Optics in Croke–Kleiner Spaces
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Summary. We explore the interior geometry of the CAT(0) spaces \( \{ X_\alpha : 0 < \alpha \leq \pi / 2 \} \), constructed by Croke and Kleiner [Topology 39 (2000)]. In particular, we describe a diffraction effect experienced by the family of geodesic rays that emanate from a basepoint and pass through a certain singular point called a triple point, and we describe the shadow this family casts on the boundary. This diffraction effect is codified in the Transformation Rules stated in Section 3 of this paper. The Transformation Rules have various applications. The earliest of these, described in Section 4, establishes a topological invariant of the boundaries of all the \( X_\alpha \)’s for which \( \alpha \) lies in the interval \( [\pi / 2(n + 1), \pi / 2n] \), where \( n \) is a positive integer. Since the invariant changes when \( n \) changes, it provides a partition of the topological types of the boundaries of Croke–Kleiner spaces into a countable infinity of distinct classes. This countably infinite partition extends the original result of Croke and Kleiner which partitioned the topological types of the Croke–Kleiner boundaries into two distinct classes. After this countably infinite partition was proved, a finer partition of the topological types of the Croke–Kleiner boundaries into uncountably many distinct classes was established by the second author [J. Group Theory 8 (2005)], together with other applications of the Transformation Rules.

1. Introduction. The CAT(0) spaces \( X_\alpha \) \( (0 < \alpha \leq \pi / 2) \) constructed in [4] and known as Croke–Kleiner spaces are remarkable for the fact that each space admits a geometric action (i.e. cocompact, properly discontinuous, and by isometries) by the same group \( G \), and yet the boundaries of these spaces are not necessarily homeomorphic. In [4] it was proved that at least two of the Croke–Kleiner spaces have topologically distinct boundaries. Corollary 4.6 of this paper implies that there is at least a countable infinity of distinct topological types among the boundaries of the Croke–Kleiner spaces \( X_\alpha \). After the proofs in this paper were completed, it was shown in [7] that there are uncountably many distinct topological types among the

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boundaries of the Croke–Kleiner spaces. Until recently\(^\text{[1]}\) the Croke–Kleiner spaces provided the only known example of this phenomenon, and it marks a significant deviation from the strict boundary rigidity enjoyed by hyperbolic groups. A hyperbolic group has a unique boundary, in the sense that if a finitely generated group \(G\) acts geometrically on two hyperbolic spaces \(X\) and \(X'\), then there is an induced quasi-isometry from \(X\) to \(X'\) that extends to a \(G\)-equivariant homeomorphism between their boundaries. (For a full treatment of the uniqueness of hyperbolic group boundaries, see \([6]\).) Clearly, the connection between the large-scale algebraic structure of the group and the large-scale geometric structure of a space on which it acts is considerably looser in the CAT(0) theory.

Despite the apparent simplicity of the Croke–Kleiner spaces, their boundaries are complex objects whose properties are not completely understood at this juncture. One source of this complexity is the way that a geodesic ray can diffract upon passage through a \textit{triple point} (a point which is the intersection of three planes). Most points cast 0-dimensional shadows on the boundary in the sense that the collection of geodesic rays issuing from a common basepoint and passing through the point comprises a 0-dimensional subset of the boundary. However, a geodesic passing through a triple point can thereafter proceed in a one-dimensional set of directions. This phenomenon is quantified by the Transformation Rules (Proposition 3.1), and exploited in Section 4 to demonstrate that Croke–Kleiner boundaries exhibit at least a countable infinity of distinct topological types.

2. Preliminaries. Before reviewing the Croke–Kleiner construction, we recall some preliminary definitions; see \([2]\) for a full treatment. A CAT(0) space \((X, d)\) is a complete metric space in which any two points can be joined by a geodesic arc, and whose metric \(d\) satisfies the following condition. Given any geodesic triangle \(\triangle abc\) in \(X\) and a comparison triangle \(\triangle a'b'c'\) in the Euclidean plane, then \(d(p, q) \leq d_{\mathbb{R}^2}(p', q')\) for every pair of points \(p, q\) in \(\triangle abc\) and their counterparts \(p', q'\) in \(\triangle a'b'c'\). An important consequence of this property is that the metric \(d\) is \textit{convex}, which roughly means that once a pair of geodesics in \(X\) begins to diverge, they must diverge at a rate that is at least linear. In particular, every pair of points is joined by a unique geodesic. The boundary of a CAT(0) space \(X\) is defined as follows. Choose a basepoint \(p \in X\) and let \(\mathcal{R}(p)\) denote the set of unit-speed geodesic rays issuing from \(p\). For simplicity, assume that every ray in \(\mathcal{R}(p)\) has domain \([0, \infty)\). To every ray in \(\mathcal{R}(p)\), associate an endpoint \(\sigma(\infty)\). For \(r > 0\) and \(\varepsilon > 0\), two rays \(\sigma, \tau \in \mathcal{R}(p)\) are \((r, \varepsilon)\)-\textit{close} if \(d(\sigma(r), \tau(r)) < \varepsilon\). This defines a topology, called the \textit{cone topology}, on the set \(\{\sigma(\infty) : \sigma \in \mathcal{R}(p)\}\). This space

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is the boundary $\partial X$ of $X$, and up to homeomorphism it is independent of the choice of basepoint $p$. See [2, II.8.8].

For any two geodesic segments or rays $\sigma$ and $\tau$ with a common point $x$ in a CAT(0) space $X$, we denote by $\angle_x(\sigma, \tau)$ the Aleksandrov angle at $x$ between $\sigma$ and $\tau$. Note that a set of the form $\{\tau(\infty) : \tau \in \mathcal{R}(p) \text{ and } \angle_p(\sigma, \tau) < \delta\}$, where $\sigma \in \mathcal{R}(p)$ and $\delta > 0$, is open in $\partial X$. (See [2], particularly II.1.6 and II.1.7.)

The space $X_\alpha$ constructed by Croke–Kleiner is the universal cover of a torus complex $\overline{X}_\alpha$, constructed as follows. Let $T_0, T_1, T_2$ be three flat tori. Let $a_1, a_2$ be a pair of geodesic loops in $T_0$ that generate $\pi_1(T_0)$ and that meet in a single point in $T_0$ at an angle $\alpha$, $0 < \alpha \leq \pi/2$. Note that any angle $\alpha \in (0, \pi/2]$ and any choice of lengths for the geodesic loops $a_1$ and $a_2$ can be realized by taking the orbit space of the action on $\mathbb{E}^2$ by the group generated by two translations whose axes cross at an angle of $\alpha$ and which move points through distances equal to the prescribed lengths of the two geodesics. Next choose closed geodesics $b_i \subset T_i$, $i = 1, 2$, such that $\text{length}(b_i) = \text{length}(a_i)$, and let $\overline{X}_\alpha$ be the union of $T_0$, $T_1$, and $T_2$, with $a_i$ identified with $b_i$. For $i = 1, 2$, let $\overline{Y}_i = T_0 \cup T_i \subset \overline{X}_\alpha$. The space $X_\alpha$ is the universal cover of $\overline{X}_\alpha$. By a plane of $X_\alpha$, we mean a component of the preimage of the $T_0, T_1,$ or $T_2$.

The following facts about $X_\alpha$ were established in [4]:

1. **Blocks**: A component of the preimage of $\overline{Y}_i$ in $X_\alpha$ is called a block. Each block is a copy of the universal cover of $\overline{Y}_i$, and hence is isometric to the metric product of a simplicial valence-4 tree with $\mathbb{R}$.

2. **Walls**: Each block is a tree of planes of two types. A plane of the type that covers $T_0$ is referred to as a wall of the block. Each wall is common to exactly two blocks (one covering $\overline{Y}_1$ and the other covering $\overline{Y}_2$), which are called adjacent blocks. Any two blocks of $X_\alpha$ are either disjoint or adjacent with a wall as their only intersection. Blocks and walls are convex subsets of $X_\alpha$. The nerve of $X_\alpha$ is the (non-locally finite) graph that has one vertex for every block of $X_\alpha$ and which has the property that vertices are adjacent exactly when the corresponding blocks are adjacent. The nerve of $X_\alpha$ is in fact a tree. We can equip the nerve with a metric topology by defining each edge to have length one.

3. **Block boundaries**: Given a block $B$, its boundary $\partial B$ embeds in $\partial X_\alpha$ and is homeomorphic to the suspension of a Cantor set. The two suspension points, called poles of the block, are the common endpoints of the lifts in $B$ of either $a_1$ or $a_2$. A longitude of the block is an arc in $\partial B$ joining the two poles, i.e. the suspension of a point in the Cantor set. If two blocks are adjacent with common wall $W$, then their boundaries meet exactly in $\partial W$. If they are at distance two in
the nerve, then their boundaries meet exactly in the two poles of the block between them. If they are at distance three or more in the nerve, then their boundaries are disjoint.

Let \( \sigma : J \to X_\alpha \) be an injective map whose domain \( J \) is a closed, connected subset of \( \mathbb{R} \). We say that \( \sigma \) enters a plane \( V \) if there are values \( a < b \) in \( J \) such that \( \sigma([a, b]) \subset V \), and that \( \sigma \) enters a block \( B \) if it enters a nonwall plane of \( B \).

