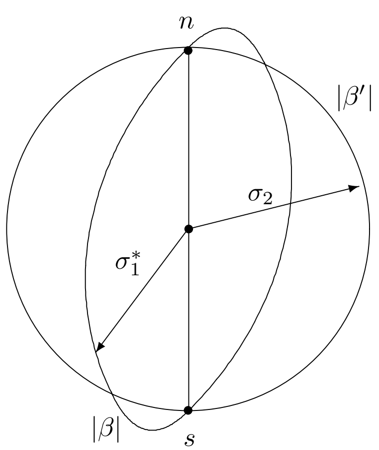

Fig. 2. Link(p) in the case of Rule 1

We will now compute $A(g(\sigma_1^*), g(\sigma_2))$. There are two candidates for the shortest path in $\mathrm{Link}(p)$ between $g(\sigma_1^*)$ and $g(\sigma_2)$. The first is via n, with length $L_1 = A(g(\sigma_1^*), n) + A(n, g(\sigma_2)) = (\pi - |\beta|) + |\beta'|$. The second is via s, with length $L_2 = A(g(\sigma_1^*), s) + A(s, g(\sigma_2)) = |\beta| + (\pi - |\beta'|)$. Thus $A(g(\sigma_1^*), g(\sigma_2)) = \min\{L_1, L_2\}$. We conclude that σ is a local geodesic at p if and only if $L_1 \geq \pi$ and $L_2 \geq \pi$, i.e. if and only if $|\beta'| - |\beta| \geq 0$ and $|\beta| - |\beta'| \geq 0$, i.e. if and only if $\beta = \pm\beta'$.

Now suppose we are still in the case of Rule 1 but p is a triple point. Then $\mathrm{Link}(p)$ is of the third type, namely the union of three unit circles C, D, and C' where C and D consist of germs of geodesic rays emanating from p into the planes V and V', and C' consists of germs of geodesic rays emanating from p into a nonwall plane of a block B' adjacent to B. Furthermore, C and D meet at diametrically opposed points n and s that are the germs of geodesic rays emanating from p toward the north and south poles of B, and D and C' meet at diametrically opposed points n' and s' that are the germs of geodesic rays emanating from p towards the north and south poles of B'. With σ_1, σ_1^*, σ_2, β, and β' as above, we have $g(\sigma_1^*)$ and $g(\sigma_2)$ in $C \cup D$. Observe that any two points of $C \cup D$ are joined by a shortest path in $C \cup D \cup C'$ that is entirely contained in $C \cup D$. Consequently, the calculation of $A(g(\sigma_1^*), g(\sigma_2))$ performed in the previous paragraph also applies here and gives the same conclusion: σ is a local geodesic at p if and only if $\beta = \pm\beta'$.

Finally we consider the case of Rule 3, in which there are real numbers $a < b < c$ and nonwall planes V and V' belonging to adjacent blocks B and B' respectively, such that $\sigma([a, b]) \subset V$, $\sigma([b, c]) \subset V'$, and $\sigma(b) = p$ is a triple point. Then B and B' meet in a wall W such that $V \cap V' = V \cap W \cap V' = \{p\}$. In this case, $\mathrm{Link}(p)$ is of the third type, namely the union of three unit circles C, D, and C' consisting of germs of geodesic rays emanating from p into the planes V, W, and V' respectively. Furthermore, C and D meet at diametrically opposed points n and s that are the germs of geodesic rays emanating from p toward the north and south poles of B, and D and C' meet at diametrically opposed points n' and s' that are the germs of rays emanating from p toward the north and south poles of B'. Let σ_1, σ_1^*, and σ_2 be as above. Then $g(\sigma_1^*) \in C$ and $g(\sigma_2) \in C'$. Let β and β' be representatives of $\theta(\sigma, V, B)$ and $\theta(\sigma, V', B')$ that lie in the interval $[-\pi, \pi]$. Then $|\beta| = \pi - A(g(\sigma_1^*), n) = A(s, g(\sigma_1^*))$, and $|\beta'| = A(n', g(\sigma_2)) = \pi - A(g(\sigma_2), s')$. (See Figure 3.) As before, σ is a local geodesic at p if and only if $A(g(\sigma_1^*), g(\sigma_2)) \geq \pi$.

If a and b are points lying on the same circular arc J of diameter less than π in $\mathrm{Link}(p)$, let ab denote the subarc of J joining a to b. A shortest path in $\mathrm{Link}(p)$ from $g(\sigma_1^*)$ to $g(\sigma_2)$ must contain one of the four arcs nn', ss', ns', or sn'. Hence there are four candidates for such a path:

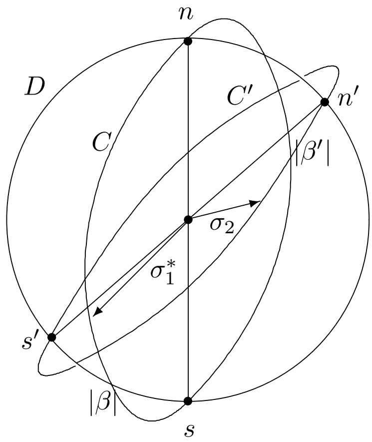

Fig. 3. Link(p) in the case of Rule 3

1. $g(\sigma_1^*)n \cup nn' \cup n'g(\sigma_2)$, with length $L_1 = A(g(\sigma_1^*), n) + A(n, n') + A(n', g(\sigma_2)) = (\pi - |\beta|) + \alpha + |\beta'|$;
2. $g(\sigma_1^*)s \cup ss' \cup s'g(\sigma_2)$, with length $L_2 = A(g(\sigma_1^*), s) + A(s, s') + A(s', g(\sigma_2)) = |\beta| + \alpha + (\pi - |\beta'|)$;
3. $g(\sigma_1^*)n \cup ns' \cup s'g(\sigma_2)$, with length $L_3 = A(g(\sigma_1^*), n) + A(n, s') + A(s', g(\sigma_2)) = (\pi - |\beta|) + (\pi - \alpha) + (\pi - |\beta'|)$;
4. $g(\sigma_1^*)s \cup sn' \cup n'g(\sigma_2)$, with length $L_4 = A(g(\sigma_1^*), s) + A(s, n') + A(n', g(\sigma_2)) = |\beta| + (\pi - \alpha) + |\beta'|$.

