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MANIFOLDS AND CELL COMPLEXES

On the Signatures of Torus Knots

by

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Summary. We study properties of the signature function of the torus knot $T_{p,q}$. First we provide a very elementary proof of the formula for the integral of the signature over the circle. We also obtain a closed formula for the Tristram–Levine signature of a torus knot in terms of Dedekind sums.

1. Preliminaries. Let K be a knot in S^3 with a Seifert matrix S. Let also $z \in S^1$, $z \neq 1$, be a complex number. The *Tristram–Levine signature* $\sigma(z)$ is the signature of the hermitian form

$$(1-z)S + (1-\bar{z})S^T.$$

This is obviously an integer-valued piecewise constant function. It does not depend on the particular choice of Seifert matrix. For $z = -1$ we get an invariant σ_{ord}, which is called the *(ordinary) signature.* We also define the integral

$$I_K = \int_0^1 \sigma(e^{2\pi i x})\, dx.$$

Signatures are very strong knot cobordism invariants, which can be used to bound the four-genus and the unknotting number of K. The integral I_K of the signature function is one of the so called ρ invariants of knots (see [COT1, COT2]) and is of independent interest.

For a torus knot $T_{p,q}$, where $\gcd(p,q) = 1$, the signature function can be expressed in the following nice way (see [Li] or [Kau, Chapter XII]):

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PROPOSITION 1.1. *Let*

$$\Sigma = \left\{ \frac{k}{p} + \frac{l}{q} : 1 \le k \le p-1,\ 1 \le l \le q-1 \right\}. \tag{1.1}$$

Then for any $x \in (0,1) \setminus \Sigma$ *we have*

$$\sigma(e^{2\pi i x}) = |\Sigma \setminus (x, x+1)| - |\Sigma \cap (x, x+1)|, \tag{1.2}$$

where $|\cdot|$ *denotes cardinality. In particular*

$$\sigma_{\mathrm{ord}} = |\Sigma \setminus (1/2, 3/2)| - |\Sigma \cap (1/2, 3/2)|.$$

Explicit formulae for σ_{ord} and I_K of torus knots have been known in the literature for quite a long time. In fact, by a result of Viro (see (2.4)) σ_{ord} is equal to τ_2, which was computed in [HZ] for p and q odd, and (denoted as $\sigma(f + z^2)$) in [Nem] in the general case. On the other hand, Kirby and Melvin [KM, Remark 3.9] and [Nem, Example 4.3] provided a formula for I_K. Nevertheless, all the above-mentioned results are related more to singularity theory and low-dimensional topology than to knot theory itself.

After the discovery of ρ invariants, the interest in computing I_K for various families of knots grew significantly. Two independent new proofs of the formula for I_K of torus knots [Bo, Co] appeared in 2009. In particular [Bo] provided a bridge between the I_K and invariants of cuspidal singularities of complex plane curves.

In this paper we present an elementary proof of the formula for I_K (Proposition 2.1). We also cite a formula of Némethi and draw some consequences from it. In Section 4 we use a theorem of Rosen to obtain the explicit value of the signature $\sigma(z)$ of a torus knot not only for $z = -1$, but also for almost every $z \in S^1 \setminus \{1\}$ (Proposition 4.3). This result seems to be new. In Section 5 we show that the formula for $\sigma_{\mathrm{ord}}(T_{p,q})$ cannot be written as a rational function of p and q.

2. Formula for the integral

PROPOSITION 2.1. *For a torus knot* $T_{p,q}$ *we have*

$$I = -\frac{1}{3}\left(p - \frac{1}{p}\right)\left(q - \frac{1}{q}\right). \tag{2.1}$$

This proposition was first proved in [KM, Remark 3.9]. We refer to [Nem, Bo, Co] for other proofs.

Proof. Let $f(x) = -\sigma(e^{2\pi i x})$ and $J = \int_0^1 f(x)\,dx = -I$. Then

$$f(x) = \sum_{y \in \Sigma} \mathbf{1}_{(x,x+1)}(y) - \sum_{y \in \Sigma} \mathbf{1}_{\mathbb{R}\setminus(x,x+1)}(y).$$

(Here, for $A \subset \mathbb{R}$, $\mathbf{1}_A$ denotes the function which is equal to 1 on A and 0 away from A.) Hence

$$J = \sum_{y \in \Sigma} \int_0^1 (\mathbf{1}_{(y-1,y)}(x) - \mathbf{1}_{\mathbb{R} \setminus (y-1,y)}(x))\, dx = \sum_{y \in \Sigma} (1 - 2|y-1|).$$