4. **Block itineraries**: Choose a basepoint \( p_0 \) that lies in a nonwall plane \( V_0 \) of a block \( B_0 \), away from all walls of \( B_0 \). Then for every \( \sigma \in \mathcal{R}(p_0) \) we can unambiguously define the block itinerary \( \{ B_0, B_2, B_3, \ldots \} \) of \( \sigma \), where \( B_i \) is the \( i \)th block that \( \sigma \) enters. Since blocks are convex, a block itinerary can contain no repetitions.

5. **Topological invariance**: Note that the metric on \( X_\alpha \) depends on \( \alpha \). Thus for \( \alpha, \beta \in (0, \pi/2] \), \( X_\alpha \) and \( X_\beta \) are homeomorphic but not isometric as CAT(0) spaces, and hence their boundaries are not necessarily homeomorphic. However, any homeomorphism from \( \partial X_\alpha \) to \( \partial X_\beta \) must take block boundaries to block boundaries, poles to poles, and longitudes to longitudes.

At a point \( p \in X_\alpha \), the **link** of \( p \) is the metric space \( \text{Link}(p) \) of unit tangent vectors or germs of geodesic rays that emanate from \( p \). If \( \sigma \) is such a ray, let \( g(\sigma) \in \text{Link}(p) \) denote its germ. More generally, if \( \sigma \) is a geodesic such that \( \sigma(a) = p \) and \( \sigma([a, b]) \subset V \) for some plane \( V \), let \( \hat{\sigma} \) be the unique geodesic ray in \( V \) emanating from \( p \) and containing \( \sigma([a, b]) \), and define \( g(\sigma) \) to be \( g(\hat{\sigma}) \). If two elements \( a \) and \( b \) of \( \text{Link}(p) \) point into the same plane of \( X_\alpha \), then the distance between them, \( A(a, b) \), is simply the angle between them. If \( a \) and \( b \) point into different planes of \( X_\alpha \), then the distance \( A(a, b) \) between them is defined to be the minimum value of all sums of the form \( A(c_0, c_1) + A(c_1, c_2) + \ldots + A(c_{k-1}, c_k) \) where \( a = c_0, c_1, \ldots, c_k = b \) are elements of \( \text{Link}(p) \) such that \( c_{i-1} \) and \( c_i \) point into the same plane of \( X_\alpha \) for \( 1 \leq i \leq k \).

Points of \( X_\alpha \) exhibit three types of links. If \( p \) belongs to only one plane of \( X_\alpha \), then its link is a circle of radius one. If \( p \) lies on the intersection of exactly two planes, its link is the union of two circles of radius one intersecting each other in a pair of points that are diametrically opposed on each circle. If \( p \) is the intersection of three planes, then \( \text{Link}(p) \) is a union of three circles \( C, D, \) and \( C' \) of radius one that intersect in the following way. \( C \) and \( D \) intersect in a pair of diametrically opposed points \( a \) and \( b \). Similarly, \( D \) and \( C' \) intersect in a pair of diametrically opposed points \( x \) and \( y \). In \( D \), the distance between \( a \) and \( x \) is \( \alpha \). In each case, the diametrically opposed intersection points are germs of rays issuing from \( p \) and asymptotic with a pair of poles of a block. See Figure 1.
We shall use the following fundamental principle regarding links in a CAT(0) space. Let \( a < b < c \) be real numbers, and suppose \( \sigma : [a, c] \to X_\alpha \) is a map with \( \sigma(b) = p \) such that \( \sigma|_{[a,b]} \) and \( \sigma|_{[b,c]} \) are geodesics. Then \( \sigma \) is a geodesic exactly when the angle between the two geodesics at \( p \) is greater than or equal to \( \pi \). More precisely, define a geodesic \( \sigma^* : [a,b] \to X \) by \( \sigma^*(t) = \sigma(a + b - t) \). (\( \sigma^* \) is \( \sigma|_{[a,b]} \) run backwards.) Then \( \sigma \) is a geodesic if and only if \( A(g(\sigma^*), g(\sigma|_{[b,c]})) \geq \pi \). (See [5, Lemma 1.39, p. 386].)

We want to label the poles of each block “north” and “south” in a coherent way so that the Aleksandrov angle between north poles of adjacent blocks is \( \alpha \) and the angle between the north pole of one block and the south pole of an adjacent block is \( \pi - \alpha \), where the angle is computed in the wall common to the two blocks. This can be done by arbitrarily labeling the poles of our base block \( B_0 \) “north” and “south”, and then inductively labeling poles of blocks at successively larger distances from \( B_0 \) in the nerve. Note that to compute the Aleksandrov angle between two points \( Q \) and \( Q' \) lying in a plane boundary \( \partial V \), one can take geodesic rays emanating from an arbitrary basepoint \( p_V \in V \) and asymptotic with \( Q \) and \( Q' \) in the cone topology associated to \( \mathcal{R}(p_0) \).

Let \( \sigma : J \to X_\alpha \) be an injective map whose domain \( J \) is a closed, connected subset of \( \mathbb{R} \), and whose image \( \sigma(J) \) intersects each plane of \( X_\alpha \) in either the empty set, a single point, a line segment, a ray, or a line. Suppose that \( \alpha \) enters a plane \( V \) of a block \( B \). Then we assign an angle of inclination \( \theta(\sigma, V, B) \) to \( \sigma \) in \( V \) relative to \( B \). This angle is a real number that is determined modulo \( 2\pi \). Suppose \( 0 \leq a < b \) are values such that \( \sigma([a,b]) \subset V \). Let \( \tau \) be the ray emanating from \( \sigma(a) \) that terminates at the north pole of \( B \). Then the angle between \( \sigma|_{[a,b]} \) and \( \tau \) equals \( |\theta(\sigma, V, B)| \mod 2\pi \). In every plane \( V \), there are two continuous choices for the function \( \theta(\sigma, V, B) \); one is the negative of the other. If \( V \) is a nonwall of \( B \), then we are free to make either choice for \( \theta(\sigma, V, B) \). However, if \( V \) is a wall of \( B \), with \( V = B \cap B' \), then we impose the following restriction on the choice of \( \theta(\sigma, V, B) \) that determines it uniquely. If the ray \( \tau' \) terminates at the north pole of \( B' \), then \( \theta(\tau', V, B) \equiv \alpha \mod 2\pi \). (The incorrect choice would result in \( \theta(\tau, V, B) \equiv -\alpha \mod 2\pi \).) By continuity, the angles of inclination of all other
rays traveling in \( V \) are now determined. Note that if a wall \( W \) is common to blocks \( B \) and \( B' \) and \( \sigma \) travels in \( W \), then \( \theta(\sigma, W, B) = \alpha - \theta(\sigma, W, B') \).

3. **Transformation rules.** In general, when a geodesic ray passes from one plane to another in \( X_\alpha \), the intersection of these planes is a line and the angle of inclination that the ray makes with this line in the successive planes is preserved. However, a ray may travel from a nonwall \( V \) of a block \( B \) directly into a nonwall \( V' \) of an adjacent block \( B' \) without traveling for positive time in the intervening wall \( W = B \cap B' \). To do this, it must travel through a *triple point*, a singular point which is the only intersection of \( V \) and \( V' \). A triple point is a lift of the intersection of the geodesic loops \( a_1 \) and \( a_2 \) in the torus \( T_0 \). In this case, the ray may continue in a one-dimensional array of different directions in \( V' \). Rule 3 of the following proposition details the transformation of this angle of inclination when the ray passes through a triple point.

**Proposition 3.1 (Transformation Rules).** Let \( \sigma : J \to X_\alpha \) be a unit speed injective map whose domain \( J \) is a closed, connected subset of \( \mathbb{R} \), and whose image \( \sigma(J) \) intersects each plane of \( X_\alpha \) in either the empty set, a single point, a line segment, a ray, or a line. Then \( \sigma \) is a geodesic if and only if it satisfies the following three rules.

1. If \( \sigma \) enters planes \( V \) and \( V' \) successively, both of which lie in the same block \( B \), then \( \theta(\sigma, V, B) \equiv \pm \theta(\sigma, V', B) \mod 2\pi \).
2. If \( \sigma \) enters a wall \( W \) that is the intersection of adjacent blocks \( B \) and \( B' \), then \( \theta(\sigma, W, B) \equiv \alpha - \theta(\sigma, W, B') \mod 2\pi \).
3. Suppose \( \sigma \) enters planes \( V \) and \( V' \) successively, where \( V \) and \( V' \) are nonwall planes of adjacent blocks \( B \) and \( B' \) respectively, and \( \sigma \) passes from \( V \) to \( V' \) via a triple point \( p \). Let \( \beta \) and \( \beta' \) be representatives of \( \theta(\sigma, V, B) \) and \( \theta(\sigma, V', B') \) respectively that are chosen to lie in the interval \([\pi, \pi]\).

- If \( |\beta| \leq \alpha \), then \( |\beta'| \in [\alpha - |\beta|, \alpha + |\beta|] \).
- If \( \alpha \leq |\beta| \leq \pi - \alpha \), then \( |\beta'| \in [(|\beta| - \alpha, |\beta| + \alpha] \).
- If \( \pi - \alpha \leq |\beta| \leq \pi \), then \( |\beta'| \in [(|\pi - \beta|) - (\pi - |\beta|), (\pi - \alpha) + (\pi - |\beta|)] \).