We know that σ is a geodesic exactly when $A(g(\sigma_1^*), g(\sigma_2)) = \min\{L_1, L_2, L_3, L_4\} \geq \pi$, i.e. when $L_i \geq \pi$ for all i. The three subcases listed in Rule 3 ($|\beta| \leq \alpha$, $\alpha \leq |\beta| \leq \pi - \alpha$, and $\pi - \alpha \leq |\beta| \leq \pi$) exhaust all possibilities. We now argue that the condition stated under each of these subcases is equivalent to the restriction that $L_i \geq \pi$ for all i.

- Assume $|\beta| \leq \alpha$. Then $\alpha - |\beta| \geq 0$, and hence $L_1 \geq \pi$. Furthermore, since $|\beta| \leq \alpha \leq \pi/2$, it follows that $|\beta| \leq \pi - \alpha$, and thus $L_3 \geq L_2$. Thus $L_i \geq \pi$ for all i if and only if $L_2 \geq \pi$ and $L_4 \geq \pi$. These last two inequalities are equivalent to saying that $|\beta'| \leq \alpha + |\beta|$ and $|\beta'| \geq \alpha - |\beta|$, i.e. $|\beta'| \in [\alpha - |\beta|, \alpha + |\beta|]$, as required.
- Assume $\alpha \leq |\beta| \leq \pi - \alpha$. Then $\alpha - |\beta| \leq |\beta| - \alpha$, which implies $L_4 \geq L_1$, and $\alpha + |\beta| \leq \pi \leq 2\pi - (\alpha + |\beta|)$, so that $L_3 \geq L_2$. Thus $L_i \geq \pi$ for all i if and only if $L_1 \geq \pi$ and $L_2 \geq \pi$. These last inequalities are equivalent to $|\beta'| \geq |\beta| - \alpha$ and $|\beta'| \leq |\beta| + \alpha$, i.e. $|\beta'| \in [|\beta| - \alpha, |\beta| + \alpha]$, as required.
- Finally, assume $\pi - \alpha \leq |\beta| \leq \pi$. Then $\alpha + |\beta| \geq \pi$, and so $L_2 \geq \pi$. Also, $\pi - |\beta| \leq \alpha \leq \pi - \alpha \leq |\beta|$, implying that $L_4 \geq L_1$. Thus $L_i \geq \pi$ for all i if and only if $L_1 \geq \pi$ and $L_3 \geq \pi$. These two inequalities are

equivalent to $|\beta'| \geq |\beta| - \alpha = (\pi - \alpha) - (\pi - |\beta|)$ and $|\beta'| \leq 2\pi - |\beta| - \alpha = (\pi - \alpha) + (\pi - |\beta|)$, i.e. $\beta' \in [(\pi - \alpha) - (\pi - |\beta|), (\pi - \alpha) + (\pi - |\beta|)]$, as required. ■

Note that the angle transformation described in Rule 2 is simply a special case of Rule 3: For suppose σ enters the wall W between adjacent blocks B and B', and $\beta = \theta(\sigma, W, B)$ and $\beta' = \theta(\sigma, W, B')$. According to Rule 2, $\beta' \equiv \alpha - \beta \bmod 2\pi$. It is straightforward to verify the following:

- if $|\beta| \leq \alpha$, then $|\beta'| = |\alpha - \beta| = \alpha - |\beta|$ or $\alpha + |\beta|$;
- if $\alpha \leq |\beta| \leq \pi - \alpha$, then $|\beta'| = |\alpha - \beta| = |\beta| - \alpha$ or $|\beta| + \alpha$;
- if $\pi - \alpha \leq |\beta| \leq \pi$, then $|\beta'| = |\alpha - \beta| = (\pi - \alpha) - (\pi - |\beta|)$ or $(\pi - \alpha) + (\pi - |\beta|)$,

in keeping with Rule 3. Thus if the weaker result of Rule 3 suffices for β', it may be used instead of Rule 2.

The Transformation Rules have various applications. In the next section we describe the earliest of these applications which leads to a partition of the topological types of the boundaries of Croke–Kleiner spaces into a countable infinity of distinct classes, thereby extending the main result of [4]. Subsequent instances of the use of the Transformation Rules appear in [7] and [1].

4. A property of the boundary. In this section we will see that the diffraction effect experienced by geodesic rays that pass through triple points (as described in Transformation Rule 3) has an impact on the topology of ∂X_α. The size of this effect, which is controlled by the angle α, determines how quickly poles of nearby blocks accumulate on the boundary of the base block B_0. Specifically, there is a partition of $(0, \pi/2]$ into countably many disjoint subintervals associated with distinct positive integers with the property that if α lies in the interval associated with a positive integer n, then n is the smallest integer such that every point of ∂B_0 is a limit of poles whose distance from B_0 is no greater than $n + 1$. (This is the content of Corollary 4.5.) This property is a topological invariant of ∂X_α. It follows (Corollary 4.6) that as α ranges over $(0, \pi/2]$, at least countably infinitely many topologically distinct ∂X_α's are encountered.

For each $n \geq 0$, let $\mathcal{P}(n)$ denote the set of poles of blocks which are at distance less than or equal to n from B_0 in the nerve.

THEOREM 4.1. *If $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$, then $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(n))$. Also, if $\alpha = \pi/2$, then $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(1))$.*

Proof. First assume $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$. Note that for all $0 < k \leq n$, $\alpha < \pi/2n \leq \pi/2k$, and so $k\alpha < \pi/2$. Thus $k\alpha < \pi - k\alpha$ for all $0 \leq k \leq n$. Let Ω be the set of all $\sigma(\infty)$ such that $\sigma \in \mathcal{R}(p_0)$ and

$\theta(\sigma, V_0, B_0) \in (n\alpha, \pi - n\alpha)$. Note that Ω is an open subset of ∂X_α. Clearly Ω contains points of $\partial V_0 \subset \partial B_0$. We will show that $\Omega \cap \mathcal{P}(n) = \emptyset$, and hence that $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(n))$.