It follows that

$$J = \sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \left(1 - 2\left|\frac{k}{p} + \frac{l}{q} - 1\right|\right).$$

As for any $u, v \in \mathbb{R}$ we have

$$1 - 2|u + v - 1| = 2\min(1-u, v) + 2\min(u, 1-v) - 1,$$

it follows that

$$\begin{aligned} J &= 2\sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \min\left(\frac{p-k}{p}, \frac{l}{q}\right) + 2\sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \min\left(\frac{k}{p}, \frac{q-l}{q}\right) - (p-1)(q-1) \\ &= 4\sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \min\left(\frac{k}{p}, \frac{l}{q}\right) - (p-1)(q-1) \\ &= \frac{4}{pq} \sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \min(qk, pl) - (p-1)(q-1). \end{aligned}$$

Now, obviously,

$$\begin{aligned} &\sum_{k=1}^{p-1} \sum_{l=1}^{q-1} \min(qk, pl) \\ &\quad = \sum_{s=0}^{\infty} |\{(k,l) \in \{1, \ldots, p-1\} \times \{1, \ldots, q-1\} \colon qk > s \text{ and } pl > s\}| \\ &\quad = \sum_{s=0}^{pq-1} (p - 1 - \lfloor s/q \rfloor)(q - 1 - \lfloor s/p \rfloor). \end{aligned}$$

We can multiply the expressions in parentheses. Then, as

$$\sum_{s=0}^{pq-1} \lfloor s/p \rfloor = p \sum_{l=0}^{q-1} l = \frac{1}{2} pq(q-1),$$

we get

$$\sum_{s=0}^{pq-1}(p-1-\lfloor s/q\rfloor)(q-1-\lfloor s/p\rfloor)$$
$$=pq(p-1)(q-1)-\frac{1}{2}pq(p-1)(q-1)-\frac{1}{2}pq(p-1)(q-1)+\sum_{s=0}^{pq-1}\lfloor s/p\rfloor\lfloor s/q\rfloor$$
$$=\sum_{s=0}^{pq-1}\lfloor s/p\rfloor\lfloor s/q\rfloor.$$

It remains to compute the last sum. To this end denote by $R_p(s)$ the remainder of s modulo p. Then

$$\begin{aligned}\sum_{s=0}^{pq-1}\lfloor s/p\rfloor\lfloor s/q\rfloor &= \sum_{s=0}^{pq-1}\left(\frac{s-R_p(s)}{p}\cdot\frac{s-R_q(s)}{q}\right)\\ &=\frac{1}{pq}\left(\sum_{s=0}^{pq-1}s^2-\sum_{s=0}^{pq-1}sR_p(s)-\sum_{s=0}^{pq-1}sR_q(s)+\sum_{s=0}^{pq-1}R_p(s)R_q(s)\right)\\ &=\frac{1}{3}p^2q^2+\frac{1}{4}pq-\frac{1}{4}p^2q-\frac{1}{4}pq^2-\frac{1}{12}p^2-\frac{1}{12}q^2+\frac{1}{12},\end{aligned}$$

where we used the fact that

$$\sum_{s=0}^{pq-1}R_p(s)R_q(s)=\sum_{k=0}^{p-1}\sum_{l=0}^{q-1}kl$$

by the Chinese remainder theorem.

Putting all the pieces together we obtain the desired formula. ■

Let us now present another proof, due to Némethi [Nem] (see also [Br, HZ]). Before we do this, we recall some facts from topology.

Assume that the knot K is drawn on $S^3=\partial B^4$ and consider a Seifert surface F of K. Let us push it slightly into B^4 and, for an integer m, let N_m be the m-fold cyclic cover of B^4 branched along F. Then the quantity $\tau_m=\sigma(N_m)$ (here σ is the signature of a four-manifold with boundary) is independent of the choices made. We have the following formula essentially due to Viro (see [GLM, Section 2] or [Vi]):

$$\tau_m=\sum_{k=1}^{m-1}\sigma_K(\xi^k), \tag{2.2}$$

where ξ is a primitive root of unity of order m. In particular, since σ is a

Riemann integrable function, we have

$$I = \int_0^1 \sigma(e^{2\pi i x})\,dx = \lim_{m\to\infty} \frac{1}{m}\tau_m. \tag{2.3}$$