Transformation Rule 1 tells us that if a geodesic ray travels from a plane \( V \) to another plane \( V' \) in the same block, then the geodesic ray may proceed in either of two directions in \( V' \). As an illustration of Transformation Rule 3, consider a geodesic segment \( \sigma : [0, t] \to X_\alpha \) from the basepoint \( p_0 \) to a triple point \( p \) that is the intersection of the nonwall planes \( V \) and \( V' \) of adjacent blocks \( B \) and \( B' \) respectively. Suppose \( \beta = \theta(\sigma, V, B) \) satisfies \( |\beta| \leq \alpha \). Then for every real number \( \beta' \) satisfying \( |\beta'| \in [\alpha - |\beta|, \alpha + |\beta|] \), there is an extension of \( \sigma \) to a geodesic ray that enters \( V' \) with angle of inclination \( \beta' \).
Proof of Proposition 3.1. Rule 2 holds for any oriented straight line segment in a wall by virtue of the manner in which we labeled poles and angles of inclination. Consequently, Rule 2 holds for $\sigma$ regardless of whether it is a geodesic.

We will prove that Rules 1 and 3 provide a necessary and sufficient condition for $\sigma$ to be a geodesic. Since local geodesics are geodesics in CAT(0) spaces (see [2, Prop. 1.4, p. 160]), it suffices to prove that $\sigma$ is a local geodesic at the points where it moves between planes if and only if it satisfies Rules 1 and 3 at these points. Recall that $\sigma$ is a local geodesic at a point $p$ where it moves between two planes if and only if the distance in $\text{Link}(p)$ between the germs at $p$ of the reversed incoming segment of $\sigma$ and the outgoing segment of $\sigma$ is at least $\pi$ (see [5, Lemma 1.39, p. 386]).

Rule 1 pertains to the case in which there are real numbers $a < b < c$ and planes $V$ and $V'$ lying in the same block $B$ such that $\sigma([a, b]) \subset V$, $\sigma([b, c]) \subset V'$, and $\sigma(b) = p$. In this case, $\text{Link}(p)$ is of the second type or third type depending on whether or not $p$ is a triple point.

First suppose $p$ is not a triple point and $\text{Link}(p)$ is of the second type, namely a union of two unit circles $C$ and $C'$ which meet at diametrically opposed points $n$ and $s$ that are the germs of geodesic rays emanating from $p$ toward the north and south poles of $B$ respectively. Let $\sigma_1 = \sigma|_{[a, b]}$, let $\sigma_1^*$ denote $\sigma_1$ run backwards, and let $\sigma_2 = \sigma|_{[b, c]}$. Let $\beta$ and $\beta'$ be representatives of $\theta(\sigma, V, B)$ and $\theta(\sigma, V', B)$ that are chosen to lie in the interval $[-\pi, \pi]$. Recall that $|\beta|$ is the Aleksandrov angle between $\sigma_1$ and a ray approaching the north pole of $B$. Thus $|\beta| = \pi - A(g(\sigma_1^*), n) = A(s, g(\sigma_1^*))$. Also, $|\beta'| = A(n, g(\sigma_2)) = \pi - A(g(\sigma_2), s)$. (See Figure 2) Recall that $\sigma$ is a local geodesic at $p$ if and only if the distance $A(g(\sigma_1^*), g(\sigma_2))$ in $\text{Link}(p)$ is at least $\pi$.

![Fig. 2. Link(p) in the case of Rule 1](image-url)
We will now compute \(A(g(\sigma_1^*), g(\sigma_2))\). There are two candidates for the shortest path in \(\text{Link}(p)\) between \(g(\sigma_1^*)\) and \(g(\sigma_2)\). The first is via \(n\), with length \(L_1 = A(g(\sigma_1^*), n) + A(n, g(\sigma_2)) = (\pi - |\beta|) + |\beta'|\). The second is via \(s\), with length \(L_2 = A(g(\sigma_1^*), s) + A(s, g(\sigma_2)) = |\beta| + (\pi - |\beta'|)\). Thus \(A(g(\sigma_1^*), g(\sigma_2)) = \min\{L_1, L_2\}\). We conclude that \(\sigma\) is a local geodesic at \(p\) if and only if \(L_1 \geq \pi\) and \(L_2 \geq \pi\), i.e. if and only if \(|\beta'| - |\beta| \geq 0\) and \(|\beta| - |\beta'| \geq 0\), i.e. if and only if \(\beta = \pm \beta'\).

Now suppose we are still in the case of Rule 1 but \(p\) is a triple point. Then \(\text{Link}(p)\) is of the third type, namely the union of three unit circles \(C, D, \) and \(C'\) where \(C\) and \(D\) consist of germs of geodesic rays emanating from \(p\) into the planes \(V\) and \(V'\), and \(C'\) consists of germs of geodesic rays emanating from \(p\) into a nonwall plane of a block \(B'\) adjacent to \(B\). Furthermore, \(C\) and \(D\) meet at diametrically opposed points \(n\) and \(s\) that are the germs of geodesic rays emanating from \(p\) toward the north and south poles of \(B\), and \(D\) and \(C'\) meet at diametrically opposed points \(n'\) and \(s'\) that are the germs of geodesic rays emanating from \(p\) towards the north and south poles of \(B'\). With \(\sigma_1, \sigma_1^*, \sigma_2, \beta, \) and \(\beta'\) as above, we have \(g(\sigma_1^*)\) and \(g(\sigma_2)\) in \(C \cup D\). Observe that any two points of \(C \cup D\) are joined by a shortest path in \(C \cup D \cup C'\) that is entirely contained in \(C \cup D\). Consequently, the calculation of \(A(g(\sigma_1^*), g(\sigma_2))\) performed in the previous paragraph also applies here and gives the same conclusion: \(\sigma\) is a local geodesic at \(p\) if and only if \(\beta = \pm \beta'\).

Finally we consider the case of Rule 3, in which there are real numbers \(a < b < c\) and nonwall planes \(V\) and \(V'\) belonging to adjacent blocks \(B\) and \(B'\) respectively, such that \(\sigma([a, b]) \subset V\), \(\sigma([b, c]) \subset V'\), and \(\sigma(b) = p\) is a triple point. Then \(B\) and \(B'\) meet in a wall \(W\) such that \(V \cap V' = V \cap W \cap V' = \{p\}\). In this case, \(\text{Link}(p)\) is of the third type, namely the union of three unit circles \(C, D, \) and \(C'\) consisting of germs of geodesic rays emanating from \(p\) into the planes \(V, W,\) and \(V'\) respectively. Furthermore, \(C\) and \(D\) meet at diametrically opposed points \(n\) and \(s\) that are the germs of geodesic rays emanating from \(p\) toward the north and south poles of \(B\), and \(D\) and \(C'\) meet at diametrically opposed points \(n'\) and \(s'\) that are the germs of rays emanating from \(p\) toward the north and south poles of \(B'\). Let \(\sigma_1, \sigma_1^*, \) and \(\sigma_2\) be as above. Then \(g(\sigma_1^*) \in C\) and \(g(\sigma_2) \in C'\). Let \(\beta\) and \(\beta'\) be representatives of \(\theta(\sigma, V, B)\) and \(\theta(\sigma, V', B')\) that lie in the interval \([-\pi, \pi]\). Then \(|\beta| = \pi - A(g(\sigma_1^*), n) = A(s, g(\sigma_1^*))\), and \(|\beta'| = A(n', g(\sigma_2)) = \pi - A(g(\sigma_2), s').\) (See Figure 3.) As before, \(\sigma\) is a local geodesic at \(p\) if and only if \(A(g(\sigma_1^*), g(\sigma_2)) \geq \pi\).

If \(a\) and \(b\) are points lying on the same circular arc \(J\) of diameter less than \(\pi\) in \(\text{Link}(p)\), let \(ab\) denote the subarc of \(J\) joining \(a\) to \(b\). A shortest path in \(\text{Link}(p)\) from \(g(\sigma_1^*)\) to \(g(\sigma_2)\) must contain one of the four arcs \(nn'\), \(ss'\), \(ns'\), or \(sn'\). Hence there are four candidates for such a path:
1. \( g(\sigma^*_1)n \cup nn' \cup n'g(\sigma_2) \), with length \( L_1 = A(g(\sigma^*_1), n) + A(n, n') + A(n', g(\sigma_2)) = (\pi - |\beta|) + \alpha + |\beta'|; \)
2. \( g(\sigma^*_1)s \cup ss' \cup s'g(\sigma_2) \), with length \( L_2 = A(g(\sigma^*_1), s) + A(s, s') + A(s', g(\sigma_2)) = |\beta| + \alpha + (\pi - |\beta'|); \)
3. \( g(\sigma^*_1)n \cup ns' \cup s'g(\sigma_2) \), with length \( L_3 = A(g(\sigma^*_1), n) + A(n, s') + A(s', g(\sigma_2)) = (\pi - |\beta|) + (\pi - \alpha) + (\pi - |\beta'|); \)
4. \( g(\sigma^*_1)s \cup sn' \cup n'g(\sigma_2) \), with length \( L_4 = A(g(\sigma^*_1), s) + A(s, n') + A(n', g(\sigma_2)) = |\beta| + (\pi - \alpha) + |\beta'|. \)

We know that \( \sigma \) is a geodesic exactly when \( A(g(\sigma^*_1), g(\sigma_2)) = \min\{L_1, L_2, L_3, L_4\} \geq \pi \), i.e. when \( L_i \geq \pi \) for all \( i \). The three subcases listed in Rule 3 \((|\beta| \leq \alpha, \alpha \leq |\beta| \leq \pi - \alpha, \text{ and } \pi - \alpha \leq |\beta| \leq \pi)\) exhaust all possibilities.