Suppose instead that $\sigma \in \mathcal{R}(p_0)$ is a ray such that $\sigma(\infty) \in \Omega \cap \mathcal{P}(n)$. Then $\sigma(\infty)$ is a pole of a block B_m at distance m from B_0 in the nerve, for some $m \leq n$. Note that the angle of inclination of σ in any plane of B_m must equal a multiple of π. Recall that σ is only said to *enter* a block if it travels for positive time in the block away from all walls. Thus σ can never enter B_m in this sense. For suppose σ first encounters B_m at a point p. Then p must belong to a wall W_m of B_m, even if p is a triple point. If p is not a triple point, then σ must move in W_m parallel to and disjoint from the lines of intersection of W_m with the nonwalls of B_m, because $\theta(\sigma, W_m, B_m)$ is a multiple of π. On the other hand, if p is a triple point that lies in the intersection of W_m with a nonwall V of B_m, then σ cannot leave W_m to *enter* V because to do so would require $\theta(\sigma, W_m, B_m)$ to take on a value that is not a multiple of π. Hence the itinerary of σ is of the form $B_0, B_1, \ldots, B_{m-1}$. Furthermore, if $W_m = B_{m-1} \cap B_m$ is the common wall between these two blocks, then σ terminates in W_m in the sense that $\sigma([t,\infty)) \subset W_m$ for some $t > 0$. Moreover, $\sigma([t,\infty))$ is a ray in W_m that is parallel to the lines of intersection of W_m with the nonwalls of B_m.

Set $P_0 = V_0$ and $P_m = W_m$, and for all $0 < i < m$, let P_i be a plane of B_i that σ enters. For $0 \leq i \leq m$, let $\beta_i = \theta(\sigma, P_i, B_i)$, and assume $\beta_i \in [-\pi, \pi]$. Note that if P_i' is any other plane of B_i that σ enters, then $|\theta(\sigma, P_i', B_i)| = |\beta_i|$, by Transformation Rule 1. Also note that $|\beta_m| = 0$ or π by an earlier remark.

Since $\sigma(\infty) \in \Omega$, it follows that $\beta_0 \in (n\alpha, \pi - n\alpha)$. We assert that in fact

$$|\beta_i| \in ((n-i)\alpha, \pi - (n-i)\alpha) \tag{1}$$

for all $0 \leq i \leq m$. Assume this is true for some i between 0 and $m-1$. Since $i \leq m-1 \leq n-1$, it follows that $1 \leq n-i$; thus $\alpha \leq (n-i)\alpha$ and $\pi - (n-i)\alpha \leq \pi - \alpha$. Also, $(n-i)\alpha < \pi - (n-i)\alpha$, as noted at the beginning of the proof. Thus $((n-i)\alpha, \pi - (n-i)\alpha) \subset (\alpha, \pi - \alpha)$, and hence $\alpha < |\beta_i| < \pi - \alpha$. The relation between $|\beta_i|$ and $|\beta_{i+1}|$ is governed by Transformation Rule 2 or 3, and Rule 3 subsumes Rule 2. Thus $|\beta_{i+1}| \in [|\beta_i| - \alpha, |\beta_i| + \alpha]$. From our inductive hypothesis (1), $(n-(i+1))\alpha < |\beta_i| - \alpha$ and $|\beta_i| + \alpha < \pi - (n-(i+1))\alpha$, from which it follows that $|\beta_{i+1}| \in ((n-(i+1))\alpha, \pi - (n-(i+1))\alpha)$. This proves our assertion.

Since $m \leq n$, the assertion implies that $|\beta_m| \in (0, \pi)$, contradicting the fact that $|\beta_m| = 0$ or π. Thus $\Omega \cap \mathcal{P}(n) = \emptyset$, and so ∂B_0 is not contained in $\mathrm{cl}(\mathcal{P}(n))$.

The proof of this theorem in the case that $\alpha = \pi/2$ is very similar to the previous argument. When $\alpha = \pi/2$, let Ω be the set of all $\sigma(\infty)$ such that $\sigma \in \mathcal{R}(p_0)$ and $\theta(\sigma, V_0, B_0) \in (0, \pi/2)$. Again Ω is an open subset of ∂X_α that contains points of ∂B_0. In this case, we argue that Ω is disjoint from $\mathcal{P}(1)$, proving that $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(1))$. To this end, assume that there is a $\sigma \in \mathcal{R}(p_0)$ such that $\sigma(\infty) \in \Omega \cap \mathcal{P}(1)$. Then, as we argued previously, there is a block B_1 that shares a wall W_1 with B_0 such that $\sigma([0,\infty)) \subset B_0$, σ terminates in W_1, and $\sigma(\infty)$ is a pole of B_1. Let $\beta_0 = \theta(\sigma, V_0, B_0)$ and $\beta_1 = \theta(\sigma, W_1, B_1)$. Then $|\theta(\sigma, W_1, B_0)| = |\beta_0| \in (0, \pi/2)$ by Transformation Rule 1, and β_1 is a multiple of π. The relationship between $|\beta_0|$ and $|\beta_1|$ is governed by Transformation Rule 2 or 3, hence by Transformation Rule 3. Since $0 < |\beta_0| < \pi/2 = \alpha$, it follows that $|\beta_1| \in [\alpha - |\beta_0|, \alpha + |\beta_0|] \subset (0, \pi)$. Therefore, β_1 is not a multiple of π. We conclude that $\Omega \cap \partial X_\alpha = \emptyset$, and thus $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(1))$. ∎

THEOREM 4.2. *If $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$, then $\partial B_0 \subset \mathrm{cl}(\mathcal{P}(n+1))$. Also, if $\alpha = \pi/2$, then $\partial B_0 \subset \mathrm{cl}(\mathcal{P}(2))$.*

Proof. First assume $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$. Let Φ be a nonempty open subset of ∂X_α that contains a point of ∂B_0. We will prove that Φ intersects $\mathcal{P}(n+1)$.