On the other hand

$$\tau_2(K) = \sigma_{\mathrm{ord}}(K). \tag{2.4}$$

If K is a torus knot $T_{p,q}$ and m, p, q are pairwise coprime, then the m-fold cover of S^3 branched along K is diffeomorphic to the Brieskorn homology sphere $B(p, q, m)$ (see [Br], [GLM, Section 5]). Then τ_m turns out [HZ, Sections 10.2 and 11] to be the signature of the manifold $X_{p,q,m}$ defined as the intersection of $z_1^p + z_2^q + z_3^m = \varepsilon$ with $B(0, 1) \subset \mathbb{C}^3$. In this context τ_m was computed in [HZ, Formula (11), p. 122] and [Nem, Example 4.3]. Especially the last formula is worth citing (Némethi uses $m(S(f))$ to denote the limit (2.3)):

$$I = -4(s(p, q) + s(q, p) + s(1, pq)). \tag{2.5}$$

Here $s(a, b)$ is the Dedekind sum (see Section 3). As by elementary computations

$$s(1, pq) = \frac{(pq-1)(pq-2)}{12pq},$$

we get

$$s(p, q) + s(q, p) = -\frac{I}{4} - \frac{(pq-1)(pq-2)}{12pq}.$$

Now we can view the above equation as defining I in terms of $s(p, q)+s(q, p)$; but if we know I, we know $s(p, q)+s(q, p)$. In other words we get the following observation.

Corollary 2.2. *Any proof of Proposition* 2.1 *which does not involve Dedekind sums provides a proof of the Dedekind reciprocity law.*

3. Lattice points in the triangle. Let us recall basic definitions. For a real number x, $\lfloor x \rfloor$ denotes the integer part and $\{x\} = x - \lfloor x \rfloor$ the fractional part. The *sawtooth function* is defined by

$$\langle x \rangle = \begin{cases} \{x\} - 1/2, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Sometimes $\langle x \rangle$ is denoted $((x))$. We prefer the former notation because it does not lead to confusion with ordinary parentheses. We can now define the following functions (below p, q and m are integers and x, y are real numbers):

$$s(p,q) = \sum_{j=0}^{p-1} \left\langle \frac{j}{q} \right\rangle \left\langle \frac{pj}{q} \right\rangle,$$

$$s(p,q;x,y) = \sum_{j=0}^{p-1} \left\langle \frac{j+y}{q} \right\rangle \left\langle p\frac{j+y}{q} + x \right\rangle.$$

$s(p,q)$ is called the (ordinary) *Dedekind sum*, while $s(p,q;x,y)$ is a *generalized Dedekind sum*. These functions satisfy the following reciprocity laws (see [RG, HZ]). If m, p and q are pairwise coprime, then

$$s(p,q) + s(q,p) = \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right) - \frac{1}{4}, \tag{3.1}$$

$$\begin{aligned} s(p,q;x,y) + s(q,p;y,x) = &-\frac{1}{4}d(x)d(y) + \langle x\rangle\langle y\rangle \\ &+ \frac{1}{2}\left(\frac{q}{p}\Psi_2(y) + \frac{1}{pq}\Psi_2(py+qx) + \frac{p}{q}\Psi_2(x)\right). \end{aligned} \tag{3.2}$$

Here

$$d(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Psi_2(x) = B_2(\{x\}) = \{x\}^2 - \{x\} + \frac{1}{6}$$

is the second Bernoulli polynomial. Now for a fixed $C \in [0,1)$ and p, q coprime, let

$$A(p,q;C) = \left\{(k,l) \in \mathbb{Z}^2_{\geq 0} \colon 0 \leq \frac{k}{p} + \frac{l}{q} < 1 - C\right\}$$

and

$$N(p,q;C) = |A(p,q;C)|.$$

We have the following result due to Rosen [Ro, Theorem 3.4].

PROPOSITION 3.1. *In this case*

$$\begin{aligned} N(p,q;C) = &\frac{(1-C)^2}{2}pq + \frac{1-C}{2}(p+q) + \frac{q}{12p} + \frac{p}{12q} + K \\ &- s(p,q;Cp,0) - s(q,p;Cq,0) + \langle Cp\rangle + \langle Cq\rangle \\ &+ (1-C)\langle Cpq\rangle - \left(\frac{7}{8}\delta_0 + \frac{3}{8}\delta_1 - \frac{1}{8}\delta_2\right) + \frac{1}{4}, \end{aligned} \tag{3.3}$$

where

$$K = \begin{cases} \dfrac{1}{12pq} - \dfrac{1}{8} & \textit{if } Cpq \in \mathbb{Z}, \\ \dfrac{1}{2pq}\Psi_2(Cpq) & \textit{otherwise,} \end{cases}$$

and for $r = 0, 1, 2$, δ_r *is the number of non-negative integers* k, l *such that* $k/p + l/q + C = r$.

This proposition has an important corollary [Ro, Corollary 3.5].