We now argue that the condition stated under each of these subcases is equivalent to the restriction that \( L_i \geq \pi \) for all \( i \).

- Assume \( |\beta| \leq \alpha \). Then \( \alpha - |\beta| \geq 0 \), and hence \( L_1 \geq \pi \). Furthermore, since \( |\beta| \leq \alpha \leq \pi/2 \), it follows that \( |\beta| \leq \pi - \alpha \), and thus \( L_3 \geq L_2 \).

- Assume \( \alpha \leq |\beta| \leq \pi - \alpha \). Then \( \alpha - |\beta| \leq |\beta| - \alpha \), which implies \( L_4 \geq L_1 \), and \( \alpha + |\beta| \leq \pi \leq 2\pi - (\alpha + |\beta|) \), so that \( L_3 \geq L_2 \).

- Finally, assume \( \pi - \alpha \leq |\beta| \leq \pi \). Then \( \alpha + |\beta| \geq \pi \), and so \( L_2 \geq \pi \). Also, \( \pi - |\beta| \leq \alpha \leq \pi - \alpha \leq |\beta| \), implying that \( L_4 \geq L_1 \). Thus \( L_i \geq \pi \) for all \( i \) if and only if \( L_1 \geq \pi \) and \( L_3 \geq \pi \). These two inequalities are
equivalent to $|\beta'| \geq |\beta| - \alpha = (\pi - \alpha) - (\pi - |\beta|)$ and $|\beta'| \leq 2\pi - |\beta| - \alpha = (\pi - \alpha) + (\pi - |\beta|)$, i.e. $\beta' \in [(\pi - \alpha) - (\pi - |\beta|), (\pi - \alpha) + (\pi - |\beta|)]$, as required. ■

Note that the angle transformation described in Rule 2 is simply a special case of Rule 3: For suppose $\sigma$ enters the wall $W$ between adjacent blocks $B$ and $B'$, and $\beta = \theta(\sigma, W, B)$ and $\beta' = \theta(\sigma, W, B')$. According to Rule 2, $\beta' \equiv \alpha - \beta \mod 2\pi$. It is straightforward to verify the following:

- if $|\beta| \leq \alpha$, then $|\beta'| = |\alpha - \beta| = \alpha - |\beta|$ or $\alpha + |\beta|$;
- if $\alpha \leq |\beta| \leq \pi - \alpha$, then $|\beta'| = |\alpha - \beta| = |\beta| - \alpha$ or $|\beta| + \alpha$;
- if $\pi - \alpha \leq |\beta| \leq \pi$, then $|\beta'| = |\alpha - \beta| = (\pi - \alpha) - (\pi - |\beta|)$ or $(\pi - \alpha) + (\pi - |\beta|)$,

in keeping with Rule 3. Thus if the weaker result of Rule 3 suffices for $\beta'$, it may be used instead of Rule 2.

The Transformation Rules have various applications. In the next section we describe the earliest of these applications which leads to a partition of the topological types of the boundaries of Croke–Kleiner spaces into a countable infinity of distinct classes, thereby extending the main result of [4]. Subsequent instances of the use of the Transformation Rules appear in [7] and [11].

4. A property of the boundary. In this section we will see that the diffraction effect experienced by geodesic rays that pass through triple points (as described in Transformation Rule 3) has an impact on the topology of $\partial X_\alpha$. The size of this effect, which is controlled by the angle $\alpha$, determines how quickly poles of nearby blocks accumulate on the boundary of the base block $B_0$. Specifically, there is a partition of $(0, \pi/2]$ into countably many disjoint subintervals associated with distinct positive integers with the property that if $\alpha$ lies in the interval associated with a positive integer $n$, then $n$ is the smallest integer such that every point of $\partial B_0$ is a limit of poles whose distance from $B_0$ is no greater than $n + 1$. (This is the content of Corollary 4.5.) This property is a topological invariant of $\partial X_\alpha$. It follows (Corollary 4.6) that as $\alpha$ ranges over $(0, \pi/2]$, at least countably infinitely many topologically distinct $\partial X_\alpha$’s are encountered.

For each $n \geq 0$, let $\mathcal{P}(n)$ denote the set of poles of blocks which are at distance less than or equal to $n$ from $B_0$ in the nerve.

**Theorem 4.1.** If $n \geq 1$ and $\pi/2(n + 1) \leq \alpha < \pi/2n$, then $\partial B_0 \not\subset \text{cl}(\mathcal{P}(n))$. Also, if $\alpha = \pi/2$, then $\partial B_0 \not\subset \text{cl}(\mathcal{P}(1))$.

**Proof.** First assume $n \geq 1$ and $\pi/2(n + 1) \leq \alpha < \pi/2n$. Note that for all $0 < k \leq n$, $\alpha < \pi/2n \leq \pi/2k$, and so $k\alpha < \pi/2$. Thus $k\alpha < \pi - k\alpha$ for all $0 \leq k \leq n$. Let $\Omega$ be the set of all $\sigma(\infty)$ such that $\sigma \in \mathcal{R}(p_0)$ and
\[\theta(\sigma, V_0, B_0) \in (n\alpha, \pi - n\alpha).\] Note that \(\Omega\) is an open subset of \(\partial X\). Clearly \(\Omega\) contains points of \(\partial V_0 \subset \partial B_0\). We will show that \(\Omega \cap \mathcal{P}(n) = \emptyset\), and hence that \(\partial B_0 \not\subseteq \text{cl}(\mathcal{P}(n))\).

Suppose instead that \(\sigma \in \mathcal{R}(p_0)\) is a ray such that \(\sigma(\infty) \in \Omega \cap \mathcal{P}(n)\). Then \(\sigma(\infty)\) is a pole of a block \(B_m\) at distance \(m\) from \(B_0\) in the nerve, for some \(m \leq n\). Note that the angle of inclination of \(\sigma\) in any plane of \(B_m\) must equal a multiple of \(\pi\). Recall that \(\sigma\) is only said to enter a block if it travels for positive time in the block away from all walls. Thus \(\sigma\) can never enter \(B_m\) in this sense. For suppose \(\sigma\) first encounters \(B_m\) at a point \(p\). Then \(p\) must belong to a wall \(W_m\) of \(B_m\), even if \(p\) is a triple point. If \(p\) is not a triple point, then \(\sigma\) must move in \(W_m\) parallel to and disjoint from the lines of intersection of \(W_m\) with the nonwalls of \(B_m\), because \(\theta(\sigma, W_m, B_m)\) is a multiple of \(\pi\). On the other hand, if \(p\) is a triple point that lies in the intersection of \(W_m\) with a nonwall \(V\) of \(B_m\), then \(\sigma\) cannot leave \(W_m\) to enter \(V\) because to do so would require \(\theta(\sigma, W_m, B_m)\) to take on a value that is not a multiple of \(\pi\). Hence the itinerary of \(\sigma\) is of the form \(B_0, B_1, \ldots, B_{m-1}\).

Furthermore, if \(W_m = B_{m-1} \cap B_m\) is the common wall between these two blocks, then \(\sigma\) terminates in \(W_m\) in the sense that \(\sigma([t, \infty)) \subseteq W_m\) for some \(t > 0\). Moreover, \(\sigma([t, \infty))\) is a ray in \(W_m\) that is parallel to the lines of intersection of \(W_m\) with the nonwalls of \(B_m\).

Set \(P_0 = V_0\) and \(P_m = W_m\), and for all \(0 < i < m\), let \(P_i\) be a plane of \(B_i\) that \(\sigma\) enters. For \(0 \leq i \leq m\), let \(\beta_i = \theta(\sigma, P_i, B_i)\), and assume \(\beta_i \in [-\pi, \pi]\). Note that if \(P'_i\) is any other plane of \(B_i\) that \(\sigma\) enters, then \(|\theta(\sigma, P'_i, B_i)| = |\beta_i|\), by Transformation Rule 1. Also note that \(|\beta_m| = 0\) or \(\pi\) by an earlier remark.

Since \(\sigma(\infty) \in \Omega\), it follows that \(\beta_0 \in (n\alpha, \pi - n\alpha)\). We assert that in fact

\[(1) \quad |\beta_i| \in ((n-i)\alpha, \pi - (n-i)\alpha)\]

for all \(0 \leq i \leq m\). Assume this is true for some \(i\) between 0 and \(m-1\). Since \(i \leq m-1 \leq n-1\), it follows that \(1 \leq n-i\); thus \(\alpha \leq (n-i)\alpha\) and \(\pi - (n-i)\alpha \leq \pi - \alpha\). Also, \((n-i)\alpha < \pi - (n-i)\alpha\), as noted at the beginning of the proof. Thus \(((n-i)\alpha, \pi - (n-i)\alpha) \subset (\alpha, \pi - \alpha)\), and hence \(\alpha < |\beta_i| < \pi - \alpha\). The relation between \(|\beta_i|\) and \(|\beta_{i+1}|\) is governed by Transformation Rule 2 or 3, and Rule 3 subsumes Rule 2. Thus \(|\beta_{i+1}| \in [|\beta_i| - \alpha, |\beta_i| + \alpha]\). From our inductive hypothesis (1), \((n-(i+1))\alpha < |\beta_i| - \alpha\) and \(|\beta_i| + \alpha < \pi - (n-(i+1))\alpha\), from which it follows that \(|\beta_{i+1}| \in ((n-(i+1))\alpha, \pi - (n-(i+1))\alpha)\). This proves our assertion.