The union of the boundaries of the nonwall planes of B_0 minus the poles of B_0 is dense in ∂B_0. Hence, there is a nonwall plane U_0 of B_0 and a point $Q \in \partial U_0 \cap \Phi$ such that Q is not a pole of B_0. Then there is a geodesic ray σ in B_0 that emanates from p_0 and *terminates* in U_0 (i.e., $\sigma([t,\infty)) \subset U_0$ for some $t > 0$), such that $\sigma(\infty) = Q$. Let A denote the finite set $\{k\alpha : 1 \leq k \leq n\} \cup \{-\pi + k\alpha : 1 \leq k \leq n\}$.

The remainder of the proof is accomplished in two steps, utilizing the two lemmas below. In the first step, σ is approximated by a ray $\sigma' \in \mathcal{R}(p_0)$ so that $\sigma'(\infty) \in \Phi$, σ' passes from the block B_0 to a block B_1 via a triple point, and σ' enters and terminates in a nonwall plane U_1 of B_1 such that the angle of inclination of σ' in U_1 is an element of the set A. In the second step of the proof, σ' is approximated by a ray $\sigma'' \in \mathcal{R}(p_0)$ so that $\sigma''(\infty) \in \Phi$ and σ'' follows an itinerary $B_0, B_1, \ldots, B_m$, where $1 \leq m \leq n$, such that the successive angles of inclination of σ'' in these blocks progress through the elements of set A by jumps of size α, moving toward either α or $-\pi + \alpha$, with the result that $\sigma''(\infty)$ is a pole of a block B_{m+1} that is adjacent to B_m.

LEMMA 4.3. *Suppose $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$, U_0 is a nonwall plane of B_0 and $\sigma \in \mathcal{R}(p_0)$ is such that σ terminates in U_0, $\sigma(\infty)$ is not a pole of B_0, and Φ is a neighborhood of $\sigma(\infty)$ in ∂X_α. Then there is a ray $\sigma' \in \mathcal{R}(p_0)$ such that $\sigma'(\infty) \in \Phi$, and there is a block B_1 adjacent to B_0, such that either $\sigma'([0,\infty)) \subset B_0$ and $\sigma'(\infty)$ is a pole of B_1, or σ' has itinerary B_0, B_1, there is a nonwall plane U_1 of B_1 such that σ' terminates in U_1, and $\theta(\sigma', U_1, B_1) \in A$.*

Proof. Since σ terminates in U_0 and $\sigma(\infty) \in \Phi$, it follows that there is a $t_0 > 0$ and a $\delta > 0$ such that $\sigma([t_0, \infty)) \subset U_0$ and such that if $\sigma' \in \mathcal{R}(p_0)$ and $d(\sigma'(t_0), \sigma(t_0)) < \delta$, then $\sigma'(\infty) \in \Phi$. We assume that $\theta(\sigma, U_0, B_0)$ and all other angles of inclination mentioned in this proof lie in the interval $[-\pi, \pi]$. Since $\sigma(\infty)$ is not a pole of B_0, it follows that $|\theta(\sigma, U_0, B_0)| \in (0, \pi)$. Since $\pi/2(n+1) \leq \alpha < \pi/2n$, we see that $(0, \pi) = (0, \pi/2] \cup (\pi/2, \pi) \subset (0, (n+1)\alpha] \cup [\pi - (n+1)\alpha, \pi)$. Hence, there is an integer m such that $0 \leq m \leq n$ and either

$$|\theta(\sigma, U_0, B_0)| \in (0, \pi/2] \quad \text{and} \quad |\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha],$$

or

$$|\theta(\sigma, U_0, B_0)| \in (\pi/2, \pi) \text{ and } |\theta(\sigma, U_0, B_0)| \in [\pi - (m+1)\alpha, \pi - m\alpha).$$

The proof now divides into three mutually exclusive cases:

- $|\theta(\sigma, U_0, B_0)| \in \{\alpha, \pi - \alpha\}$,
- $|\theta(\sigma, U_0, B_0)| \in \{(m+1)\alpha, \pi - (m+1)\alpha\}$, where $1 \leq m \leq n$,
- $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha) \cup (\pi - (m+1)\alpha, \pi - m\alpha)$, where $0 \leq m \leq n$.

First, assume $|\theta(\sigma, U_0, B_0)| \in \{\alpha, \pi - \alpha\}$. In this case we construct a ray $\sigma' \in \mathcal{R}(p_0)$ that coincides with σ on $[0, t_0 + 1]$ and then enters and terminates in a wall W_1 of B_0. Furthermore, if $|\theta(\sigma, U_0, B_0)| = \alpha$, then we insist that σ' enters the *positive half* of W_1, so that $\theta(\sigma', W_1, B_0) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - \alpha$, then we insist that σ' enters the *negative half* of W_1, so that $\theta(\sigma', W_1, B_0) < 0$. Clearly, $\sigma'([0, \infty)) \subset B_0$ and $\sigma'(\infty) \in \Phi$, because σ' and σ coincide at t_0. Transformation Rule 1 implies that $|\theta(\sigma', W_1, B_0)| = |\theta(\sigma', U_0, B_0)| = |\theta(\sigma, U_0, B_0)|$. Hence, $\theta(\sigma', W_1, B_0) = \alpha$ if $|\theta(\sigma, U_0, B_0)| = \alpha$, and $\theta(\sigma', W_1, B_0) = -\pi + \alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - \alpha$. Let B_1 be the block of X_α that intersects B_0 in the wall W_1. Transformation Rule 2 implies that $\theta(\sigma', W_1, B_1) \equiv \alpha - \theta(\sigma', W_1, B_0) \mod 2\pi$. Consequently, $\theta(\sigma', W_1, B_1) = 0$ if $|\theta(\sigma, U_0, B_0)| = \alpha$, and $\theta(\sigma', W_1, B_1) = \pi$ if $|\theta(\sigma, U_0, B_0) = \pi - \alpha$. Hence, $\sigma'(\infty)$ is a pole of B_1.