COROLLARY 3.2. *If* p *and* q *are odd and coprime, then*

$$N\left(p, q; \frac{1}{2}\right) = \frac{pq}{8} + \frac{p+q}{4} + \frac{q}{6p} + \frac{p}{6q} + \frac{1}{24pq} - s(2p, q) - s(2q, p). \tag{3.4}$$

If p *and* q *are coprime and* q *is even, then*

$$N\left(p, q; \frac{1}{2}\right) = \frac{pq}{8} + \frac{p+q}{4} - s(2p, q) + 2s(p, q). \tag{3.5}$$

We shall use these results to compute the signature of the torus knots. We need the following trivial lemma:

LEMMA 3.3. *The number of points* $(k, l) \in A(p, q; C)$ *such that* $kl = 0$ *is equal to*

$$Z(p, q; C) = \lfloor (1 - C)p \rfloor + \lfloor (1 - C)q \rfloor + 1 - d((1 - C)p) - d((1 - C)q),$$

where $d(x)$ *is again* 1 *if* $x \in \mathbb{Z}$, *and* 0 *otherwise.*

If Cp and Cq are not integers, Lemma 3.3 says that

$$Z(p, q; C) = (1 - C)(p + q) - \langle (1 - C)p \rangle - \langle (1 - C)q \rangle.$$

4. Explicit formulae for the signatures. We begin by computing the value of the ordinary signature. As already mentioned, $\sigma_{\mathrm{ord}} = \tau_2$ (see (2.4)) so the first result below is in general known [HZ, Nem], but not necessarily in the context of knot theory.

PROPOSITION 4.1. *If* p *and* q *are both odd and coprime, then the ordinary signature of the torus knot* $T_{p,q}$ *satisfies*

$$\sigma_{\mathrm{ord}}(T_{p,q}) = -\frac{pq}{2} + \frac{2p}{3q} + \frac{2q}{3p} + \frac{1}{6pq} - 4(s(2p, q) + s(2q, p)) - 1,$$

where $s(x, y)$ *is the Dedekind sum* (see Section 3 or [RG]; cf. [HZ, Formula (11), p. 122]). *If* p *is odd and* $q > 2$ *is even, then*

$$\sigma_{\mathrm{ord}}(T_{p,q}) = -\frac{pq}{2} + 1 + 4s(2p, q) - 8s(p, q).$$

Proof. Let Σ be as in (1.1). We can write σ_{ord} as

$$\sigma_{\mathrm{ord}} = 4|\Sigma \cap (0, 1/2)| - |\Sigma|. \tag{4.1}$$

Since $|\Sigma| = (p-1)(q-1)$, we need to find a closed formula for

$$S(p,q) = |\Sigma \cap (0,1/2)| = \left| \left\{ x = \frac{k}{p} + \frac{l}{q} : x < \frac{1}{2},\ 1 \le k \le p-1,\ 1 \le l \le q-1 \right\} \right|. \tag{4.2}$$

From the definition we get immediately

$$S(p,q) = N(p,q;1/2) - Z(p,q;1/2).$$

Now $Z(p,q;1/2) = \frac{1}{2}(p+q)$ if p and q are both odd, and $\frac{1}{2}(p+q-1)$ if q is even and $q > 2$. Hence, for p and q odd we have, by (3.4),

$$S(p,q) = \frac{pq}{8} - \frac{p+q}{4} - s(2p,q) + 2s(p,q),$$

while for q even we have, by (3.5),

$$S(p,q) = \frac{pq}{8} - \frac{p+q}{4} + \frac{1}{2} - s(2p,q) + 2s(p,q),$$

and using (4.1) we complete the proof. ∎

Remark 4.2. Formula (4.2) can be rewritten as

$$S(p,q) = \sum_{k=1}^{\lfloor p/2 \rfloor} \left\lfloor \frac{qp - 2kq}{2p} \right\rfloor, \tag{4.3}$$

which gives a formula for σ_{ord} using ordinary sums, not Dedekind sums. On the other hand [NY, Proposition 2.1] provides a different formula for σ_{ord} using ordinary sums. The latter is especially useful for providing explicit formulae for $\sigma_{\mathrm{ord}}(T_{p,p+r})$ for small values of r.