Since \(m \leq n\), the assertion implies that \(|\beta_m| \in (0, \pi)\), contradicting the fact that \(|\beta_m| = 0\) or \(\pi\). Thus \(\Omega \cap \mathcal{P}(n) = \emptyset\), and so \(\partial B_0\) is not contained in \(\text{cl}(\mathcal{P}(n))\).

The proof of this theorem in the case that \(\alpha = \pi/2\) is very similar to the previous argument. When \(\alpha = \pi/2\), let \(\Omega\) be the set of all \(\sigma(\infty)\) such
that \( \sigma \in \mathcal{R}(p_0) \) and \( \theta(\sigma, V_0, B_0) \in (0, \pi/2) \). Again \( \Omega \) is an open subset of \( \partial X_\alpha \) that contains points of \( \partial B_0 \). In this case, we argue that \( \Omega \) is disjoint from \( \mathcal{P}(1) \), proving that \( \partial B_0 \not\subseteq \text{cl}(\mathcal{P}(1)) \). To this end, assume that there is a \( \sigma \in \mathcal{R}(p_0) \) such that \( \sigma(\infty) \in \Omega \cap \mathcal{P}(1) \). Then, as we argued previously, there is a block \( B_1 \) that shares a wall \( W_1 \) with \( B_0 \) such that \( \sigma([0, \infty)) \subset B_0 \), \( \sigma \) terminates in \( W_1 \), and \( \sigma(\infty) \) is a pole of \( B_1 \). Let \( \beta_0 = \theta(\sigma, V_0, B_0) \) and \( \beta_1 = \theta(\sigma, W_1, B_1) \). Then \( |\theta(\sigma, W_1, B_0)| = |\beta_0| \in (0, \pi/2) \) by Transformation Rule 1, and \( \beta_1 \) is a multiple of \( \pi \). The relationship between \( |\beta_0| \) and \( |\beta_1| \) is governed by Transformation Rule 2 or 3, hence by Transformation Rule 3. Since \( 0 < |\beta_0| < \pi/2 = \alpha \), it follows that \( |\beta_1| \in [\alpha - |\beta_0|, \alpha + |\beta_0|] \subset (0, \pi) \). Therefore, \( \beta_1 \) is not a multiple of \( \pi \). We conclude that \( \Omega \cap \partial X_\alpha = \emptyset \), and thus \( \partial B_0 \not\subseteq \text{cl}(\mathcal{P}(1)) \).

**Theorem 4.2.** If \( n \geq 1 \) and \( \pi/2(n+1) \leq \alpha < \pi/2n \), then \( \partial B_0 \subset \text{cl}(\mathcal{P}(n+1)) \). Also, if \( \alpha = \pi/2 \), then \( \partial B_0 \subset \text{cl}(\mathcal{P}(2)) \).

**Proof.** First assume \( n \geq 1 \) and \( \pi/2(n+1) \leq \alpha < \pi/2n \). Let \( \Phi \) be a nonempty open subset of \( \partial X_\alpha \) that contains a point of \( \partial B_0 \). We will prove that \( \Phi \) intersects \( \mathcal{P}(n+1) \).

The union of the boundaries of the nonwall planes of \( B_0 \) minus the poles of \( B_0 \) is dense in \( \partial B_0 \). Hence, there is a nonwall plane \( U_0 \) of \( B_0 \) and a point \( Q \in \partial U_0 \cap \Phi \) such that \( Q \) is not a pole of \( B_0 \). Then there is a geodesic ray \( \sigma \) in \( B_0 \) that emanates from \( p_0 \) and terminates in \( U_0 \) (i.e., \( \sigma([t, \infty)) \subset U_0 \) for some \( t > 0 \)), such that \( \sigma(\infty) = Q \). Let \( A \) denote the finite set \( \{k \alpha : 1 \leq k \leq n\} \cup \{-\pi + k\alpha : 1 \leq k \leq n\} \).

The remainder of the proof is accomplished in two steps, utilizing the two lemmas below. In the first step, \( \sigma \) is approximated by a ray \( \sigma' \in \mathcal{R}(p_0) \) so that \( \sigma'(\infty) \in \Phi \), \( \sigma' \) passes from the block \( B_0 \) to a block \( B_1 \) via a triple point, and \( \sigma' \) enters and terminates in a nonwall plane \( U_1 \) of \( B_1 \) such that the angle of inclination of \( \sigma' \) in \( U_1 \) is an element of the set \( A \). In the second step of the proof, \( \sigma' \) is approximated by a ray \( \sigma'' \in \mathcal{R}(p_0) \) so that \( \sigma''(\infty) \in \Phi \) and \( \sigma'' \) follows an itinerary \( B_0, B_1, \ldots, B_m \), where \( 1 \leq m \leq n \), such that the successive angles of inclination of \( \sigma'' \) in these blocks progress through the elements of set \( A \) by jumps of size \( \alpha \), moving toward either \( \alpha \) or \( -\pi + \alpha \), with the result that \( \sigma''(\infty) \) is a pole of a block \( B_{m+1} \) that is adjacent to \( B_m \).

**Lemma 4.3.** Suppose \( n \geq 1 \) and \( \pi/2(n+1) \leq \alpha < \pi/2n \), \( U_0 \) is a nonwall plane of \( B_0 \) and \( \sigma \in \mathcal{R}(p_0) \) is such that \( \sigma \) terminates in \( U_0 \), \( \sigma(\infty) \) is not a pole of \( B_0 \), and \( \Phi \) is a neighborhood of \( \sigma(\infty) \) in \( \partial X_\alpha \). Then there is a ray \( \sigma' \in \mathcal{R}(p_0) \) such that \( \sigma'(\infty) \in \Phi \), and there is a block \( B_1 \) adjacent to \( B_0 \), such that either \( \sigma'([0, \infty)) \subset B_0 \) and \( \sigma'(\infty) \) is a pole of \( B_1 \), or \( \sigma' \) has itinerary \( B_0, B_1 \), there is a nonwall plane \( U_1 \) of \( B_1 \) such that \( \sigma' \) terminates in \( U_1 \), and \( \theta(\sigma', U_1, B_1) \in A \).
Proof. Since $\sigma$ terminates in $U_0$ and $\sigma(\infty) \in \Phi$, it follows that there is a $t_0 > 0$ and a $\delta > 0$ such that $\sigma((t_0, \infty)) \subset U_0$ and such that if $\sigma' \in \mathcal{R}(p_0)$ and $d(\sigma'(t_0), \sigma(t_0)) < \delta$, then $\sigma'(\infty) \in \Phi$. We assume that $\theta(\sigma, U_0, B_0)$ and all other angles of inclination mentioned in this proof lie in the interval $[-\pi, \pi]$. Since $\sigma(\infty)$ is not a pole of $B_0$, it follows that $|\theta(\sigma, U_0, B_0)| \in (0, \pi)$. Since $\pi/2(n + 1) \leq \alpha < \pi/2n$, we see that $(0, \pi) = (0, \pi/2) \cup (\pi/2, \pi) \subset (0, (n + 1)\alpha] \cup [\pi - (n + 1)\alpha, \pi)$. Hence, there is an integer $m$ such that $0 \leq m \leq n$ and either

$$|\theta(\sigma, U_0, B_0)| \in (0, \pi/2) \quad \text{and} \quad |\theta(\sigma, U_0, B_0)| \in (m\alpha, (m + 1)\alpha],$$

or

$$|\theta(\sigma, U_0, B_0)| \in (\pi/2, \pi) \quad \text{and} \quad |\theta(\sigma, U_0, B_0)| \in [\pi - (m + 1)\alpha, \pi - m\alpha).$$

The proof now divides into three mutually exclusive cases:

- $|\theta(\sigma, U_0, B_0)| \in \{\alpha, \pi - \alpha\}$,
- $|\theta(\sigma, U_0, B_0)| \in \{(m + 1)\alpha, \pi - (m + 1)\alpha\}$, where $1 \leq m \leq n$,
- $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m + 1)\alpha) \cup (\pi - (m + 1)\alpha, \pi - m\alpha)$, where $0 \leq m \leq n$.