Second, assume $|\theta(\sigma, U_0, B_0)| \in \{(m+1)\alpha, \pi - (m+1)\alpha\}$, where $1 \leq m \leq n$. In this case we construct a ray $\sigma' \in \mathcal{R}(p_0)$ that coincides with σ on $[0, t_0 + 1]$ and then enters a wall W_1 of B_0. Let B_1 be the block of X_α that intersects B_0 in the wall W_1. After spending a positive amount of time in W_1, σ' leaves W_1 and enters and terminates in a nonwall plane U_1 of B_1. Furthermore, if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, then we insist that σ' enters the positive half of W_1, so that $\theta(\sigma', W_1, B_0) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$, then we insist that σ enters the negative half of W_1, so that $\theta(\sigma', W_1, B_0) < 0$. In addition, if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, then we insist that σ' enters the positive half of U_1, so that $\theta(\sigma', U_1, B_1) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$, then we insist that σ' enters the negative half of U_1, so that $\theta(\sigma', U_1, B_1) < 0$. Clearly σ' has itinerary B_0, B_1, and $\sigma'(\infty) \in \Phi$ because σ' and σ coincide at t_0. Transformation Rule 1 implies that $|\theta(\sigma', W_1, B_0)| = |\theta(\sigma', U_0, B_0)| = |\theta(\sigma, U_0, B_0)|$. Hence $\theta(\sigma', W_1, B_0) = (m+1)\alpha$ if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, whereas $\theta(\sigma', W_1, B_0) = -\pi+(m+1)\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi-(m+1)\alpha$. Transformation Rule 2 implies $\theta(\sigma', W_1, B_1) \equiv \alpha - \theta(\sigma', W_1, B_0) \bmod 2\pi$. Consequently, $\theta(\sigma', W_1, B_1) = -m\alpha$ if $|\theta(,\sigma, U_0, B_0)| = (m+1)\alpha$, and $\theta(\sigma', W_1, B_1) = \pi - m\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$. Again, Transformation Rule 1 implies that $|\theta(\sigma', U_1, B_1)| = |\theta(\sigma', W_1, B_1)|$. Therefore, $\theta(\sigma', U_1, B_1) = m\alpha$ if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, and $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$. Thus, $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$, where $1 \le m \le n$. Consequently, $\theta(\sigma', U_1, B_1) \in A$.

Third, assume $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha) \cup (\pi - (m+1)\alpha, \pi - m\alpha)$, where $0 \le m \le n$. We remark that if τ is a ray in X_α such that $\tau([t_0, t_0 + 1]) \subset U_0$, then $\theta(\tau, U_0, B_0)$ depends continuously on the two points $\tau(t_0)$ and $\tau(t_0 + 1)$. We also note that if $\tau \in \mathcal{R}(p_0)$, then $d(\sigma(t_0), \tau(t_0)) \le d(\tau(t_0 + 1), \sigma(t_0 + 1))$ because X_α is a CAT(0) space. Consequently, we can assume that $\delta > 0$ has been chosen so small that if $\tau \in \mathcal{R}(p_0)$ and $d(\tau(t_0 + 1), \sigma(t_0 + 1)) < \delta$, then both $|\theta(\tau, U_0, B_0)|$ and $\theta(\sigma, U_0, B_0)|$ belong either to the interval $(m\alpha, (m+1)\alpha)$ or to the interval $(\pi - (m+1)\alpha, \pi - m\alpha)$. We observe that the triple points in U_0 form a quasi-dense subset. Indeed, the distance from a point of U_0 to the nearest triple point is no greater than the geodesic diameter of the torus T_i which is covered by U_0, since the triple points in U_0 are simply the preimages of the triple intersection point $T_0 \cap T_1 \cap T_2$ under the locally isometric covering map $U_0 \to T_i$. Hence, there is an $r > 0$ such that the r-neighborhood of every point of U_0 contains a triple point. Choose $t_1 > (\max\{1, 2r/\delta\})(t_0 + 1)$. Let p be a triple point in U_0 such that $d(\sigma(t_1), p) < r$. Let $\tau \in \mathcal{R}(p_0)$ be the geodesic ray that passes through the triple point p and terminates in U_0. Let $u = d(p_0, p)$, so that $\tau(u) = p$. Hence, $|t_1 - u| = |d(p_0, \sigma(t_1)) - d(p_0, p)| \le d(\sigma(t_1), p) < r$. Therefore

$$\begin{aligned} d(\sigma(t_1), \tau(t_1)) &\le d(\sigma(t_1), \tau(u)) + d(\tau(u), \tau(t_1)) \\ &= d(\sigma(t_1), p) + |t_1 - u| < 2r. \end{aligned}$$

It follows by the CAT(0) condition that

$$d(\sigma(t_0 + 1), \tau(t_0 + 1)) \le \frac{t_0 + 1}{t_1}(2r) < \delta.$$

Therefore $d(\sigma(t_0), \tau(t_0) < \delta$. Hence, both $|\theta(\tau, U_0, B_0)|$ and $|\theta(\sigma, U_0, B_0)|$ belong either to $(m\alpha, (m+1)\alpha)$ or to the interval $(\pi - (m+1)\alpha, \pi - m\alpha)$. Since p is a triple point that belongs to U_0, it follows that p also belongs to a wall W_1 of B_0, there is a block B_1 that meets B_0 in the wall W_1, and p also belongs to a nonwall plane U_1 of B_1.