To express explicitly the values of Tristram–Levine signatures at other points let us assume that Cpq is not an integer (in particular $C \notin \Sigma$). Define

$$\begin{aligned} M(p,q;C) &= N(p,q;C) - Z(p,q;C) \\ &= \frac{(1-C)^2}{2}pq - \frac{1-C}{2}(p+q) \\ &\quad + \frac{q}{12p} + \frac{p}{12q} - s(p,q;Cp,0) - s(q,p;Cq,0) + \frac{1}{4} \\ &\quad - \frac{1}{2}(\langle Cp \rangle + \langle Cq \rangle) + (1-C)\langle Cpq \rangle + \frac{1}{2pq}\Psi_2(Cpq). \end{aligned}$$

Now it is a trivial consequence of Proposition 1.1 that if $C \in [0,1)$ and $e^{2\pi i C} = z$, then

$$\sigma(z) = -(p-1)(q-1) + 2M(p,q;C) + 2M(p,q;1-C).$$

Since for any integer k and real x we have $\langle (1-x)k \rangle + \langle xk \rangle = 0$, the formula for $M(p,q;C)+M(p,q;1-C)$ can be simplified to

$$\frac{1-2C+2C^2}{2}pq-\frac{1}{2}(p+q)+\frac{q}{6p}+\frac{p}{6q}+(1-2C)\langle Cpq\rangle+\frac{1}{pq}\left(\langle Cpq\rangle^2-\frac{1}{12}\right)+\frac{1}{2}$$
$$-s(p,q;Cp,0)-s(q,p;Cq,0)-s(p,q;(1-C)p,0)-s(q,p;(1-C)q,0).$$

Hence we obtain the following result.

Proposition 4.3. *If $z=e^{2\pi iC}$ where $C\in[0,1)$ is such that Cpq is not an integer, then the signature of the torus knot $T_{p,q}$ can be given by the following formula:*

$$\sigma(z)=-2(C-C^2)pq+\frac{q}{3p}+\frac{p}{3q}+(2-4C)\langle Cpq\rangle+\frac{2}{pq}\left(\langle Cpq\rangle^2-\frac{1}{12}\right)$$
$$-2(s(p,q;Cp,0)+s(q,p;Cq,0)+s(p,q;(1-C)p,0)+s(q,p;(1-C)q,0)).$$

In particular we see rigorously that for large p and q the shape of the function $\sigma(e^{2\pi ix})$ resembles that of the function $2pq(x^2-x)$.

Remark 4.4. We can integrate the above formula over the interval $[0,1]$ with respect to C. The term $-2(C-C^2)pq$ contributes $-pq/3$, and the next two terms contribute $q/3p$ and $p/3q$, respectively. A straightforward computation gives

$$\int_0^1 C\langle Cpq\rangle\,dC=\frac{1}{12pq}\quad\text{and}\quad\int_0^1\left(\langle Cpq\rangle^2-\frac{1}{12}\right)dC=0.$$

All other integrals trivially vanish. We thus recover formula (2.1) from Proposition 4.3.

5. Expressing $\sigma_{\mathrm{ord}}(T_{p,q})$ as a rational function

Proposition 5.1. *There* does not *exist a rational function $R(p,q)$ such that for all odd and coprime positive integers p,q,*

$$R(p,q)=\sigma_{\mathrm{ord}}(T_{p,q}).$$

Proof. Assume the contrary. Then $S(p,q)=\frac{1}{4}(R(p,q)+(p-1)(q-1))$ is also a rational function.

If $p\,|\,(q-1)$ and p,q are both odd, the value of $S(p,q)$ can be easily computed using (4.3):

$$S(p,q)=\sum_{k=1}^{(p-1)/2}\left\lfloor\frac{q}{2}-\frac{qk}{p}\right\rfloor=\sum_{k=1}^{(p-1)/2}\left\lfloor\frac{q-1}{2}-\frac{(q-1)k}{p}+\frac{p-2k}{2p}\right\rfloor$$
$$=\sum_{k=1}^{(p-1)/2}\left(\frac{q-1}{2}-k\frac{q-1}{p}\right)=\frac{(q-1)(p-1)^2}{8p}.$$

Since for infinitely many values (p,q) with $q = np+1$ with p odd and n even, we have $p \mid (q-1)$, it follows that

$$S(p,q) = \frac{(q-1)(p-1)^2}{8p}$$

on each line $q = np+1$. Since these rational functions agree on infinitely many lines, they must be equal.

Now assume that $p = nq+1$ for some even n. Similar arguments to those above show that $S(p,q)$ must also be identical to the function

$$\frac{(p-1)(q-1)^2}{8q}.$$

This is a contradiction, since these two rational functions are different. ■

REMARK 5.2. We can also compute values of $S(p,q)$ in many other cases, like $q = np-1$, $q = p+2$. With more care we can prove that e.g. $S(p,q) - \lfloor q/p \rfloor$ is not a rational function.

The proof carries over to show that no such rational function exists for the case p even and q odd. We leave the obvious details to the reader.

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