First, assume $|\theta(\sigma, U_0, B_0)| \in \{\alpha, \pi - \alpha\}$. In this case we construct a ray $\sigma' \in \mathcal{R}(p_0)$ that coincides with $\sigma$ on $[0, t_0 + 1]$ and then enters and terminates in a wall $W_1$ of $B_0$. Furthermore, if $|\theta(\sigma, U_0, B_0)| = \alpha$, then we insist that $\sigma'$ enters the positive half of $W_1$, so that $\theta(\sigma', W_1, B_0) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - \alpha$, then we insist that $\sigma'$ enters the negative half of $W_1$, so that $\theta(\sigma', W_1, B_0) < 0$. Clearly, $\sigma'([0, \infty)) \subset B_0$ and $\sigma'(\infty) \in \Phi$, because $\sigma'$ and $\sigma$ coincide at $t_0$. Transformation Rule 1 implies that $|\theta(\sigma', W_1, B_0)| = |\theta(\sigma', U_0, B_0)| = |\theta(\sigma, U_0, B_0)|$. Hence, $\theta(\sigma', W_1, B_0) = \alpha$ if $|\theta(\sigma, U_0, B_0)| = \alpha$, and $\theta(\sigma', W_1, B_0) = -\pi + \alpha$ if $|\theta(\sigma, U_0, B_0)| = -\pi - \alpha$. Let $B_1$ be the block of $X_\alpha$ that intersects $B_0$ in the wall $W_1$. Transformation Rule 2 implies that $\theta(\sigma', W_1, B_1) = \alpha - \theta(\sigma', W_1, B_0) \mod 2\pi$. Consequently, $\theta(\sigma', W_1, B_1) = 0$ if $|\theta(\sigma, U_0, B_0)| = \alpha$, and $\theta(\sigma', W_1, B_1) = \pi$ if $|\theta(\sigma, U_0, B_0)| = -\pi - \alpha$. Hence, $\sigma'(\infty)$ is a pole of $B_1$.

Second, assume $|\theta(\sigma, U_0, B_0)| \in \{(m + 1)\alpha, \pi - (m + 1)\alpha\}$, where $1 \leq m \leq n$. In this case we construct a ray $\sigma' \in \mathcal{R}(p_0)$ that coincides with $\sigma$ on $[0, t_0 + 1]$ and then enters a wall $W_1$ of $B_0$. Let $B_1$ be the block of $X_\alpha$ that intersects $B_0$ in the wall $W_1$. After spending a positive amount of time in $W_1$, $\sigma'$ leaves $W_1$ and enters and terminates in a nonwall plane $U_1$ of $B_1$. Furthermore, if $|\theta(\sigma, U_0, B_0)| = (m + 1)\alpha$, then we insist that $\sigma'$ enters the positive half of $W_1$, so that $\theta(\sigma', W_1, B_0) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - (m + 1)\alpha$, then we insist that $\sigma$ enters the negative half of $W_1$, so that $\theta(\sigma', W_1, B_0) < 0$. In addition, if $|\theta(\sigma, U_0, B_0)| = (m + 1)\alpha$, then we insist that $\sigma'$ enters the positive half of $U_1$, so that $\theta(\sigma', U_1, B_1) > 0$. However, if $|\theta(\sigma, U_0, B_0)| = \pi - (m + 1)\alpha$, then we insist that $\sigma'$ enters
the negative half of $U_1$, so that $\theta(\sigma', U_1, B_1) < 0$. Clearly $\sigma'$ has itinerary $B_0, B_1$, and $\sigma'(-\infty) \in \Phi$ because $\sigma'$ and $\sigma$ coincide at $t_0$. Transformation Rule 1 implies that $|\theta(\sigma', U_1, B_0)| = |\theta(\sigma', U_0, B_0)| = |\theta(\sigma, U_0, B_0)|$. Hence $\theta(\sigma', W_1, B_0) = (m+1)\alpha$ if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, whereas $\theta(\sigma', W_1, B_0) = -\pi + (m+1)\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$. Transformation Rule 2 implies $\theta(\sigma', W_1, B_1) \equiv \alpha - \theta(\sigma', W_1, B_0) \mod 2\pi$. Consequently, $\theta(\sigma', W_1, B_1) = -m\alpha$ if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, and $\theta(\sigma', W_1, B_1) = \pi - m\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$. Again, Transformation Rule 1 implies that $|\theta(\sigma', U_1, B_1)| = |\theta(\sigma', W_1, B_1)|$. Therefore, $\theta(\sigma', U_1, B_1) = m\alpha$ if $|\theta(\sigma, U_0, B_0)| = (m+1)\alpha$, and $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$ if $|\theta(\sigma, U_0, B_0)| = \pi - (m+1)\alpha$. Thus, $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$, where $1 \leq m \leq n$. Consequently, $\theta(\sigma', U_1, B_1) \in A$.

Third, assume $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha) \cup (\pi - (m+1)\alpha, \pi - m\alpha)$, where $0 \leq m \leq n$. We remark that if $\tau$ is a ray in $X_\alpha$ such that $\tau([t_0, t_0 + 1]) \subset U_0$, then $\theta(\tau, U_0, B_0)$ depends continuously on the two points $t(t_0)$ and $t(t_0 + 1)$. We also note that if $\tau \in \mathcal{R}(p_0)$, then $d(\sigma(t_0), \tau(t_0)) \leq d(\tau(t_0 + 1), \sigma(t_0 + 1))$ because $X_\alpha$ is a CAT(0) space. Consequently, we can assume that $\delta > 0$ has been chosen so small that if $\tau \in \mathcal{R}(p_0)$ and $d(\tau(t_0 + 1), \sigma(t_0 + 1)) < \delta$, then both $|\theta(\tau, U_0, B_0)|$ and $\theta(\sigma, U_0, B_0)$ belong either to the interval $(m\alpha, (m+1)\alpha)$ or to the interval $(\pi - (m+1)\alpha, \pi - m\alpha)$. We observe that the triple points in $U_0$ form a quasi-dense subset. Indeed, the distance from a point of $U_0$ to the nearest triple point is no greater than the geodesic diameter of the torus $T_i$ which is covered by $U_0$, since the triple points in $U_0$ are simply the preimages of the triple intersection point $T_0 \cap T_1 \cap T_2$ under the locally isometric covering map $U_0 \to T_i$. Hence, there is an $r > 0$ such that the $r$-neighborhood of every point of $U_0$ contains a triple point. Choose $t_1 > (\max\{1, 2r/\delta\})(t_0 + 1)$. Let $p$ be a triple point in $U_0$ such that $d(\sigma(t_1), p) < r$. Let $\tau \in \mathcal{R}(p_0)$ be the geodesic ray that passes through the triple point $p$ and terminates in $U_0$. Let $u = d(p_0, p)$, so that $\tau(u) = p$. Hence, $|t_1 - u| = |d(p_0, \sigma(t_1)) - d(p_0, p)| \leq d(\sigma(t_1), p) < r$. Therefore

$$d(\sigma(t_1), \tau(t_1)) \leq d(\sigma(t_1), \tau(u)) + d(\tau(u), \tau(t_1)) = d(\sigma(t_1), p) + |t_1 - u| < 2r.$$ 

It follows by the CAT(0) condition that

$$d(\sigma(t_0 + 1), \tau(t_0 + 1)) \leq \frac{t_0 + 1}{t_1} (2r) < \delta.$$ 

Therefore $d(\sigma(t_0), \tau(t_0)) < \delta$. Hence, both $|\theta(\tau, U_0, B_0)|$ and $|\theta(\sigma, U_0, B_0)|$ belong either to $(m\alpha, (m+1)\alpha)$ or to the interval $(\pi - (m+1)\alpha, \pi - m\alpha)$. Since $p$ is a triple point that belongs to $U_0$, it follows that $p$ also belongs to a wall $W_1$ of $B_0$, there is a block $B_1$ that meets $B_0$ in the wall $W_1$, and $p$ also belongs to a nonwall plane $U_1$ of $B_1$. 
Let $\sigma'$ be the union of the geodesic segment in $X_\alpha$ from $p_0$ to $p$ together with the geodesic ray in $U_1$ that emanates from the triple point $p$ and satisfies the following:

- $\theta(\sigma', U_1, B_1) = \alpha$ if $m = 0$ and $|\theta(\sigma, U_0, B_0)| \in (0, \alpha)$;
- $\theta(\sigma', U_1, B_1) = -\pi + \alpha$ if $m = 0$ and $|\theta(\sigma, U_0, B_0)| \in (\pi - \alpha, \pi)$;
- $\theta(\sigma', U_1, B_1) = m\alpha$ if $1 \leq m \leq n$ and $|\theta(\sigma, U_0, B_0)| \in (m\alpha, (m+1)\alpha)$;
- $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$ if $1 \leq m \leq n$ and $|\theta(\sigma, U_0, B_0)| \in (\pi - (m+1)\alpha, \pi - m\alpha)$.

Clearly, $\sigma'$ is a geodesic ray. To verify this, we must show that $\beta = \theta(\tau, U_0, B_0)$ and $\beta' = \theta(\sigma', U_1, B_1)$ are related as prescribed by Transformation Rule 3. Indeed:

- if $m = 0$ and $|\beta| \in (0, \alpha)$, then $|\beta'| = \alpha \in [\alpha - |\beta|, \alpha + |\beta|]$;
- if $m = 0$ and $|\beta| \in (\pi - \alpha, \pi)$, then $|\beta'| = \pi - \alpha \in [(\pi - \alpha) - (\pi - \beta), (\pi - \alpha) + (\pi - \beta)]$;
- if $1 \leq m \leq n$ and $|\beta| \in (m\alpha, (m+1)\alpha)$, then $|\beta| - \alpha < m\alpha < |\beta|$, whence $|\beta'| = m\alpha \in [|\beta| - \alpha, |\beta| + \alpha]$;
- if $1 \leq m \leq n$ and $|\beta| \in (\pi - (m+1)\alpha, \pi - m\alpha)$, then $|\beta| < \pi - m\alpha < |\beta| + \alpha$, whence $|\beta'| = \pi - m\alpha \in [|\beta| - \alpha, |\beta| + \alpha]$.