Let σ' be the union of the geodesic segment in X_α from p_0 to p together with the geodesic ray in U_1 that emanates from the triple point p and satisfies the following:

- $\theta(\sigma', U_1, B_1) = \alpha$ if $m = 0$ and $|\theta(\sigma, U_0, B_0)| \in (0, \alpha)$;
- $\theta(\sigma', U_1, B_1) = -\pi + \alpha$ if $m = 0$ and $|\theta(\sigma, U_0, B_0)| \in (\pi - \alpha, \pi)$;
- $\theta(\sigma', U_1, B_1) = m\alpha$ if $1 \leq m \leq n$ and $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha)$;
- $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$ if $1 \leq m \leq n$ and $|\theta(\sigma, U_0, B_0)| \in (\pi - (m+1)\alpha, \pi - m\alpha)$.

Clearly, σ' coincides with τ on $[0, u]$. We assert that σ' is a geodesic ray. To verify this, we must show that $\beta = \theta(\tau, U_0, B_0)$ and $\beta' = \theta(\sigma', U_1, B_1)$ are related as prescribed by Transformation Rule 3. Indeed:

- if $m = 0$ and $|\beta| \in (0, \alpha)$, then $|\beta'| = \alpha \in [\alpha - |\beta|, \alpha + |\beta|]$;
- if $m = 0$ and $|\beta| \in (\pi - \alpha, \pi)$, then $|\beta'| = \pi - \alpha \in [(\pi - \alpha) - (\pi - \beta), (\pi - \alpha) + (\pi - \beta)]$;
- if $1 \leq m \leq n$ and $|\beta| \in (m\alpha, (m+1)\alpha)$, then $|\beta| - \alpha < m\alpha < |\beta|$, whence $|\beta'| = m\alpha \in [|\beta| - \alpha, |\beta| + \alpha]$;
- if $1 \leq m \leq n$ and $|\beta| \in (\pi - (m+1)\alpha, \pi - m\alpha)$, then $|\beta| < \pi - m\alpha < |\beta| + \alpha$, whence $|\beta'| = \pi - m\alpha \in [|\beta| - \alpha, |\beta| + \alpha]$.

Thus σ' is a geodesic ray. Hence, $\sigma' \in \mathcal{R}(p_0)$. Since σ' coincides with τ on $[0, u]$, it follows that $\sigma'(t_0) = \tau(t_0)$. Therefore, $d(\sigma'(t_0), \sigma(t_0)) = d(\tau(t_0, \sigma(t_0)) < \delta$. Hence, $\sigma'(\infty) \in \Phi$. Also, σ' enters and terminates in the nonwall plane U_1 of B_1. Furthermore, $\theta(\sigma', U_1, B_1) \in \{\alpha, -\pi + \alpha\}$ if $m = 0$, and $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$ if $1 \leq m \leq n$. Thus $\theta(\sigma', U_1, B_1) \in A$. ∎

Lemma 4.4. *Suppose $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$, $\sigma' \in \mathcal{R}(p_0)$ has itinerary B_0, B_1 where B_1 is a block of X_α that is adjacent to B_0, there is a nonwall plane U_1 of B_1 such that σ' terminates in U_1 and $\theta(\sigma', U_1, B_1) \in A$, and Φ is a neighborhood of $\sigma'(\infty)$ in ∂X_α. Then there is a ray $\sigma'' \in \mathcal{R}(p_0)$ such that $\sigma''(\infty) \in \Phi$ and there is an integer m such that $1 \leq m \leq n$ and there are blocks $B_2, B_3, \ldots, B_m, B_{m+1}$ of X_α where $B_0, B_1, B_2, \ldots, B_m, B_{m+1}$ are distinct, $B_0, B_1, \ldots, B_m$ is the itinerary of σ'', B_m is adjacent to B_{m+1}, and $\sigma''(\infty)$ is a pole of B_{m+1}.*

Proof. Since σ' terminates in U_1 and $\sigma'(\infty) \in \Phi$, it follows that there is a $t_0 > 0$ such that $\sigma'([t_0, \infty)) \subset U_1$ and such that if $\tau \in \mathcal{R}(p_0)$ and $\tau(t_0) = \sigma'(t_0)$, then $\tau(\infty) \in \Phi$. We again assume that $\theta(\sigma', U_1, B_1)$ and all other angles of inclination mentioned in this proof lie in the interval $[-\pi, \pi]$.

Since $\theta(\sigma', U_1, B_1) \in A$, it follows that there is an integer m such that $1 \leq m \leq n$ and $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$. We will construct a ray $\sigma'' \in \mathcal{R}(p_0)$ and a sequence $B_2, \ldots, B_m, B_{m+1}$ of blocks of X_α with the following properties:

- σ'' coincides with σ' on $[0, t_0 + 1]$;
- $B_0, B_1, B_2, \ldots, B_m, B_{m+1}$ are distinct, $B_0, B_1, B_2, \ldots, B_m$ is the itinerary of σ'', and B_m is adjacent to B_{m+1};
- B_i and B_{i+1} meet in a common wall W_{i+1} for $1 \leq i \leq m$;
- σ'' enters W_{i+1} for $1 \leq i \leq m$, and σ'' terminates in W_{m+1};
- if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\sigma'', W_{i+1}, B_i) = (m + 1 - i)\alpha$ for $1 \leq i \leq m$;
- if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\sigma'', W_{i+1}, B_i) = -\pi + (m + 1 - i)\alpha$ for $1 \leq i \leq m$.

The construction of σ'' is an inductive process in which a sequence $\tau_1, \tau_2, \ldots, \tau_m$ of progressively longer geodesic segments emanating from p_0 are constructed.

The first step of the process is to construct a geodesic segment τ_1 that coincides with σ' on $[0, t_0 + 1]$ and then enters and terminates in a wall W_2 of B_1 that intersects U_1. Furthermore, if $\theta(\sigma', U_1, B_1) = m\alpha$, then we insist that τ_1 enters the positive half of W_2, so that $\theta(\tau_1, W_2, B_1) > 0$. However, if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then we insist that τ_1 enters the negative half of W_2, so that $\theta(\tau_1, W_2, B_1) < 0$. Transformation Rule 1 implies that $|\theta(\tau_1, W_2, B_1)| = |\theta(\tau_1, U_1, B_1)| = |\theta(\sigma', U_1, B_1)|$. Hence, if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_1, W_2, B_2) = m\alpha$; whereas if $\theta(\sigma', U_1, B_1) = -\pi + ma\alpha$, then $\theta(\tau_1, W_2, B_1) = -\pi + m\alpha$.

For the inductive step of the process, let $1 \leq k < m$ and assume that a geodesic segment τ_k has been constructed and that $B_2, \ldots, B_k$ of X_α have been chosen with the following properties:

- τ_k coincides with σ' on $[0, t_0 + 1]$;
- $B_0, B_1, B_2, \ldots, B_k$ are distinct and form the itinerary of τ_k;
- B_i and B_{i+1} meet in a common wall W_{i+1} for $1 \leq i < k$ and W_{k+1} is a wall of B_k that is disjoint from B_{k-1};
- τ_k enters W_{i+1} for $1 \leq i \leq k$, and τ_k terminates in W_{k+1};
- if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_k, W_{i+1}, B_i) = (m+1-i)\alpha$ for $1 \leq i \leq k$;
- if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\tau_k, W_{i+1}, B_i) = -\pi + (m + 1 - i)\alpha$ for $1 \leq i \leq k$.

Let B_{k+1} be the block that meets B_k in the wall W_{k+1}. Observe that $\theta(\tau_k, W_{k+1}, B_k) = (m + 1 - k)\alpha$ whenever $\theta(\sigma', U_1, B_1) = m\alpha$, and that $\theta(\tau_k, W_{k+1}, B_k) = -\pi + (m + 1 - k)\alpha$ whenever $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$. Transformation Rule 2 implies that $\theta(\tau_k, W_{k+1}, B_{k+1}) \equiv \alpha - \theta(\tau_k, W_{k+1}, B_k) \mod 2\pi$. Consequently, if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_k, W_{k+1}, B_{k+1}) = -(m - k)\alpha$; whereas if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\tau_k, W_{k+1}, B_{k+1}) = \pi - (m - k)\alpha$. We now extend τ_k to a longer geodesic segment τ_{k+1} by first making τ_{k+1} enter a nonwall plane U_{k+1} of B_{k+1} that intersects W_{k+1}. Then we make τ_{k+1} enter and terminate in a wall W_{k+2} of B_{k+1} that intersects

U_{k+1} and is disjoint from B_k. Furthermore, if $\theta(\sigma', U_1, B_1) = m\alpha$, then we insist that τ_{k+1} enters the positive half of W_{k+2} so that $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) > 0$. However, if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then we insist that τ_{k+1} enters the negative half of W_{k+2} so that $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) < 0$. Transformation Rule 1 implies that $|\theta(\tau_{k+1}, W_{k+2}, B_{k+1})| = |\theta(\tau_{k+1}, W_{k+1}, B_{k+1})| = |\theta(\tau_k, W_{k+1}, B_{k+1})|$. Hence, if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_{k+1}, W_{k+1}, B_{k+1}) = (m-k)\alpha$; whereas if $\theta(\sigma', U_1, B_1) = -\pi+m\alpha$, then $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) = -\pi + (m-k)\alpha$. Also, since τ_{k+1} is an extension of τ_k, it follows that $\theta(\tau_{k+1}, W_{i+1}, B_k) = \theta(\tau_k, W_{i+1}, B_i)$ for $1 \le i \le k$. This establishes the inductive step.

The final result of the inductive process is a geodesic segment τ_m and blocks $B_2, \ldots, B_m$ of X_α with the following properties:

- τ_m coincides with σ' on $[0, t_0 + 1]$;
- $B_0, B_1, B_2, \ldots, B_m$ are distinct and form the itinerary of τ_m;
- B_i and B_{i+1} meet in a common wall W_{i+1} for $1 \le i < m$ and W_{m+1} is a wall of B_m that is disjoint from B_{m-1};
- τ_m enters W_{i+1} for $1 \le i \le m$, and τ_m terminates in W_{m+1};
- if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_m, W_{i+1}, B_i) = (m + 1 - i)\alpha$ for $1 \le i \le m$;
- if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\tau_m, W_{i+1}, B_i) = -\pi + (m+1-i)\alpha$ for $1 \le i \le m$.

In particular, $\theta(\tau_m, W_{m+1}, B_m) \in \{\alpha, \pi - \alpha\}$. To complete the proof of Lemma 4.4, we simply extend the geodesic segment τ_m to a geodesic ray σ'' that terminates in W_{m+1}. Since σ'', τ_m, and σ' all coincide on $[0, t_0]$, it follows that $\sigma''(\infty) \in \Phi$. Clearly, $B_0, B_2, \ldots, B_m$ is the itinerary of σ''. Let B_{m+1} be the block that meets B_m in the wall W_{m+1}. Then Transformation Rule 2 implies that $\theta(\sigma'', W_{m+1}, B_{m+1}) = \theta(\tau_m, W_{m+1}, B_{m+1}) \equiv \alpha - \theta(\tau_m, W_{m+1}, B_m) \bmod 2\pi$. Hence, $\theta(\sigma'', W_{m+1}, B_{m+1}) \in \{0, \pi\}$. Consequently, $\sigma''(\infty)$ is a pole of B_{m+1}. ∎