Thus $\sigma'$ is a geodesic ray. Hence, $\sigma' \in \mathcal{R}(p_0)$. Since $\sigma'$ coincides with $\tau$ on $[0, u]$, it follows that $\sigma'(t_0) = \tau(t_0)$. Therefore, $d(\sigma'(t_0), \sigma(t_0)) = d(\tau(t_0), \sigma(t_0)) < \delta$. Hence, $\sigma'(\infty) \in \Phi$. Also, $\sigma'$ enters and terminates in the nonwall plane $U_1$ of $B_1$. Furthermore, $\theta(\sigma', U_1, B_1) \in \{\alpha, -\pi + \alpha\}$ if $m = 0$, and $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$ if $1 \leq m \leq n$. Thus $\theta(\sigma', U_1, B_1) \in A$. ■

**Lemma 4.4.** Suppose $n \geq 1$ and $\pi/2(n+1) \leq \alpha < \pi/2n$, $\sigma' \in \mathcal{R}(p_0)$ has itinerary $B_0, B_1$ where $B_1$ is a block of $X_\alpha$ that is adjacent to $B_0$, there is a nonwall plane $U_1$ of $B_1$ such that $\sigma'$ terminates in $U_1$ and $\theta(\sigma', U_1, B_1) \in A$, and $\Phi$ is a neighborhood of $\sigma'(\infty)$ in $\partial X_\alpha$. Then there is a ray $\sigma'' \in \mathcal{R}(p_0)$ such that $\sigma''(\infty) \in \Phi$ and there is an integer $m$ such that $1 \leq m \leq n$ and there are blocks $B_2, B_3, \ldots, B_m, B_{m+1}$ of $X_\alpha$ where $B_0, B_1, B_2, \ldots, B_m, B_{m+1}$ are distinct, $B_0, B_1, \ldots, B_m$ is the itinerary of $\sigma''$, $B_m$ is adjacent to $B_{m+1}$, and $\sigma''(\infty)$ is a pole of $B_{m+1}$.

**Proof.** Since $\sigma'$ terminates in $U_1$ and $\sigma'(\infty) \in \Phi$, it follows that there is a $t_0 > 0$ such that $\sigma'([t_0, \infty)) \subset U_1$ and such that if $\tau \in \mathcal{R}(p_0)$ and $\tau(t_0) = \sigma'(t_0)$, then $\tau(\infty) \in \Phi$. We again assume that $\theta(\sigma', U_1, B_1)$ and all other angles of inclination mentioned in this proof lie in the interval $[-\pi, \pi]$.

Since $\theta(\sigma', U_1, B_1) \in A$, it follows that there is an integer $m$ such that $1 \leq m \leq n$ and $\theta(\sigma', U_1, B_1) \in \{m\alpha, -\pi + m\alpha\}$. We will construct a ray $\sigma'' \in \mathcal{R}(p_0)$ and a sequence $B_2, \ldots, B_m, B_{m+1}$ of blocks of $X_\alpha$ with the following properties:
\begin{itemize}
  \item \(\sigma''\) coincides with \(\sigma'\) on \([0, t_0 + 1]\);
  \item \(B_0, B_1, B_2, \ldots, B_m, B_{m+1}\) are distinct, \(B_0, B_1, B_2, \ldots, B_m\) is the itinerary of \(\sigma''\), and \(B_m\) is adjacent to \(B_{m+1}\);
  \item \(B_i\) and \(B_{i+1}\) meet in a common wall \(W_{i+1}\) for \(1 \leq i \leq m\);
  \item \(\sigma''\) enters \(W_{i+1}\) for \(1 \leq i \leq m\), and \(\sigma''\) terminates in \(W_{m+1}\);
  \item if \(\theta(\sigma', U_1, B_1) = m\alpha\), then \(\theta(\sigma'', W_{i+1}, B_i) = (m + 1 - i)\alpha\) for \(1 \leq i \leq m\);
  \item if \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\), then \(\theta(\sigma'', W_{i+1}, B_i) = -\pi + (m + 1 - i)\alpha\) for \(1 \leq i \leq m\).
\end{itemize}

The construction of \(\sigma''\) is an inductive process in which a sequence \(\tau_1, \tau_2, \ldots, \tau_m\) of progressively longer geodesic segments emanating from \(p_0\) are constructed.

The first step of the process is to construct a geodesic segment \(\tau_1\) that coincides with \(\sigma'\) on \([0, t_0 + 1]\) and then enters and terminates in a wall \(W_2\) of \(B_1\) that intersects \(U_1\). Furthermore, if \(\theta(\sigma', U_1, B_1) = m\alpha\), then we insist that \(\tau_1\) enters the positive half of \(W_2\), so that \(\theta(\tau_1, W_2, B_1) > 0\). However, if \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\), then we insist that \(\tau_1\) enters the negative half of \(W_2\), so that \(\theta(\tau_1, W_2, B_1) < 0\). Transformation Rule 1 implies that \(\theta(\tau_1, W_2, B_1) = \theta(\tau_1, W_1, B_1) = \theta(\sigma', U_1, B_1)\). Hence, if \(\theta(\sigma', U_1, B_1) = m\alpha\), then \(\theta(\tau_1, W_2, B_2) = m\alpha\); whereas if \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\), then \(\theta(\tau_1, W_2, B_1) = -\pi + m\alpha\).

For the inductive step of the process, let \(1 \leq k < m\) and assume that a geodesic segment \(\tau_k\) has been constructed and that \(B_2, \ldots, B_k\) of \(X_{\alpha}\) have been chosen with the following properties:

\begin{itemize}
  \item \(\tau_k\) coincides with \(\sigma'\) on \([0, t_0 + 1]\);
  \item \(B_0, B_1, B_2, \ldots, B_k\) are distinct and form the itinerary of \(\tau_k\);
  \item \(B_i\) and \(B_{i+1}\) meet in a common wall \(W_{i+1}\) for \(1 \leq i < k\) and \(W_{k+1}\) is a wall of \(B_k\) that is disjoint from \(B_{k-1}\);
  \item \(\tau_k\) enters \(W_{i+1}\) for \(1 \leq i \leq k\), and \(\tau_k\) terminates in \(W_{k+1}\);
  \item if \(\theta(\sigma', U_1, B_1) = m\alpha\), then \(\theta(\tau_k, W_{i+1}, B_i) = (m+1-i)\alpha\) for \(1 \leq i \leq k\);
  \item if \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\), then \(\theta(\tau_k, W_{i+1}, B_i) = -\pi + (m + 1 - i)\alpha\) for \(1 \leq i \leq k\).
\end{itemize}

Let \(B_{k+1}\) be the block that meets \(B_k\) in the wall \(W_{k+1}\). Observe that \(\theta(\tau_k, W_{k+1}, B_k) = (m + 1 - k)\alpha\) whenever \(\theta(\sigma', U_1, B_1) = m\alpha\), and that \(\theta(\tau_k, W_{k+1}, B_k) = -\pi + (m + 1 - k)\alpha\) whenever \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\). Transformation Rule 2 implies that \(\theta(\tau_k, W_{k+1}, B_{k+1}) \equiv \alpha - \theta(\tau_k, W_{k+1}, B_k) \mod 2\pi\). Consequently, if \(\theta(\sigma', U_1, B_1) = m\alpha\), then \(\theta(\tau_k, W_{k+1}, B_{k+1}) = -(m - k)\alpha\); whereas if \(\theta(\sigma', U_1, B_1) = -\pi + m\alpha\), then \(\theta(\tau_k, W_{k+1}, B_{k+1}) = \pi - (m - k)\alpha\). We now extend \(\tau_k\) to a longer geodesic segment \(\tau_{k+1}\) by first making \(\tau_{k+1}\) enter a nonwall plane \(U_{k+1}\) of \(B_{k+1}\) that intersects \(W_{k+1}\). Then we make \(\tau_{k+1}\) enter and terminate in a wall \(W_{k+2}\) of \(B_{k+1}\) that intersects
it follows that Lemma 4.4, we simply extend the geodesic segment $\alpha$. Transformation Rule 1 implies that $\theta(\sigma', U_1, B_1) = m\alpha$, then we insist that $\tau_{k+1}$ enters the positive half of $W_{k+2}$ so that $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) > 0$. However, if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then we insist that $\tau_{k+1}$ enters the negative half of $W_{k+2}$ so that $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) < 0$. Transformation Rule 2 implies that $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) = (m-k)\alpha$; whereas if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\tau_{k+1}, W_{k+2}, B_{k+1}) = -\pi + (m-k)\alpha$. Also, since $\tau_{k+1}$ is an extension of $\tau_k$, it follows that $\theta(\tau_{k+1}, W_{i+1}, B_k) = \theta(\tau_k, W_{i+1}, B_i)$ for $1 \leq i \leq k$. This establishes the inductive step.