We now complete the proof of Theorem 4.2 in the case that $n \ge 1$ and $\pi/2(n+1) \le \alpha \le \pi/2n$. Recall that Φ is a nonempty open subset of ∂X_α that contains a point of ∂B_0. We must prove that Φ intersects $\mathcal{P}(n+1)$. Let $\sigma \in \mathcal{R}(p_0)$ be such that $\sigma(\infty) \in \Phi$. If $\sigma(\infty)$ is a pole of B_0, then $\sigma(\infty) \in \Phi \cap \mathcal{P}(n+1)$, and the proof is over. So assume that $\sigma(\infty)$ is not a pole of B_0. Then Lemma 4.3 provides a ray $\sigma' \in \mathcal{R}(p_0)$ and a block B_1 adjacent to B_0 such that $\sigma'(\infty) \in \Phi$ and either $\sigma'(\infty)$ is a pole of B_1 or σ' has itinerary B_0, B_1, σ' terminates in a nonwall plane U_1 of B_1, and $\theta(\sigma', U_1, B_1)$ lies in the set $A = \{k\alpha : 1 \le k \le n\} \cup \{-\pi + k\alpha : 1 \le k \le n\}$. If $\sigma'(\infty)$ is a pole of B_1, then $\sigma'(\infty) \in \Phi \cap \mathcal{P}(n+1)$, and the proof is over. So assume $\sigma'(\infty)$ is not a pole of B_1. Then Lemma 4.4 provides a ray $\sigma'' \in \mathcal{R}(p_0)$, an integer m such that $1 \le m \le n$, and blocks $B_2, B_3, \ldots, B_m, B_{m+1}$ of X_α

such that $\sigma''(\infty) \in \Phi$, $B_0, B_1, B_2, \ldots, B_m, B_{m+1}$ are distinct, $B_0, B_1, \ldots, B_m$ is the itinerary of σ'', B_m is adjacent to B_{m+1}, and $\sigma''(\infty)$ is a pole of B_{m+1}. Since $m+1 \leq n+1$, it follows that $\sigma''(\infty) \in \Phi \cap \mathcal{P}(n+1)$. This concludes the case that $n \geq 1$ and $\pi/2(n+1) \leq \alpha \leq \pi/2n$.

The proof of this theorem in the case that $\alpha = \pi/2$ parallels the previous proof. We assume that Φ is a nonempty open subset of ∂X_α that contains a point of ∂B_0, and we must prove that Φ intersects $\mathcal{P}(2)$. As we argued previously, there is a geodesic ray $\sigma \in \mathcal{R}(p_0)$ that terminates in a nonwall plane U_0 of B_0 such that $\sigma(\infty) \in \Phi$. If $\sigma(\infty)$ is a pole of B_0, then the proof is over. So we assume that $\sigma(\infty)$ is not a pole of B_0. Then $|\theta(\sigma, U_0, B_0)|$ lies in one of the three sets $(0, \pi/2)$, $\{\pi/2\}$, or $(\pi/2, \pi)$. If $|\theta(\sigma, U_0, B_0)| = \pi/2$, then following the proof of Lemma 4.3, we can approximate σ by a geodesic ray $\sigma' \in \mathcal{R}(p_0)$ such that $\sigma'(\infty) \in \Phi$ and $\sigma'(\infty)$ is a pole of a block B_1 that is adjacent to B_0. In this situation, $\sigma'(\infty) \in \Phi \cap \mathcal{P}(2)$, and again the proof is over. So we assume that $|\theta((\sigma, U_0, B_0)| \neq \pi/2$. Then $|\theta((\sigma, U_0, B_0)|$ is an element of one of the two sets $(0, \pi/2)$ or $\pi/2, \pi)$. We follow the proof of Lemma 4.3 in this case as well, approximating σ by a geodesic ray $\sigma' \in \mathcal{R}(p_0)$ such that $\sigma'(\infty) \in \Phi$, and σ' passes from B_0 via a triple point into a nonwall plane U_1 of an adjacent block B_1, σ' terminates in U_1, and $\theta(\sigma', U_1, B_1) = \pi/2$. (Since $\alpha = \pi/2$, Transformation Rule 3 will allow a geodesic ray entering a triple point with angle of inclination in either of the intervals $(0, \pi/2)$ or $(\pi/2, \pi)$ to leave the triple point with angle of inclination equal to $\pi/2$.) Finally, we follow the proof of Lemma 4.4, approximating σ' by a geodesic ray $\sigma'' \in \mathcal{R}(p_0)$ such that $\sigma''(\infty) \in \Phi$ and $\sigma''(\infty)$ is a pole of a block B_2 that is adjacent to B_1. Thus $\sigma''(\infty) \in \Phi \cap \mathcal{P}(2)$. ∎

Combining Theorems 4.1 and 4.2, we obtain:

COROLLARY 4.5. *If $n \geq 1$ and $\pi/2(n+1) \leq \alpha \leq \pi/2n$, then $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(n))$ and $\partial B_0 \subset \mathrm{cl}(\mathcal{P}(n+1))$. Also, if $\alpha = \pi/2$, then $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(1))$ and $\partial B_0 \subset \mathrm{cl}(\mathcal{P}(2))$.*

The principal theorem of [4] established that ∂X_α is not homeomorphic to $\partial X_{\pi/2}$ whenever $0 < \alpha < \pi/2$. Furthermore, it was shown in [4] that for $\alpha, \beta \in (0, \pi/2]$, any homeomorphism between ∂X_α and ∂X_β preserves block boundaries and block adjacency. Thus, distance in the nerve is preserved by such a homeomorphism, as are the sets $\mathcal{P}(n)$. Consequently, any homeomorphism between ∂X_α and ∂X_β preserves the condition that $\partial B_0 \not\subset \mathrm{cl}(\mathcal{P}(n))$ and $\partial B_0 \subset \mathrm{cl}(\mathcal{P}(n+1))$. These observations together with Corollary 4.5 give rise to the following topological differentiation of the boundaries of the Croke–Kleiner spaces into countably many distinct classes:

COROLLARY 4.6. *If $\alpha, \beta \in (0, \pi/2]$ and there is an integer $n \geq 1$ such that $\alpha < \pi/2n \leq \beta$, then ∂X_α is not homeomorphic to ∂X_β.*

We note that a much finer definitive topological differentiation of the boundaries of the Croke–Kleiner spaces was proved after Corollary 4.6. Specifically, if $\alpha, \beta \in (0, \pi/2]$ and $\alpha \neq \beta$, then ∂X_α is not homeomorphic to ∂X_β. See [7].

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