The final result of the inductive process is a geodesic segment $\tau_m$ and blocks $B_2, \ldots, B_m$ of $X_\alpha$ with the following properties:

- $\tau_m$ coincides with $\sigma'$ on $[0, t_0 + 1]$;
- $B_0, B_1, B_2, \ldots, B_m$ are distinct and form the itinerary of $\tau_m$;
- $B_i$ and $B_{i+1}$ meet in a common wall $W_{i+1}$ for $1 \leq i < m$ and $W_{m+1}$ is a wall of $B_m$ that is disjoint from $B_{m-1}$;
- $\tau_m$ enters $W_{i+1}$ for $1 \leq i \leq m$, and $\tau_m$ terminates in $W_{m+1}$;
- if $\theta(\sigma', U_1, B_1) = m\alpha$, then $\theta(\tau_m, W_{i+1}, B_i) = (m + 1 - i)\alpha$ for $1 \leq i \leq m$;
- if $\theta(\sigma', U_1, B_1) = -\pi + m\alpha$, then $\theta(\tau_m, W_{i+1}, B_i) = -\pi + (m + 1 - i)\alpha$ for $1 \leq i \leq m$.

In particular, $\theta(\tau_m, W_{m+1}, B_m) \in \{\alpha, \pi - \alpha\}$. To complete the proof of Lemma 4.4, we simply extend the geodesic segment $\tau_m$ to a geodesic ray $\sigma''$ that terminates in $W_{m+1}$. Since $\sigma''$, $\tau_m$, and $\sigma'$ all coincide on $[0, t_0]$, it follows that $\sigma''(\infty) \in \Phi$. Clearly, $B_0, B_2, \ldots, B_m$ is the itinerary of $\sigma''$. Let $B_{m+1}$ be the block that meets $B_m$ in the wall $W_{m+1}$. Then Transformation Rule 2 implies that $\theta(\sigma'', W_{m+1}, B_{m+1}) = \theta(\tau_m, W_{m+1}, B_{m+1}) \equiv \alpha - \theta(\tau_m, W_{m+1}, B_m) \mod 2\pi$. Hence, $\theta(\sigma'', W_{m+1}, B_{m+1}) \in \{0, \pi\}$. Consequently, $\sigma''(\infty)$ is a pole of $B_{m+1}$.

We now complete the proof of Theorem 4.2 in the case that $n \geq 1$ and $\pi/2(n+1) \leq \alpha \leq \pi/2n$. Recall that $\Phi$ is a nonempty open subset of $\partial X_\alpha$ that contains a point of $\partial B_0$. We must prove that $\Phi$ intersects $P(n + 1)$. Let $\sigma \in \mathcal{R}(p_0)$ be such that $\sigma(\infty) \in \Phi$. If $\sigma(\infty)$ is a pole of $B_0$, then $\sigma(\infty) \in \Phi \cap \mathcal{P}(n+1)$, and the proof is over. So assume that $\sigma(\infty)$ is not a pole of $B_0$. Then Lemma 4.3 provides a ray $\sigma' \in \mathcal{R}(p_0)$ and a block $B_1$ adjacent to $B_0$ such that $\sigma'(\infty) \in \Phi$ and either $\sigma'(\infty)$ is a pole of $B_1$ or $\sigma'$ has itinerary $B_0, B_1$, $\sigma'$ terminates in a nonwall plane $U_1$ of $B_1$, and $\theta(\sigma', U_1, B_1)$ lies in the set $A = \{k\alpha : 1 \leq k \leq n\} \cup \{-\pi + k\alpha : 1 \leq k \leq n\}$. If $\sigma'(\infty)$ is a pole of $B_1$, then $\sigma'(\infty) \in \Phi \cap \mathcal{P}(n+1)$, and the proof is over. So assume $\sigma'(\infty)$ is not a pole of $B_1$. Then Lemma 4.4 provides a ray $\sigma'' \in \mathcal{R}(p_0)$, an integer $m$ such that $1 \leq m \leq n$, and blocks $B_2, B_3, \ldots, B_m, B_{m+1}$ of $X_\alpha$.
such that \( \sigma''(\infty) \in \Phi, B_0, B_1, B_2, \ldots, B_m, B_{m+1} \) are distinct, \( B_0, B_1, \ldots, B_m \) is the itinerary of \( \sigma'' \), \( B_m \) is adjacent to \( B_{m+1} \), and \( \sigma''(\infty) \) is a pole of \( B_{m+1} \). Since \( m + 1 \leq n + 1 \), it follows that \( \sigma''(\infty) \in \Phi \cap \mathcal{P}(n + 1) \). This concludes the case that \( n \geq 1 \) and \( \pi/2(n + 1) \leq \alpha \leq \pi/2n \).

The proof of this theorem in the case that \( \alpha = \pi/2 \) parallels the previous proof. We assume that \( \Phi \) is a nonempty open subset of \( \partial X_\alpha \) that contains a point of \( \partial B_0 \), and we must prove that \( \Phi \) intersects \( \mathcal{P}(2) \). As we argued previously, there is a geodesic ray \( \sigma \in \mathcal{R}(p_0) \) that terminates in a nonwall plane \( U_0 \) of \( B_0 \) such that \( \sigma(\infty) \in \Phi \). If \( \sigma(\infty) \) is a pole of \( B_0 \), then the proof is over. So we assume that \( \sigma(\infty) \) is not a pole of \( B_0 \). Then \( |\theta(\sigma, U_0, B_0)| \) lies in one of the three sets \( (0, \pi/2), \{\pi/2\}, \text{ or } (\pi/2, \pi) \). If \( |\theta(\sigma, U_0, B_0)| = \pi/2 \), then following the proof of Lemma 4.3, we can approximate \( \sigma \) by a geodesic ray \( \sigma' \in \mathcal{R}(p_0) \) such that \( \sigma'(\infty) \in \Phi \) and \( \sigma'(\infty) \) is a pole of a block \( B_1 \) that is adjacent to \( B_0 \). In this situation, \( \sigma'(\infty) \in \Phi \cap \mathcal{P}(2) \), and again the proof is over. So we assume that \( |\theta((\sigma, U_0, B_0)| \neq \pi/2 \). Then \( |\theta((\sigma, U_0, B_0)| \) is an element of one of the two sets \( (0, \pi/2) \) or \( \pi/2, \pi) \). We follow the proof of Lemma 4.3 in this case as well, approximating \( \sigma \) by a geodesic ray \( \sigma' \in \mathcal{R}(p_0) \) such that \( \sigma'(\infty) \in \Phi \), and \( \sigma' \) passes from \( B_0 \) via a triple point into a nonwall plane \( U_1 \) of an adjacent block \( B_1 \), \( \sigma' \) terminates in \( U_1 \), and \( \theta(\sigma', U_1, B_1) = \pi/2 \). (Since \( \alpha = \pi/2 \), Transformation Rule 3 will allow a geodesic ray entering a triple point with angle of inclination in either of the intervals \( (0, \pi/2) \) or \( (\pi/2, \pi) \) to leave the triple point with angle of inclination equal to \( \pi/2 \).) Finally, we follow the proof of Lemma 4.4, approximating \( \sigma' \) by a geodesic ray \( \sigma'' \in \mathcal{R}(p_0) \) such that \( \sigma''(\infty) \in \Phi \) and \( \sigma''(\infty) \) is a pole of a block \( B_2 \) that is adjacent to \( B_1 \). Thus \( \sigma''(\infty) \in \Phi \cap \mathcal{P}(2) \). \( \blacksquare \)

Combining Theorems 4.1 and 4.2, we obtain:

**Corollary 4.5.** If \( n \geq 1 \) and \( \pi/2(n + 1) \leq \alpha \leq \pi/2n \), then \( \partial B_0 \nsubseteq \text{cl}(\mathcal{P}(n)) \) and \( \partial B_0 \subset \text{cl}(\mathcal{P}(n + 1)) \). Also, if \( \alpha = \pi/2 \), then \( \partial B_0 \nsubseteq \text{cl}(\mathcal{P}(1)) \) and \( \partial B_0 \subset \text{cl}(\mathcal{P}(2)) \).

The principal theorem of [4] established that \( \partial X_\alpha \) is not homeomorphic to \( \partial X_{\pi/2} \) whenever \( 0 < \alpha < \pi/2 \). Furthermore, it was shown in [4] that for \( \alpha, \beta \in (0, \pi/2) \), any homeomorphism between \( \partial X_\alpha \) and \( \partial X_\beta \) preserves block boundaries and block adjacency. Thus, distance in the nerve is preserved by such a homeomorphism, as are the sets \( \mathcal{P}(n) \). Consequently, any homeomorphism between \( \partial X_\alpha \) and \( \partial X_\beta \) preserves the condition that \( \partial B_0 \nsubseteq \text{cl}(\mathcal{P}(n)) \) and \( \partial B_0 \subset \text{cl}(\mathcal{P}(n + 1)) \). These observations together with Corollary 4.5 give rise to the following topological differentiation of the boundaries of the Croke–Kleiner spaces into countably many distinct classes:
Corollary 4.6. If $\alpha, \beta \in (0, \pi/2]$ and there is an integer $n \geq 1$ such that $\alpha < \pi/2n \leq \beta$, then $\partial X_\alpha$ is not homeomorphic to $\partial X_\beta$.

We note that a much finer definitive topological differentiation of the boundaries of the Croke–Kleiner spaces was proved after Corollary 4.6. Specifically, if $\alpha, \beta \in (0, \pi/2]$ and $\alpha \neq \beta$, then $\partial X_\alpha$ is not homeomorphic to $\partial X_\beta$. See [7].

References


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